Federal University of Alagoas
Graduate Program in Mathematics

Ph.D. Program Entrance Exam

Date: December 3rd, 2018 **Time**: 15h30 - 18h30

Candidate: ________________________________

Q1- Suppose that $f:\mathbb{R}^n\to\mathbb{R}^m$ is continuous at $a\in\mathbb{R}^n$. Prove that if $f(a)$ does not belong to the closed ball $B[b;r]\subset\mathbb{R}^m$, then there exists $\delta>0$ such that $f(x)$ does not belong to $B[b;r]$ for all $x\in\mathbb{R}^n$ satisfying $|x-a|<\delta$.

Solution: Since $f(a)\notin B[b;r]$, we have $|f(a)-b|>r$, that is, $|f(a)-b|-r>0$. Since f is continuous at a, there exists $\delta>0$ such that $|f(x)-f(a)|<|f(a)-b|-r$ for all $x\in\mathbb{R}^n$ satisfying $|x-a|<\delta$. Then, it follows from the Triangle Inequality that

$$\begin{aligned} |f(x)-b| &= |f(x)-f(a)+f(a)-b| \\ &\geq |f(a)-b|-|f(x)-f(a)| \\ &> r \end{aligned}$$

for all $x\in\mathbb{R}^n$ satisfying $|x-a|<\delta$.

Q2- Suppose that $f:\mathbb{R}^n\to\mathbb{R}$ is a differentiable function such that $f(x/2)=f(x)/2$ for each $x\in\mathbb{R}^n$. Prove that f is a linear functional.

Solution: Since $f(x/2)=f(x)/2$ for each $x\in\mathbb{R}^n$, we have $f(0)=f(0/2)=f(0)/2$, that is, $f(0)=0$. By the other hand, it follows directly from the hypothesis that

$$f\left(\frac{x}{2^k}\right)=\frac{f(x)}{2^k}$$

for all $k\in\mathbb{N}$. So, since f is differentiable, we have

$$f'(0)\cdot x=\lim_{t\to 0}\frac{f(tx)}{t}=\lim_{k\to\infty}\frac{f\left(\frac{x}{2^k}\right)}{\frac{1}{2^k}}=\lim_{k\to\infty}f(x)=f(x).$$

This proves that $f=f'(0)$, which is a linear functional.

Q3- Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ given by

$$f(x,y)=\begin{cases}\frac{x^3y^2}{x^4+y^4}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0),\end{cases}$$

a) Show that there exists $\frac{\partial f}{\partial v}(0,0)$, for all $v\in\mathbb{R}^2$.
b) Is the equality $\langle\nabla f(0,0),v\rangle=\frac{\partial f}{\partial v}(0,0)$ holds?
c) What can be said about the differentiability of f at $(0,0)$?

Solution:
a) Set $v=(a,b)$. By a straightforward computation we have that

$$\frac{\partial f}{\partial v}(0,0)=\lim_{t\to 0}\left(\frac{f(ta,tb)-f(0,0)}{t}\right)=\frac{a^3b^2}{a^4+b^4}.$$

So, there exists $\frac{\partial f}{\partial v}(0,0)$ and it is equal to $\frac{a^3b^2}{a^4+b^4}$.
b) False. Indeed, $\nabla f(0,0)=(0,0)$, however $\frac{\partial f}{\partial v}(0,0)$ is not always zero.

c) No, because of the previous item.

Q4- Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function satisfying

$$|f(x) - f(y)| \leq |x - y|^2$$

for all $x, y \in \mathbb{R}^m$. Show that f is a constant function.

Solution: Notice that f is differentiable and that $df(x) = 0$ for all $x \in \mathbb{R}^m$. Indeed,

$$f(x + h) = f(x) + O(h) + r(h),$$

where O is the identically zero map and r satisfies

$$\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = \lim_{h\to 0} \frac{\|f(x + h) - f(y) - O(h)\|}{\|h\|} \leq \limsup_{h\to 0} \frac{\|h\|^2}{\|h\|} = 0.$$

Thus, $\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = 0$ and so f is differentiable and $df(x) = 0$. Using that $\mathbb{R}^m$ is connected, we conclude that f is constant.

Q5- Let $U \subset \mathbb{R}^2$ be an open set in $\mathbb{R}^2$. Let $a(x, y)$ and $b(x, y)$ be positive functions defined on $U \cup \partial U$, where ∂U denotes the boundary of U. Suppose that the quadratic form given by the matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

is positive definite for all (x, y). Given a function of class C^2, v, defined on $U \cup \partial U$ we define the operator L by

$$Lv = a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial^2 v}{\partial y^2},$$

which with this positivity condition is called of elliptic operator. A function f is said to be strictly subharmonic relative to L if $Lv > 0$. Show that a strictly subharmonic function cannot attain its maximum value at any point of U.

Solution: In an interior point of maximum we would have $\frac{\partial^2 v}{\partial x^2} \leq 0$ e $\frac{\partial^2 v}{\partial y^2} \leq 0$. Since a and b are nonnegative, then Lv at this point of maximum is nonpositive, which contradicts the hypothesis of $Lv > 0$.