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Optimized and Quantitative Estimates for Parameter Exclusions in One-dimensional Dynamics and Periodic Orbits in Three Dimensional Differential Equations

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Orientador: Prof. Dr. Ali Golmakani.

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Abstract

In the first part of this work we give analytic techniques to get an explicit lower bound for the Lebesgue measure of stochastic parameters for the real quadratic family of one-dimensional maps. We then find some optimized relations between the constants that are being used in the mentioned analytic proof of positiveness of the measure of the set of stochastic parameters. In the second part we consider systems of nonlinear third-order differential equations depending on a small parameter such that the unperturbed system has $\mathbb{R}^3$ as a manifold of periodic solutions. We obtain a displacement function for the system in two cases. At first case we give some sufficient conditions under which some of the periodic solutions to persist after small perturbation. At the second case we reduce the dimension of displacement function to a lower dimension.

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Chapter 1

Introduction

1.0.1 Introduction

The real quadratic family of one-dimensional maps $f_a : \mathbb{R} \to \mathbb{R}$ given by

$$f_a(x) = a - x^2$$

where $x, a \in \mathbb{R}$, is perhaps one of the most intensively studied family of dynamical systems which exhibits an incredibly rich variety of dynamical behaviour and a complex pattern of bifurcations. We remind that a probability measure μ on M is invariant under f if $\mu(f^{-1}(A)) = \mu(A)$ for every μ - measurable set $A \subset M$. And we say that μ is ergodic if there does not exist a measurable set A with $f^{-1}(A) = A$ and $\mu(A) \in (0,1)$ In other words, any fully invariant set A, i.e. a set for which $f^{-1}(A) = A$, has either zero or full measure.The measure μ is absolutely continuous with respect to m if $m(A) = 0$ implies $\mu(A) = 0$ for every measurable set $A \subset M$.

Definition 1.0.1. *For $a \in [-1/4, 2]$, we say that f_a exhibits **regular** dynamics if there* *exists a unique periodic orbit which attracts Lebesgue almost every $x \in I$.*

Definition 1.0.2. *We say that f_a exhibits* ***stochastic*** *dynamics if it admits a unique ergodic invariant probability measure μ which is absolutely continuous with respect to Lebesgue.*

While lots of questions remain open, it is known that for Lebesgue almost every parameter $a \in \mathbb{R}$, the dynamics is either *regular* or *stochastic* [24] . Moreover it was proved that the set of parameters corresponding to regular dynamics is *open and dense* [13, 23], whereas that corresponding to stochastic dynamics is *nowhere dense* and contained in the interval

$$\Omega := [1.4, 2].$$

In particular, all parameters outside Ω are regular. We let

$$\Omega^+ := \{a \in \Omega : f_a \text{ is stochastic}\} \quad \text{and} \quad \Omega^- := \{a \in \Omega : f_a \text{ is regular}\}$$

denote the set of stochastic and regular parameters in Ω. For the first time Ulam and von Neumann showed that the parameter $a = 2$, exhibits stochastic behaviour and so the set of stochastic parameters is not empty. So far this is still the only explicit parameter known to be stochastic. Moreover, later Jakobson proved that the set of parameters corresponding to stochastic dynamics has positive measure in parameter space. Since the set of regular parameters is open it also has positive measure and so a natural question, which has been open for more than thirty years, regards the *relative probability* of the sets Ω^+ and Ω^-.

This has turned out to be a very difficult problem and there seems to be no heuristic arguments to even take any kind of educated guess. Numerical calculations performed rigorously in [39], and showing remarkable agreement with much earlier non-rigorous calculations of [33], have proved that for the logistic family regular parameters account for at least 10% of the corresponding relevant parameter interval. It seems likely that analogous calculations should lead to an explicit lower bound of the same order of magnitude. Let $|\cdot|$ denote Lebesgue measure.

In a recent work by Golmakani, Luzzatto and Pilarczyke [5] they could find a formula that gives a lower bound η for the proportion of stochastic parameters in every small parameter interval $\Omega \subset [1.4, 2]$. This formula outputs η as a function of a number of constants which depend on the parameter interval under consideration. These constants can be rigorously computed and are required to satisfy a certain number of formal constraints for the formula to work.

In this work we find some relations for the constants that are used to find the formula which has been gotten in their work. Indeed to find this formula they used a set of constants and conditions to prove that the set of stochastic parameters has positive measure. However to apply this formula for every subset of parameters inside Ω it is necessary to get an optimistic relation between these constants and conditions.

Before we end this section we give some more explanations about the stochastic dynamics and also mention some recent works regarding the explicit lower bound of stochastic dynamics in the real quadratic family.

1.0.2 Stochastic Dynamics

It is known that a sufficient condition for f_a to be stochastic is that its derivative along the critical orbit tends to infinity. In this work we shall construct a set $\Omega^* \subset \Omega$ with the property that

$$|(f_a^n)'(c_0)| \geq e^{\tilde{\lambda} n^{\beta}}$$

for some constant $\tilde{\lambda}, \beta > 0$ and all $n \geq 1$, where $c_0 = c_0(a) = f_a(c)$ denotes the critical value for f_a. In particular, all parameters in Ω^* are stochastic, i.e. $\Omega^* \subseteq \Omega^+$. We shall then estimate the measure of Ω^* in terms of certain computable quantities related to the family $\{f_a\}_{a\in\Omega}$.

In spite of the fact that regular dynamics is "typical" from a topological point of view, it turns out that Ω^- does not have full measure in Ω and thus its complement is not negligible. More recently Lyubich [24] proved that Lebesgue almost every $a \in \mathbb{R}$ is either regular or stochastic.

Jakobson's Theorem that stochastic parameters have positive Lebesgue measure is, on the other hand, much more "robust", and over the years, this remarkable result has been extended and generalised a number of times to more general families, including families of maps with discontinuities, infinite derivatives or even an infinite number of critical points [6, 8, 12, 16, 21, 29, 31, 32, 35, 37]. We emphasize however that Jakobson's original Theorem and its generalizations mentioned above are based on a "perturbative" argument, showing that a sufficiently small neighbourhood of a "particularly good" stochastic parameter value itself must contain a positive Lebesgue measure of stochastic parameters. None of these papers give any quantitative estimates on the size of the neighbourhood or on the specific Lebesgue measure of the set of stochastic parameters, though most of them yield asymptotic estimates which show that the chosen good parameter is a Lebesgue density point of stochastic parameters and, in the case of [37], even some estimate on the asymptotic rate of convergence of the measure to full measure as the neighbourhood shrinks.

1.0.3 Explicit estimates for the probability of stochastic dynamics

It may seem that leading to explicit bounds for the measure of stochastic parameters, is simply a technical question of keeping track of the constants, this is in fact not at all the case.

First of all there is the key issue of assuming the existence of a "good" parameter value in the chosen parameter interval Ω. Apart from the very exceptional case of the parameter value $a = 2$ there are so far no other explicitly known stochastic parameter values. Certain similarly exceptional parameters in which the critical point is pre-periodic with low period could in principle also be calculated explicitly [30], but in general it is essentially *undecidable*, both from a practical as well as a theoretical point of view [1], if a given parameter is stochastic or even if a given interval of parameter values Ω contains a stochastic parameter.

Secondly, even assuming the existence of a good parameter value, explicit estimates such as the size of the neighbourhood Ω in which the "parameter exclusion" argument can be carried out and the final estimate on the proportion of stochastic parameters, depend, as we shall below, on several quite non trivial quantitative characteristics which are required to hold uniformly in Ω. Knowing the values of these characteristics for a reference parameter in Ω is not necessarily sufficient to estimate the size of the parameter neighbourhood in which they would continue to hold.

These observations indicate that, while the basic structure of an inductive parameter exclusion argument, which is common to all the versions of these results, can still be applied, new ideas need to be developed in order to apply the argument to general parameter intervals and to obtain explicit measure bounds. We give here a very brief overview of these ideas and discuss them more in detail at relevant parts of the proof.

The first paper to set up an explicit "quantitative" parameter exclusion argument is

Jakobsons paper [15]. The first explicit lower bound for the probability of stochastic parameters was obtained in [22] where it was proved that $|\Omega^+| \geq 10^{-5000}$. More recently this estimate has been improved in two as yet unpublished works: [14] where an estimate of $|\Omega^+| \geq 10^{-22}$ was obtained, and in [34] where an estimate of $|\Omega^+| \geq 3.89 \times 10^{-5}$ was obtained. The present work is the first that approaches the problem with a systematic combination of analytic and numerical techniques in order to achieve some good global bounds.

Chapter 2

Preliminary

2.0.1 Preliminary

As we introduced in previous chapter, we consider the real quadratic family of one-dimensional maps $f_a : \mathbb{R} \to \mathbb{R}$ given by

$$f_a(x) = a - x^2$$

where $x, a \in \mathbb{R}$.

To motivate and provide a context for our results, in this part , we give a brief description of some well known properties of this map.

2.0.2 Dynamic of Quadratic family

It is clear that the map f_a has two fixed points, $q = q_a = \frac{-1+\sqrt{1+4x^2}}{2} \geq p_a = p = \frac{-1-\sqrt{1+4x^2}}{2}$ (which coincide for $a = -1/4$ and are distinct for all $a > -1/4$), and letting $I = [p, -p]$ (notice that $p < 0$ for all $a \geq -1/4$). We remark first of all that the dynamics outside the parameter interval $[-1/4, 2]$ is particularly "simple". Indeed, it is quite easy to see that for $a < -1/4$ we have $f_a(x) \to -\infty$ for all $x \in \mathbb{R}$, and less trivial but well known, that for $a > 2$ we have that $f_a(x) \to -\infty$ for all $x \in \mathbb{R}$ except for a zero Lebesgue measure Cantor set of points on which f_a is topologically conjugate to a full shift on two symbols, see e.g. [11].

On the other hand, for every $a \in [-1/4, 2]$ there exists an interval $I = I_a$ such that $f_a(I) \subseteq I$ and such that $f_a(x) \to -\infty$ for all $x \notin I$. As mensioned abow this interval can be given quite explicitly by noticing that for all $a \geq -1/4$ the map f_a has two fixed $q = q_a \geq p_a = p$, and letting $I = I_a = [p, -p]$ (notice that $p < 0$ for all $a \geq -1/4$). Thus, for parameters $a \in [-1/4, 2]$, non trivial dynamics may occur in the interval I. From now on we restrict our attention to this range of parameters and for simplicity, when discussing the dynamics of the map f_a, we will always implicity refer to f_a restricted to the corresponding interval I. In this range of parameters we can give the following formal definition.

So by definition of Regular parameters every $a \in [-1/4, 3/4)$ is clearly periodic since we have $|(f_a)'(q)| < 1$ and thus the fixed point q is *attracting*, and, it is easy to show that $f^n(x) \to q$ for all $x \in I$. At $a = a_1 = 3/4$, the family exhibits the first *period doubling* bifurcation where the fixed point q loses its stability and thus becomes a *repelling* fixed point, since $|(f_a)'(q)| > 1$ for all $a > 3/4$ and a new global attractor appears as a new *attracting periodic orbit* of period 2 which attracts every $x \in I$ except for the preimages of the two repelling fixed points p, q. A second period doubling bifurcation occurs at $a_2 = 5/4$ where this periodic orbit loses its stability and a new attracting periodic orbit of period 4 appears, and attracts every point in the interval I except for the preimages of the repelling fixed and periodic orbits. Further period doubling bifurcations occur at parameter values $a_3 \approx 1.368...$, $a_4 = 1.394...$, $a_5 \approx 1.399...$, giving rise to the so-called infinite *cascade of period doubling* bifurcations a_1, a_2, a_3.... which was first systematically studied by Feigenbaum in the 1970's. This sequence of parameters has some remarkable properties, many of which were initially conjectured by Feigenbaum, we refer the interested reader to [11, 10] for further information. For our purposes we just mention that the sequence of period doubling bifurcations a_n is monotone increasing and bounded above and satisfies $\lim_{n\to\infty} a_n = a_\infty \approx 1.4011...$ This discussion implies that all parameters $a \notin \Omega := [1.4, 2]$ are regular parameters and it is therefore sufficient to restrict our attention to the interval Ω. Simple numerical studies produce the so-called *bifurcation diagram* which shows (a numerical approximation of) the "asymptotic" distribution in phase space of some chosen initial condition, see Figure 2.0.2. This clearly

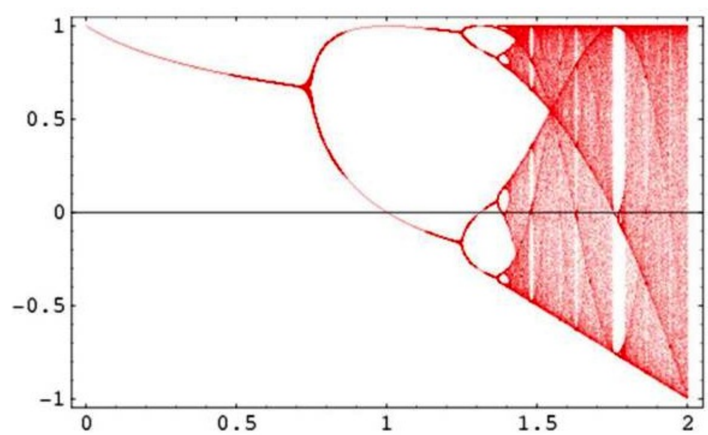

Figure 2.1: Bifurcation diagram for the quadratic family

shows the existence of a unique fixed point for smaller values of the parameter a, and the first few period doubling bifurcations as a is increased. As a approaches the limiting value a_∞ the limited resolution of the picture makes it hard to identify the attracting periodic orbit which has very high period. Similarly, apart from a small number of parameter "windows" in Ω in which an attracting periodic orbit of low period is clearly visible, it is essentially impossible to establish in any immediate or naive way the precise dynamics of any given parameter value. Indeed, numerical simulations for most parameter values in Ω indicate a very unpredictable or random-like behaviour, also sometimes referred to as chaotic behaviour, which has led to referring to the parameter a_∞ as the *onset of chaos*. The situation is in fact even more complicated and subtle that it first appears due to the remarkable fact Ω^- is open and dense in Ω. [13, 23], i.e. that periodic behaviour is *typical* from a topological point of view. More recently several extensions of this result have appeared showing that in fact regular dynamics is open and dense in essentially any reasonable topology in the space of all sufficiently smooth one-dimensional dynamical systems [17, 18, 19]

Chapter 3

Constants and conditions

3.1 Constants and conditions

We introduce and formulate now series of constants and conditions for the family of maps $\{f_a\}$, in terms of the primitive constants stated above. Before to continue we first introduce some constants, notations and definitions.

3.1.1 Some Constants

We assume that the set of *independent variables*

$$\mathcal{V} = \{\Omega, \delta, \delta^+, \alpha, \beta, \nu, \gamma\}$$

is fixed (we include the choice of the dynamical parameter interval Ω) and satisfies the following property

$$0 < \nu < \alpha < \beta < 1, \quad \gamma \in [\frac{1}{2}, 1). \tag{$\mathcal{P}1$}$$

We also assume that we have a set of *primitive* positive constants

$$\mathcal{C} = \{N_0, N, \Gamma, \lambda, \tilde{\lambda}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{B}, \{\gamma_i\}_{i=1}^{\infty}\}.$$

Let

$$\mu_\delta := (\ln \frac{1}{\delta})^{\frac{1}{\alpha}}, \quad \mu_{\delta^+} := (\ln \frac{1}{\delta^+})^{\frac{1}{\alpha}}, \quad \nu' := 1 + \text{ integer part of } (\frac{2}{\nu})$$

and

$$\eta := \exp(-\frac{\nu'!}{(1 - e^{(\gamma-1)\mu_\delta^{\nu}})(1-\gamma)^{\nu'}\mu_\delta}) \tag{3.1.1}$$

We assume without loss of generality that $\mu_\delta, \mu_{\delta^+}$ are integers and note that $\eta \in (0, 1)$. We define below *dynamical* hypotheses $\mathcal{H}$ based on dynamics of $\{f_a\}$, and a *formal* conditions $\mathcal{F}$ on the elements of $\mathcal{V}$ and $\mathcal{C}$.

3.1.2 Notations

For a given parameter a and point $x \in (-\delta^+, \delta^+)$, we let $x_0 = x_0(a) = f_a(x)$, $x_n = x_n(a) = f_a^n(x_0)$. We shall define for $n \in \mathbb{N} \cup \{0\}$, a map

$$c_n : \Omega \to [-2, 2]$$

by

$$c_n(a) := f_a^n(c_0).$$

where c_0 is the critical value. Also, for a parameter interval $\omega \subseteq \Omega$, we let $\omega_n = c_n(\omega) = \{c_n(a); a \in \omega\}$. Then we consider

$$\Delta_i^+(a) = f_a^i(\Delta_0^+(a)) = f_a^{i+1}(\Delta^+) \text{ and } |\Delta_i^+| := \text{the length of the interval } \Delta_i^+.$$

Whereas most of the constants are not exactly computable, but they are required to satisfy some lower and upper bounds, we use the notations " $:\geq$ " and " $:\leq$ ". The notation $:\geq$ is used to denote the fact that the constant on the left is required to be an upper bound for the expression on the right. Similarly for $:\leq$.

3.1.3 Dynamical hypothesis

We treat maps characterized by the following dynamically hypotheses. An important observation is that the hypotheses below are at all, at least in principle, rigorously verifiable by computational methods and algorithms. Another way of saying this is that they are essentially "computable". We start by assuming some uniform expansivity estimates outside the critical neighborhood Δ, i.e.:

Hypothesis 3.1.1. *For all $a \in \Omega, x \in I$ and $n \geq 1$ with $x, f_a(x), ..., f_a^{n-1}(x) \notin \Delta$ we have:*

$$|(f_a^n)'(x)| \geq \begin{cases} \Gamma e^{\lambda n}, & \text{if } f_a^n(x) \in \Delta^+ \\ e^{\lambda n}, & \text{if } f_a^n(x) \in \Delta \text{ and/or if } x \in f_a(\Delta^+) \end{cases}, \qquad \textbf{(H1)}$$

Our first assumption says that some uniform expansivity estimates hold for a sufficiently large region of the phase space for all parameter values. It is not immediately obvious that this assumption can be even in theory verified in a finite time, but it is indeed the case.

Hypothesis 3.1.2.

$$f_a^k(\Delta) \cap \Delta = \emptyset \quad \forall a \in \Omega \ , \ \forall k < N_0 \quad \text{and} \quad |c_{N_0}(\Omega)| \geq \delta^+ \qquad \textbf{(H2)}$$

$$|c_i(a)| \geq min\{\delta, e^{-i^\alpha}\} \quad \text{for all } a \in \Omega \text{ and for all } i = 0, ..., N_0; \qquad (3.1.2)$$

Hypothesis 3.1.3.

$$\mathcal{P}_1 \leq \inf_{a\in\Omega} \min\{\min_{1\leq k\leq N}\{|1+\sum_{i=1}^{k}\frac{1}{(f_a^i)'(c_0)}|\}, |1+\sum_{i=1}^{N}\frac{1}{(f_a^i)'(c_0)}|-\sum_{m}^{\infty}((j+1)^b-j^b)(\frac{1}{e^{\tilde{\lambda}}})^j\} \quad \textbf{(H3)}$$

Hypothesis 3.1.4.

$$\mathcal{P}_2 \geq \sup_{a\in\Omega} \max\{\max_{1\leq k\leq N}\{|1+\sum_{i=1}^{k}\frac{1}{(f_a^i)'(c_0)}|\}, |1+\sum_{i=1}^{N}\frac{1}{(f_a^i)'(c_0)}|+\sum_{m}^{\infty}((j+1)^b-j^b)(\frac{1}{e^{\tilde{\lambda}}})^j\} \quad \textbf{(H4)}$$

We observe, directly from the previous two hypotheses, that $\mathcal{P}_2 > \mathcal{P}_1$. We note that choosing N large makes the ratio $\frac{\mathcal{P}_1}{\mathcal{P}_2}$ close to 1 which is useful as this ratio enters the distortion constant and estimating the excluded parameter set.

Hypothesis 3.1.5.

$$\begin{cases} \gamma_i \geq \sup_{a\in\Omega}\frac{|\Delta_i^+(a)|}{|c_i(a)|} & \textit{if} \quad 0\leq i < N_0 \\ \gamma_i = \frac{1}{i^2}, & \textit{if} \quad i \geq N_0 \end{cases} \quad \textbf{(H5)}$$

We shall see that this sequence is used to define binding period.

Hypothesis 3.1.6.

$$\mathcal{B} \geq \exp(\sup_{a\in\Omega}\sum_{i=0}^{N_0-1}\log(\frac{|\Delta_i^+(a)|+|c_i(a)|}{|c_i(a)|}) + \frac{1}{2}(\frac{1}{N_0-1}+\frac{1}{N_0})) \quad \textbf{(H6)}$$

The constant $\mathcal{B}$ bounds the distortion of $|(f_a^k)'(x)|$ for the times k during binding period of x, see Lemma 6.1.2.

3.1.4 Formal conditions

An important part of the assumptions of our main Theorem is that the primitive constants used to formulate the hypotheses stated above, should satisfy some formal constraints.

Condition 3.1.1.

$$\Gamma \leq 1, \quad 0 < \tilde{\lambda} < \frac{2\beta 4^{\frac{\beta}{1-\beta}}}{\alpha + 2\beta 4^{\frac{\beta}{1-\beta}}}\lambda. \tag{\textbf{F1}}$$

We note that here $\tilde{\lambda} < \frac{2\beta 4^{\frac{\beta}{1-\beta}}}{\alpha+2\beta 4^{\frac{\beta}{1-\beta}}}\lambda$ will be useful in the argument of Lemma 7.1.3.

Condition 3.1.2.

$$\frac{(N_0 - 1)^2 e^{-\tilde{\lambda}}}{2\mathcal{B}} \geq 1. \tag{\textbf{F2}}$$

This is a very soft condition which will be used in the proof of lemma 6.1.3. Indeed, choosing small values of δ^+ leads to have big values for N_0 and then this condition become true.

Condition 3.1.3.

$$\mu_{\delta^+}^{(\beta-\alpha)(\frac{1}{\beta}-1)} \geq \frac{\lambda}{1-\gamma}(\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}}. \tag{\textbf{F3}}$$

This condition appears in the proof of lemmas 6.1.3, 7.1.1 and 7.1.3.

Condition 3.1.4.

$$\gamma \mu_{\delta^+}^{\alpha} - \frac{2\alpha}{\beta}\ln \mu_{\delta^+} - (\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}} \mu_{\delta^+}^{\frac{\alpha^2}{\beta}} \geq \frac{2}{\beta}\ln(\frac{2}{\tilde{\lambda}}). \tag{\textbf{F4}}$$

$(F4)$ used in the argument of the proof of lemma 6.1.4.

Condition 3.1.5.

$$(1-\gamma)\mu_{\delta^+}^{\alpha} + \ln\frac{2}{\mathcal{B}^2 e^{2^{\alpha}}} - \lambda(\mu_{\delta^+} + (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu_{\delta^+}^{\frac{\alpha}{\beta}} + 1)^{\beta} + \lambda \mu_{\delta^+}^{\beta} - \ln \Gamma \geq 0. \tag{\textbf{F5}}$$

This condition is used in the argument of the proof of Lemma 7.1.1.

Condition 3.1.6.

$$\lambda\mu_\delta^\beta - \mu_\delta^\alpha - \tilde{\lambda}(\mu_\delta + 1)^\beta + \ln\frac{2}{\mathcal{B}} \geq 0, \qquad (\mathbf{F6})$$

This condition is used in the proof of Lemma 7.1.3.

Condition 3.1.7.

$$\frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_2\mathcal{B}}{\mathcal{P}_1})^2 \leq e^{(1-\gamma)\mu_{\delta+}^\alpha} \qquad (\mathbf{F7})$$

Condition 3.1.8.

$$3 + \nu(1-\gamma)\mu_\delta^\nu - \alpha(1-\gamma)\mu_\delta^\alpha \leq \alpha. \qquad (\mathbf{F8})$$

This condition is used in the argument of the proof of Proposition 4.1.2. This condition gives us $\mathcal{B}_2 > 1$ which is used in the proof of Lemmas 8.1.4 and 9.1.2.

The statement of the last two formal conditions requires defining a principle constant D which is called bounded uniform distortion. But to do this we need to define some auxiliary constants as follows. First of all, we let

$$\mathcal{B}_1 = \mathcal{B}_1(\mathcal{P}_1, \mathcal{P}_2, \mathcal{B}, \gamma, \alpha) :> \frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}^2})e^{(1-\gamma)\mu_{\delta+}^\alpha}, \qquad (3.1.3)$$

$$\mathcal{B}_2 = \mathcal{B}_2(\mathcal{P}_1, \mathcal{P}_2, \mathcal{B}_1) :> \frac{\mathcal{P}_1}{\mathcal{P}_2}\mathcal{B}_1 \qquad (3.1.4)$$

which appears in the statement of Proposition 6.1.1. In the following, the constants $\mathcal{D}_i$, $1 \leq i \leq 7$, are fixed which are usable in the context of Section 8.1. The first one is

$$\mathcal{D}_1 = \mathcal{D}_1(\mathcal{D}_2, \mathcal{D}_3) := \mathcal{D}_2 + \mathcal{D}_3 \qquad (3.1.5)$$

where $\mathcal{D}_2$ and $\mathcal{D}_3$ are defined as:

$$\mathcal{D}_2 = \mathcal{D}_2(\mathcal{P}_1, \mathcal{P}_2, \lambda) = \frac{e\mathcal{P}_2}{\mathcal{P}_1(1-e^{-\lambda})}, \tag{3.1.6}$$

$$\mathcal{D}_3 = \mathcal{D}_3(\mathcal{P}_1, \mathcal{P}_2, \mathcal{B}, \{\gamma_i\}) := 4e\frac{\mathcal{P}_2\mathcal{B}^2}{\mathcal{P}_1}(\sum_0^{N_0-1} \frac{\gamma_s}{1-\gamma_s} + \frac{1}{2}(\frac{1}{N_0-1} + \frac{1}{N_0})). \tag{3.1.7}$$

The next two are

$$\mathcal{D}_4 = \mathcal{D}_4(\mathcal{B}_2) := \frac{\mathcal{B}_2}{\mathcal{B}_2 - 1} \quad \text{and} \quad \mathcal{D}_5 = \mathcal{D}_5(\alpha) := 2(e^{2^\alpha} - 1), \tag{3.1.8}$$

which both appear in the context of Lemma 8.1.4. The last two are

$$\mathcal{D}_6 = \mathcal{D}_6(\delta^+) := \frac{1}{2}(\frac{1}{\mu_{\delta^+} - 1} + \frac{1}{\mu_{\delta^+}}) \tag{3.1.9}$$

$$\mathcal{D}_7 = \mathcal{D}_7(\mathcal{D}_1, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6) := \mathcal{D}_1\mathcal{D}_4\mathcal{D}_5\mathcal{D}_6 \tag{3.1.10}$$

used in the proof of Lemma 8.1.5. At the end of Section 8.1 we will see that all these constants lead to the definition of the principle constant

$$\mathcal{D} = \mathcal{D}(\mathcal{P}_1, \mathcal{P}_2, \Gamma, \mathcal{D}_2, \mathcal{D}_7, \lambda) := \frac{\mathcal{P}_1}{\mathcal{P}_2}\exp(\mathcal{D}_7 + \frac{e^{-\lambda-1}}{\Gamma}\mathcal{D}_2), \tag{3.1.11}$$

where $\mathcal{D}$ is the global distortion bound, see proposition 8.1.1. Finally we formulate the last two formal conditions as following.

Condition 3.1.9.

$$\delta^+ - \delta \geq 2\mathcal{D}e^{-\mu_\delta^\alpha + (1-\gamma)\mu_\delta^\nu}. \tag{\textbf{F9}}$$

$(F9)$ gives a relation between δ and δ^+ which appears in the argument of Lemma 9.1.2.

Condition 3.1.10.

$$\ln(\frac{\mathcal{D}e^{2^\alpha}}{\alpha}(\frac{\mathcal{P}_2\mathcal{B}}{\mathcal{P}_1})^2) + (\gamma - 1)\mu_\delta^\alpha + (3 - \alpha)\ln\mu_\delta + (1 - \gamma)\mu_\delta^\nu \leq 0. \qquad (\mathbf{F10})$$

which is used to complete the proof of Proposition 4.1.2.

Chapter 4

Combinatorics and strategy of the proof

4.1 Combinatorics and strategy of the proof

The proof of main theorem is a probabilistic argument which consists of a combinatorial construction on which we carry out some analytic estimates. In this section we give the complete combinatorial construction and state precisely the required analytic estimates by them our main theorem followed. The remaining sections are devoted to the analytic estimates.

4.1.1 Partition of the critical neighborhood

For all integers $\mu > 0$ the intervals

$$I_\mu = [e^{-\mu^\alpha}, e^{-(\mu-1)^\alpha}) \quad \text{and} \quad I_{-\mu} = -I_\mu.$$

are defined. Without loss of generality let

$$\Delta = \{0\} \cup \bigcup_{|\mu|>\mu_\delta} I_\mu \quad \text{and} \quad \Delta^+ = \{0\} \cup \bigcup_{|\mu|>\mu_{\delta+}} I_\mu.$$

We further subdivide each I_μ into μ^2 subintervals $I_{\mu,m}$, for $m = 1, ..., \mu^2$, of equal length

$$l_\mu := \frac{e^{-(\mu-1)^\alpha} - e^{-\mu^\alpha}}{\mu^2}$$

This defines partitions $\mathcal{I}$, $\mathcal{I}^+$ of Δ and Δ^+ consisting of intervals of the form $I_{\mu,m}$. We also let

$$\hat{I}_{\mu,m} = I^L_{\mu,m} \cup I_{\mu,m} \cup I^R_{\mu,m}$$

where $I^L_{\mu,m}$ and $I^R_{\mu,m}$ are the left and right adjacent elements to $I_{\mu,m}$ respectively. If $I_{\mu,m}$ happens to be far left or far right subintervals of $\mathcal{I}^+$, then we put

$$I^L_{\mu,m} = (-\delta^+ - \frac{\mathcal{P}_2 e^{-\lambda}}{\mathcal{P}_1 \Gamma} d, -\delta^+] \quad \text{and} \quad I^R_{\mu,m} = [\delta^+, \delta^+ + \frac{\mathcal{P}_2 e^{-\lambda}}{\mathcal{P}_1 \Gamma} d)$$

respectively, where d is used as short writing and defined as following:

$$d :\geq 2\mathcal{D} e^{-\mu_\delta^\alpha + (1-\gamma)\mu_\delta^\nu}, \tag{4.1.1}$$

4.1.2 The binding period

For a given parameter value $a \in \Omega$ and point $x \in \Delta^+$, we define the *binding period*[1]

$$p(x) := p_a(x) := \max\{k : |c_i - x_i| \leq |c_i|/i^2 \quad \forall\ 1 \leq i \leq k-1\}. \tag{4.1.2}$$

For any given integer $\mu \geq \mu_{\delta^+}$ we can compute

$$p_\mu := \min\{p_a(e^{-(\mu-1)^\alpha}) : a \in \Omega\}.$$

Notice that $e^{-(\mu-1)^\alpha}$ is the boundary point of I_μ furthest from the critical point and so has the shortest binding period of any point in I_μ and therefore for any $a \in \Omega$ and any $x \in I_\mu$ with $\mu \geq \mu_{\delta^+}$ we have $p_a(x) \geq p_\mu$. Moreover, $p_\mu \to \infty$ monotonically as $\mu \to \infty$. For an interval $J \subseteq \Delta^+ \setminus \{0\}$, $\omega \subset \Omega$, and $a \in \omega$ we let

$$p_a(J) := \min_{x \in J}\{p_a(x)\} \quad p_\omega(J) := \min_{a \in \omega}\{p_a(J)\}$$

and for an interval $\omega \subset \Omega$ and an integer $k \geq 0$ such that $\omega_k \subseteq \Delta^+$, we let

$$p(\omega_k) := \min_{a \in \omega}\{p(c_k(a))\}. \tag{4.1.3}$$

Notice that for $p = p(x)$ we have $|c_p - x_p| > \gamma_p |c_p|$.

[1] The notion of binding period was introduced and systematically exploited by Benedicks and Carleson in [6].

4.1.3 Inductive assumptions

We fix once and for all some positive integer $n \geq 1$ and assume that there exists a nested sequence of sets

$$\Omega^{(n-1)} \subseteq \cdots \subseteq \Omega^{(N_0)} = \cdots = \Omega^{(0)} = \Omega$$

and a corresponding sequence

$$\mathcal{P}^{(n-1)} \preceq \cdots \preceq \mathcal{P}^{(N_0)} = \cdots = \mathcal{P}^{(0)} = \{\Omega^{(0)}\},$$

where for each $0 \leq k \leq n-1$, $\mathcal{P}^{(k)}$ is a partition of $\Omega^{(k)}$ into subintervals and $\preceq$ means that $\mathcal{P}^{(k)}$ is a refinement of $\mathcal{P}^{(k-1)}$ in the sense that for every $\omega^{(k)} \in \mathcal{P}^{(k)}$ there exists an $\omega^{(k-1)} \in \mathcal{P}^{(k-1)}$ such that $\omega^{(k)} \subseteq \omega^{(k-1)}$. Moreover, there exists $s = s(\omega^{(k)}) \geq 0$, a sequence of *return times* $\{r_j\}_{j=0}^{s}$ with

$$r_0 := 0 < \mu_\delta < N_0 \leq r_1 < ... < r_s \leq k \tag{4.1.4}$$

and corresponding sequences of *return depths* $\{(\mu_j, m_j)\}_{j=1}^{s}$, *binding times* $\{p_j\}_{j=0}^{s}$, with $p_0 := -1$ and, for $j = 1, ..., s$, $p_j = p(\omega_{r_j}^{(k)})$, see (4.1.3), *binding periods* $[r_j + 1, r_j + p_j]$, such that $r_j + p_j < r_{j+1}$ for every $j = 0, ..., s-1$, $\omega_{r_j}^{(k)} \subset \hat{I}_{\mu_j, m_j}$ and

$$r_j \geq \mu_j > \mu_{\delta^+} \tag{4.1.5}$$

for every $j = 1, ..., s$, and $\omega_i^{(k)} \cap \Delta = \emptyset \quad$ for all $\; i \notin \bigcup_{j=1}^{s} [r_j, r_j + p_j]$. This is the inductive description of the combinatorial structure.

4.1.4 General inductive step

We now explain how to construct the set $\Omega^{(n)}$ and the corresponding partition $\mathcal{P}^{(n)}$. The inductive assumption on the combinatorial structure at time n will be an automatic consequence of the construction, whereas the other conditions are non trivial consequences and will be proved in later sections. We will carry out the construction on each element $\omega = \omega^{(n-1)} \in \mathcal{P}^{(n-1)}$ independently. Recall that $\omega_n = \{c_n(a) : a \in \omega\}$ is the "image" of ω in phase space at time n. We distinguish two cases.

Case 1 (non-chopping time):

We say that n is a *non-chopping time* for $\omega = \omega^{(n-1)}$ if one of the following situations occur:

1. $n \leq r_s + p_s$,
2. $n > r_s + p_s$ and $\omega_n \cap \Delta^+ = \emptyset$,
3. $n > r_s + p_s$ and $\omega_n \cap \Delta^+ \neq \emptyset$ but ω_n does not contain a full interval $I_{\mu,m} \in \mathcal{I}$.

If n is a non-chopping time for $\omega = \omega^{(n-1)}$, then we let $\omega \subset \Omega^{(n)}$, i.e. we do not exclude any parameter in ω, and we let $\omega = \omega^{(n)} \in \mathcal{P}^{(n)}$, i.e. we do not refine this particular element of the partition $\mathcal{P}^{(n-1)}$. Also, $\omega = \omega^{(n)}$ inherits the sequence of returns, return depths and binding times of $\omega = \omega^{(n-1)}$. In cases 1 and 2 no additional return times or return depths or binding times are associated to ω so that n is not a return time for ω. In case 3, we distinguish two further sub cases.

(3a) $\omega_n \setminus \Delta^+ \neq \emptyset$,

(3b) $\omega_n \subset \Delta^+$.

Case (3a) means that ω_n is not completely contained in Δ^+ and thus, combined with the assumption that ω_n does not contain a full interval $I_{\mu,m} \in \mathcal{I}$ it means that ω_n only "just" intersects Δ^+ in one of the two extreme subintervals of $\mathcal{I}$ in Δ^+. In this case, we also do not regard ω as having had a return, and do not add the iterate n to the sequence of returns. On the other hand, in case (3b) we do add the iterate n to the sequence of returns, so that ω now has a new sequence of return times $N_0 < r_1 < \cdots < r_s < r_{s+1} := n$ where the first s returns are those associated to $\omega = \omega^{(n-1)} \in \mathcal{P}^{(n-1)}$ and $r_{s+1} = n$ is the new return time representing the fact that ω_n returns to Δ^+ at time n. By assumption ω_n does not contain a full interval $I_{\mu,m}$ and therefore is necessarily contained in the union $\hat{I}_{\mu,m}$ of three (in fact two) adjacent such intervals. We therefore add the index (μ_{s+1}, m_{s+1}) corresponding to one of these two intervals (it doesn't matter which one) to the list of return depths and we thus have $\omega_n = \omega_{r_{s+1}} \subseteq \hat{I}_{\mu_{s+1}, m_{s+1}}$. Finally we define the associated binding time as $p_{s+1} := p(\omega_{r_{s+1}})$.

Case 2 (chopping time):

We say that n is a *chopping time* for $\omega = \omega^{(n-1)}$ if

(4) $n > r_s + p_s$ and $\omega_n \cap \Delta^+ \neq \emptyset$ and ω_n contains at least a full interval $I_{\mu,m} \in \mathcal{I}$.

If n is a chopping time for $\omega = \omega^{(n-1)}$ then we proceed as follows. Chop ω (or more precisely $c_n^{-1}(\omega_n \setminus \{c\})$ if ω_n intersects the critical point) into a possibly countable collection of disjoint subintervals:

$$\omega = \omega^- \cup \bigcup_{(\mu,m)} \omega^{(\mu,m)} \cup \omega^+ \tag{4.1.6}$$

such that $\omega_n^{\pm}$ are components of $\omega_n \setminus \Delta^+$ with $|\omega_n^{\pm}| \geq \delta^+$ (if such components exist) and $\{(\mu, m)\}$ is some collection of indices corresponding to intervals $I_{\mu,m} \in \mathcal{I}$ (with $|\mu| > \mu_{\delta^+}$)

such that for each such (μ, m) we have

$$I_{\mu,m} \subseteq \omega_n^{(\mu,m)} \subseteq \hat{I}_{\mu,m}.$$

Notice that one or both components $\omega_n^{\pm}$ of $\omega_n \setminus \Delta^+$ may of course not satisfy the condition $|\omega_n^{\pm}| \geq \delta^+$. In this case we "glue" them to their adjacent interval and continue to denote this enlarged interval by $\omega^{(\mu,m)}$ with $\mu = \mu_{\delta^+}$.

All intervals inherit the sequence of returns, return depths and binding periods associated to the "parent interval" ω. For intervals of the form ω^+ or ω^- no new combinatorial information is added and in particular they are not considered to have a "return" at time n. On the other hand, for the intervals of the form $\omega^{(\mu,m)}$ we define a new return time $r_{s+1} = n$, a new return depth $(\mu_{s+1}, m_{s+1}) = (\mu, m)$, and a new binding period $p_{s+1} := p(\omega_n^{(\mu,m)})$.

Parameter exclusions

On the basis of the various cases mentioned above, we are now ready to decide which parameters to exclude at step n of the construction. More formally, as mentioned above, if n is a non-chopping time for ω, then we do not exclude any parameters and simply let $\omega \subseteq \Omega^{(n)}$ and $\mathcal{P}^{(n)}|_\omega = \{\omega\}$. If n is instead a chopping time, we carry out the subdivision procedure described above and exclude all parameters which intersect the prohibited neighborhood of the critical point. More formally, we let

$$\Omega^{(n)}|_\omega := \omega^- \cup \omega^+ \cup \{a : a \in \omega^{(\mu,m)} \text{ with } \omega^{(\mu,m)} \cap (-e^{-n^\alpha}, e^{-n^\alpha}) = \emptyset\} \tag{4.1.7}$$

and let

$$\mathcal{P}^{(n)}|_{\omega} = \{\omega^-\} \cup \{\omega^+\} \cup \{\omega^{(\mu,m)} : \omega^{(\mu,m)} \cap (-e^{-n^\alpha}, e^{-n^\alpha}) = \emptyset\}.$$

Repeating the construction above for each $\omega^{(n-1)}$, distinguishing whether n is a chopping or non-chopping time for $\omega^{(n-1)}$, we obtain the full set $\Omega^{(n)}$ and the corresponding partition $\mathcal{P}^{(n)}$. This completes the construction of $\Omega^{(n)}$ and $\mathcal{P}^{(n)}$.

4.1.5 The set Ω^*

Now we define a set $\Omega^* \subseteq \Omega$ as

$$\Omega^* = \bigcap^{n} \Omega^{(n)}, \tag{4.1.8}$$

where ... $\subseteq \Omega^{(n)} \subseteq \Omega^{(n-1)} \subseteq$ ... $\subseteq \Omega^{(0)} = \Omega$ is a nested sequence of subsets of the chosen parameter interval Ω. We will then prove, by two separate argument that every $a \in \Omega^*$ is stochastic and that $|\Omega^*| \geq \eta |\Omega|$. More precisely, we will prove the following two results

Proposition 4.1.1. *For any $n \geq 1$ and any $a \in \Omega^{(n)}$ we have*

$$|(f_a^n)'(f_a(c))| \geq e^{\tilde{\lambda} n^\beta}.$$

To give the statement of the second proposition we introduce the following sequence.

$$\alpha_n = \alpha_n(\nu) := \begin{cases} 0, & \text{if } n \leq \mu_\delta \\ e^{(\gamma-1)n^\nu}, & \text{if } n > \mu_\delta \end{cases} \tag{4.1.9}$$

Indeed, this defines an upper bound on the excluded parameters mentioned in the previous part.

Proposition 4.1.2. *For all $n \geq 1$ we have*

$$|\Omega^{(n-1)} \setminus \Omega^{(n)}| \leq \alpha_n |\Omega^{(n-1)}|.$$

Proof of Theorem **??** *assuming Propositions 4.1.1 and 4.1.2.* Proposition 4.1.1 together with the nesting property of the $\Omega^{(n)}$'s implies that for every $a \in \Omega^*$ we have $|(f_a^n)'(f_a(c))| \geq e^{\tilde{\lambda} n^\beta}$ for all $n \geq 1$. By standard results this implies that f_a admits an ergodic invariant absolutely continuous invariant probability measure and this results that $a \in \Omega^+$. By Proposition 4.1.2 we have

$$\frac{|\Omega^*|}{|\Omega|} = \frac{|\bigcap_n \Omega^{(n)}|}{|\Omega|} = \prod_n \frac{|\Omega^{(n)}|}{|\Omega^{(n-1)}|} \geq \prod_n (1-\alpha_n)$$

A straightforward calculation then shows that:

$$\ln(\prod_n (1-\alpha_n)) = \sum_n \ln(1-\alpha_n) = -\sum_n \ln(1+\frac{\alpha_n}{1-\alpha_n}) \geq -\sum_n \frac{\alpha_n}{1-\alpha_n} \geq \frac{-1}{1-\alpha_{\mu_\delta}} \sum_{n>\mu_\delta} \alpha_n$$

where in the last inequality we used the fact that $\alpha_n = 0$ for $n \leq \mu_\delta$ and $\alpha_n \leq \alpha_{\mu_\delta}$ for $n > \mu_\delta$. Substituting $\alpha_n = e^{(\gamma-1)n^\nu}$ in the right hand side of previous formula, we get:

$$\begin{aligned}
\ln(\prod_n (1-\alpha_n)) &\geq \frac{-1}{1-\alpha_{\mu_\delta}} \sum_{n>\mu_\delta} \frac{1}{e^{(1-\gamma)n^\nu}} \\
&\geq \frac{-1}{1-\alpha_{\mu_\delta}} \sum_{n>\mu_\delta} \frac{\nu'!}{(1-\gamma)^{\nu'} n^2} \qquad \text{by definition of } \nu' \\
&\geq -\frac{(\nu')!}{(1-\alpha_{\mu_\delta})((1-\gamma)^{\nu'})} \sum_{n>\mu_\delta} \frac{1}{n^2} \\
&\geq -\frac{(\nu')!}{(1-\alpha_{\mu_\delta})((1-\gamma)^{\nu'})\mu_\delta} = \ln(\eta)
\end{aligned}$$

and therefore we get the statement of theorem. ∎

4.1.6 Overview of the work

We have now reduced the proof of Theorem to the proofs of Proposition (4.1.1) (to ensure that all parameters in Ω^* have an absolutely continuous invariant measure) and Proposition (4.1.2) (to guarantee that Ω^* has positive Lebesgue measure). The statement of first proposition also plays an important role in the proof of the second proposition.

The proof of Proposition (4.1.1) is given in Section 7.1. Its proof relies on several estimates about binding period which are gained in Section 6.1. Proposition (4.1.2) will be proved in Section 9.1 based on the distortion bound founded in Section 8.1.First of all in Section 5.1 we find a general estimate which is useful repeatedly in the sequel and relates derivatives with respect to space variable with derivatives with respect to the parameter.

Chapter 5

Parameter dependence

5.1 Parameter dependence

In this section we prove a relatively straightforward but crucial estimate concerning the parameter dependence of iterates of the critical point. In fact, the following main lemma of this section states that the parameter and space derivatives are comparable. Before that let we introduce the shorthand $(\Phi)_n$ as following:

$$|(f_a^k)'(f_a(c))| \geq \exp(\tilde{\lambda} k^{\beta}) \quad \forall k \leq n \tag{$(\Phi)_n$}$$

Lemma 5.1.1. *Suppose $(\Phi)_n$ holds for the parameter $a \in \Omega$. Then, for every $k \leq n$ we have*

$$\mathcal{P}_1 \leq |\frac{c_k'(a)}{(f_a^k)'(c_0)}| \leq \mathcal{P}_2 \tag{5.1.1}$$

Proof. By differentiating recursively and using the fact that $c_k(a) = f_a(c_{k-1}(a)) = -c_{k-1}^2(a)+$

a we get

$$c_k'(a) = (-2)^k c_{k-1}...c_1c_0 + (-2)^{k-1}c_{k-1}...c_1 + ... + (-2)c_{k-1} + 1, \tag{5.1.2}$$

On the other hand chain rule gives us

$$(f_a^k)'(c_0) = f_a'(c_{k-1})...f_a'(c_1)f_a'(c_0). \tag{5.1.3}$$

So dividing (5.1.2) by (5.1.3) and putting $f_a'(c_i) = -2c_i$, we have

$$|\frac{c_k'(a)}{(f_a^k)'(c_0)}| = |1 + \sum_{i=1}^{k} \frac{1}{(f_a^i)'(c_0)}|$$

If $n \leq N$, then the hypotheses (3.1.3) and (3.1.4) immediately gives the statement of the Lemma. Otherwise, using the condition $(\Phi)_n$ one can write

$$\begin{aligned}
|\frac{c_k'(a)}{(f_a^k)'(c_0)}| &= |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)} + \sum_{i=N+1}^{k} \frac{1}{(f_a^i)'(c_0)}| \\
&\leq |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)}| + \sum_{i=N+1}^{k} \frac{1}{|(f_a^i)'(c_0)|} \\
&\leq |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)}| + \sum_{i=N+1}^{k} \frac{1}{e^{\tilde{\lambda} i^{\beta}}} \\
&\leq |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)}| + \sum_{i=N+1}^{\infty} \frac{1}{e^{\tilde{\lambda} i^{\beta}}}
\end{aligned}$$

And notice that

$$\sum_{i=N+1}^{\infty} \frac{1}{e^{\tilde{\lambda} i^{\beta}}} \leq \frac{1}{e^{\tilde{\lambda}(N+1)^{\frac{1}{b}}}} + \frac{1}{e^{\tilde{\lambda}(N+1)^{\frac{1}{b}}}} + ... + \frac{1}{e^{\tilde{\lambda}(N+k_i+1)^{\frac{1}{b}}}} + ...$$

where $N + k_i + 1 = (m+i)^b$. So a simple calculation shows that

$$\sum_{i=N+1}^{\infty} \frac{1}{e^{\tilde{\lambda} i^{\beta}}} \leq \frac{(m+1)^b - m^b}{e^{\tilde{\lambda} m}} + \frac{(m+2)^b - (m+1)^b}{e^{\tilde{\lambda}(m+1)}} + ... = \sum_{m}^{\infty} ((j+1)^b - j^b)(\frac{1}{e^{\tilde{\lambda}}})^j$$

Hence

$$|\frac{c_k'(a)}{(f_a^k)'(c_0)}| \leq |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)}| + \sum_{m}^{\infty} ((j+1)^b - j^b)(\frac{1}{e^{\tilde{\lambda}}})^j \leq \mathcal{P}_2$$

And by a similar calculation for $k \geq N$ we can write

$$\begin{aligned}
|\frac{c_k'(a)}{(f_a^k)'(c_0)}| &\geq |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)}| - |\sum_{i=N+1}^{k} \frac{1}{(f_a^i)'(c_0)}| \\
&\geq |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)}| - \sum_{i=N+1}^{\infty} \frac{1}{e^{\tilde{\lambda} i^{\beta}}} \\
&\geq |1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)}| - \sum_{m}^{\infty} ((j+1)^b - j^b)(\frac{1}{e^{\tilde{\lambda}}})^j \geq \mathcal{P}_1.
\end{aligned}$$

∎

Chapter 6

Binding Period

6.1 Binding Period

As we mentioned already, binding period is part of the combinatorial information used in the construction of Ω^*. In this section we obtain various analytic estimates regarding this concept. We collect the features of the binding period to get the following proposition.

Proposition 6.1.1. *Let $\omega = \omega^{(k)} \in \mathcal{P}^{(k)}$ and $(\Phi)_k$ be valid for each $a \in \omega$. Assume that $r \leq k$ is a return time for ω with $\omega_r \subseteq \hat{I}_{\mu,m}$ for some $I_{\mu,m} \in \mathcal{I}^+$. If p is the corresponding binding period of ω_r, then*

$$|\omega_{r+p+1}| \geq \frac{2}{e^{2\alpha}} \frac{\mathcal{P}_1}{\mathcal{P}_2 \mathcal{B}^2} e^{(1-\gamma)\mu^\alpha} \geq \mathcal{B}_1 |\omega_r|$$

In particular, if $r < r' \leq k$ are two consecutive returns for ω with $\omega_r \subseteq \hat{I}_{\mu,m}$ and $\omega_{r'} \subseteq \hat{I}_{\mu',m'}$ for some $I_{(\mu,m)}$ and $I_{(\mu',m')}$ in $\mathcal{I}^+$, then

$$|\omega_{r'}| \geq \mathcal{B}_2 |\omega_r|.$$

This result will be applied in the arguments of the proof of two main Propositions. A sequence of lemmas in the following subsections is given to prove this.

6.1.1 Bounded distortion during the binding period

We start by obtaining an estimate which guarantees that there is no big difference between the derivatives of iterates of f_a^n during the binding period. In fact by the following lemma we obtain a bound on the distortion of f_a^n.

Lemma 6.1.2. *Let $p = p(x)$ be the binding period of an arbitrary point x. Then for all $k \leq p$ and y with $|y| \leq |x|$, we have*

$$\frac{|(f_a^k)'(y_0)|}{|(f_a^k)'(c_0)|} \leq \mathcal{B} \tag{6.1.1}$$

*where $\mathcal{B}$ defined in ((**H6**)).*

Proof. If $k \leq N_0$ then we can write

$$\log \frac{|(f_a^k)'(y_0)|}{|(f_a^k)'(c_0)|} = \log \prod_{i=0}^{k-1} \frac{|y_i|}{|c_i|} = \sum_{i=0}^{k-1} \log \frac{|y_i|}{|c_i|} \leq \sum_{i=0}^{k-1} \log(\frac{|c_i| + |y_i - c_i|}{|c_i|}) \tag{6.1.2}$$

From the definition of binding period we have $|y_i - c_i| \leq |\Delta_i|$, and so we get

$$\frac{|(f_a^k)'(y_0)|}{|(f_a^k)'(c_0)|} \leq \exp(\sum_{i=0}^{k-1} \log(\frac{|c_i| + |\Delta_i^+|}{|c_i|})) \leq \mathcal{B},$$

which is the statement of Lemma. Otherwise when $k > N_0$, using the definition of binding period , one may write

$$
\begin{aligned}
\log \frac{|(f_a^k)'(y_0)|}{|(f_a^k)'(c_0)|} &\leq \sum_{i=0}^{k-1} \log(\frac{|c_i| + |y_i - c_i|}{|c_i|}) \quad \text{by (6.1.2)} \\
&\leq \sum_{i=0}^{N_0-1} \log(\frac{|c_i| + |\Delta_i^+|}{|c_i|}) + \sum_{i=N_0}^{k-1} \log(\frac{|c_i| + \frac{1}{i^2}|c_i|}{|c_i|}) \\
&\leq \sum_{i=0}^{N_0-1} \log(\frac{|c_i| + |\Delta_i^+|}{|c_i|}) + \sum_{i=N_0}^{k-1} \frac{1}{i^2} \quad (\text{ since } \log(1+\frac{1}{i^2}) \leq \frac{1}{i^2}) \\
&\leq \sum_{i=0}^{N_0-1} \log(\frac{|c_i| + |\Delta_i^+|}{|c_i|}) + \sum_{i=N_0}^{\infty} \frac{1}{i^2-1} \\
&= \sum_{i=0}^{N_0-1} \log(\frac{|c_i| + |\Delta_i^+|}{|c_i|}) + \frac{1}{2}\sum_{i=N_0}^{\infty} (\frac{1}{i-1} - \frac{1}{i+1}) \\
&= \sum_{i=0}^{N_0-1} \log(\frac{|c_i| + |\Delta_i^+|}{|c_i|}) + \frac{1}{2}(\frac{1}{N_0-1} + \frac{1}{N_0})
\end{aligned}
$$

and therefore we obtain:

$$
\frac{|(f_a^k)'(y_0)|}{|(f_a^k)'(c_0)|} \leq \exp(\sum_{i=0}^{N_0-1} \log(\frac{|\Delta_i^+(a)| + |c_i(a)|}{|c_i(a)|}) + \frac{1}{2}(\frac{1}{N_0-1} + \frac{1}{N_0}))
$$

which completes the proof. ∎

6.1.2 Length of binding period

The definition of binding period implies that $p_a(x) \geq N_0$ for each $x \in \Delta$. The following lemma gives us an upper bound for the length of binding period in terms of the distance $|x|$ of the point x from the critical point.

Lemma 6.1.3. *Let $a \in \Omega$ be a parameter for which $(\Phi)_\mu$ holds with $\mu \geq \max\{N_0, \mu_{\delta+}\}$. Then, for all $x \in I$ with $|x| \geq e^{-\mu^\alpha}$ we have*

$$p(x) \leq (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}} \leq \mu$$

Proof. In order to use the assumption $(\Phi)_\mu$ we let $\hat{p} = \min\{p, \mu\}$. Then the definition of binding period implies that

$$|x_{\hat{p}-1} - c_{\hat{p}-1}| \leq \gamma_{\hat{p}-1}|c_{\hat{p}-1}| \leq \frac{2}{(\hat{p}-1)^2} \quad (\text{ since } |c_{\hat{p}-1}| \leq 2). \tag{6.1.3}$$

On the other hand the Mean value theorem gives us some $\xi_0 \in [x_0, c_0]$ for which we have

$$|x_{\hat{p}-1} - c_{\hat{p}-1}| = |(f_a^{\hat{p}-1})'(\xi_0)||x_0 - c_0| \geq \frac{|(f_a^{\hat{p}-1})'(c_0)|}{\mathcal{B}}|x|^2 \geq \frac{e^{\tilde{\lambda}(\hat{p}-1)^\beta}}{\mathcal{B}} e^{-2\mu^\alpha} \tag{6.1.4}$$

where the second inequality comes from lemma 6.1.2 and in the third inequality we use the assumptions of Lemma. Comparing (6.1.3) and (6.1.4) and also since $(\hat{p}-1)^\beta \geq \hat{p}^\beta - 1$, we get

$$e^{\tilde{\lambda}\hat{p}^\beta} \leq \frac{(\hat{p}-1)^2 e^{-\tilde{\lambda}}}{2\mathcal{B}} e^{\tilde{\lambda}\hat{p}^\beta} \leq e^{2\mu^\alpha}$$

in which the first inequality comes from $(F2)$ and the fact that $\hat{p} = \min\{p, \mu\} \geq N_0$. Thus

$$\hat{p} \leq (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}}$$

Now if we can show that

$$(\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}} < \mu \tag{6.1.5}$$

then we have $p = \hat{p} \leq (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}} \leq \mu$. Recalling condition $(F3)$ we can write

$$\mu^{(\beta-\alpha)} \geq (\frac{2}{\tilde{\lambda}})^{\frac{1}{1-\beta}} (\frac{\lambda}{1-\gamma})^{\frac{\beta}{1-\beta}} \geq (\frac{2}{\tilde{\lambda}})(\frac{2\lambda}{\tilde{\lambda}(1-\gamma)})^{\frac{\beta}{1-\beta}} \geq \frac{2}{\tilde{\lambda}}$$

Therefore (6.1.5) followed and we obtain the statement of Lemma. ∎

6.1.3 Expansion during the binding period

We turn now to the proof of the main proposition of this section. Before the demonstration we formulate two technical lemmas as following.

Lemma 6.1.4. *Suppose p satisfies $p \leq (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}}$. Then for each $\mu \geq \mu_{\delta+}$ we have*

$$\frac{e^{-p^{\alpha}}}{p^2} \geq e^{-\gamma \mu^{\alpha}} \tag{6.1.6}$$

Proof. We get (6.1.6), regarding the fact that $p \leq (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}}$, if we able to show that

$$2\log(\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}} + (\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}} \mu^{\frac{\alpha^2}{\beta}} \leq \gamma \mu^{\alpha} \tag{6.1.7}$$

We define the function $M_1(\mu) = \gamma \mu^{\alpha} - (\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}} \mu^{\frac{\alpha^2}{\beta}} - \frac{2\alpha}{\beta} \log \mu - \frac{2}{\beta} \log \frac{2}{\tilde{\lambda}}$, and then (6.1.7) is equivalent to $M_1(\mu) \geq 0$. Observing condition $(F3)$ results that $M_1(\mu_{\delta+}) \geq 0$, and it is therefore sufficient to prove that M_1 is monotone increasing with μ for $\mu \geq \mu_{\delta+}$. Differentiating $M_1(\mu)$ with respect to μ, we get $M_1'(\mu) = \frac{\alpha}{\mu} M_2(\mu)$, where $M_2(\mu) = \gamma \mu^{\alpha} - (\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}} \mu^{\frac{\alpha^2}{\beta}} - \frac{2}{\beta}$. In the rest of the proof we recall again $(F3)$ and show that $M_2(\mu)$ is positive for $\mu \geq \mu_{\delta+}$. Indeed, to do this multiplying $(F3)$ by $\frac{1}{\gamma}\mu_{\delta+}^{\frac{\alpha^2}{\beta}}$, we can write

$$\begin{aligned}
\mu_{\delta+}^{\alpha-\frac{\alpha^2}{\beta}} &\geq \frac{1}{\gamma}(\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}} + \frac{2}{\gamma\beta\mu_{\delta+}^{\frac{\alpha^2}{\beta}}}(\alpha\log\mu_{\delta+} + \log\frac{2}{\tilde{\lambda}}) \\
&\geq \frac{1}{\gamma}(\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}} \quad (\text{ since } \alpha\log\mu_{\delta+} + \log\frac{2}{\tilde{\lambda}} \geq 0) \\
&> \frac{1}{\gamma}(\frac{\alpha}{\beta})^2(\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}} \quad \text{ by } \alpha < \beta
\end{aligned}$$

which results $M_2'(\mu) \geq 0$ for $\mu \geq \mu_{\delta+}$. Also since $(\frac{2}{\lambda})^{\frac{\alpha}{\beta}}\mu^{\frac{\alpha^2}{\beta}} + \frac{2\alpha}{\beta}\log\mu \geq (\frac{\alpha}{\beta})(\frac{2}{\lambda})^{\frac{\alpha}{\beta}}\mu^{\frac{\alpha^2}{\beta}}$ then $(F4)$ results that

$$\gamma\mu^\alpha - (\frac{\alpha}{\beta})(\frac{2}{\tilde{\lambda}})^{\frac{\alpha}{\beta}}\mu^{\frac{\alpha^2}{\beta}} \geq \frac{2}{\beta}\log(\frac{2}{\tilde{\lambda}}) \geq \frac{2}{\beta}$$

This proves that $M_2(\mu)$ and so $M_1'(\mu)$ are positive for $\mu \geq \mu_{\delta+}$. Therefore we could show that $M_1(\mu)$ is positive. ∎

Lemma 6.1.5. *Let $\omega \in \mathcal{P}^{(k)}$ be chosen and $r < k$ be a return time of ω with $\omega_r \subseteq \hat{I}_{\mu,m}$ for some $I_{(\mu,m)} \in \mathcal{I}^+$. Then letting $p = p(\omega_r)$ we have*

$$|(f_a^{p+1})'(c_r(a))| \geq \frac{2}{\mathcal{B}^2 e^{2\alpha}} e^{(1-\gamma)\mu^\alpha}$$

Proof. The definition of binding period and applying Lemma 6.1.2 implies that

$$\gamma_p|c_p| \leq |x_p - c_p| = |(f_a^p)'(\zeta_0)||x|^2 \leq \mathcal{B}^2|(f_a^p)'(x_0)||x|^2$$

for some $\zeta_0 \in (x_0, c_0)$, and so using bounded distortion Lemma 6.1.2 we have

$$|(f_a^p)'(x_0)| \geq \frac{\gamma_p|c_p|}{\mathcal{B}^2|x|^2}. \tag{6.1.8}$$

If we put $x = c_r(a)$ in (6.1.8), then $|(f_a^p)'(f_a(c_r(a)))| \geq \frac{\gamma_p|c_p|}{\mathcal{B}^2|c_r(a)|^2}$, and so

$$|(f_a^{p+1})'(c_r(a))| = 2|c_r(a)||(f_a^p)'(f_a(c_r(a)))| \geq \frac{2\gamma_p|c_p|}{\mathcal{B}^2|c_r(a)|} \geq \frac{2e^{-p^\alpha}}{\mathcal{B}^2p^2}e^{(\mu-2)^\alpha}$$

Hence the result follows from the previous lemma and the fact that $(\mu-2)^\alpha \geq \mu^\alpha - 2^\alpha$.

∎

[Proof of Proposition 6.1.1] Let $\omega = \omega^{(k)} \in \mathcal{P}^{(k)}$ and $r \leq k$ be a return time for ω with $\omega_r \subseteq \hat{I}_{\mu,m}$. Then

$$\begin{aligned}
|(c_{r+p+1} \circ c_r^{-1})(\omega_r)| &= |(c_{r+p+1} \circ c_r^{-1})'(\zeta)||\omega_r| \quad \text{for some } \zeta \in \omega_r \\
&= \frac{c'_{r+p+1}(a)}{c'_r(a)}|\omega_r| \quad \text{with } a = c_r^{-1}(\zeta) \\
&\geq \frac{\mathcal{P}_1|(f_a^{r+p+1})'(c_0)|}{\mathcal{P}_2|(f_a^r)'(c_0)|}|\omega_r| \quad \text{by Lemma 5.1.1} \\
&= \frac{\mathcal{P}_1}{\mathcal{P}_2}|(f_a^{p+1})'(c_r(a))||\omega_r| \\
&\geq \frac{2\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}^2e^{2^\alpha}}e^{(1-\gamma)\mu^\alpha}|\omega_r| \quad \text{by Lemma 6.1.5}
\end{aligned}$$

and therefore we get

$$|\omega_{r+p+1}| = |(c_{r+p+1} \circ c_r^{-1})(\omega_r)| \geq \frac{2\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}^2e^{2^\alpha}}e^{(1-\gamma)\mu^\alpha}|\omega_r| \geq \mathcal{B}_1|\omega_r| \qquad (6.1.9)$$

In particular if $r < r'$ are two consecutive returns less than k then, by a similar argument as above, for $\omega_{r'} = (c_{r'} \circ c_{r+p+1}^{-1})(\omega_{r+p+1})$ one may write

$$\begin{aligned}
|\omega_{r'}| &\geq \frac{\mathcal{P}_1 |(f_b^{r'})'(c_0)|}{\mathcal{P}_2 |(f_b^{r+p+1})'(c_0)|} |\omega_{r+p+1}| \quad \text{for some } b \in \omega \\
&\geq \frac{\mathcal{P}_1}{\mathcal{P}_2} |(f_b^{r'-(r+p+1)})'(c_{r+p+1})| |\omega_{r+p+1}| \\
&\geq \Gamma \frac{\mathcal{P}_1}{\mathcal{P}_2} e^{\lambda(r'-(r+p+1))} |\omega_{r+p+1}| \quad \text{by hypothesis } (H1)
\end{aligned}$$

and since $\lambda(r' - (r+p+1)) \geq 0$ then

$$|\omega_{r'}| \geq \Gamma \frac{\mathcal{P}_1}{\mathcal{P}_2} |\omega_{r+p+1}| \geq \frac{2\Gamma}{e^{2\alpha}} (\frac{\mathcal{P}_1}{\mathcal{P}_2 \mathcal{B}})^2 e^{(1-\gamma)\mu^\alpha} |\omega_r| \geq \mathcal{B}_2 |\omega_r|$$

and hence the proof of the main proposition of this section is completed.

Chapter 7

Derivative growth

7.1 Derivative growth

We are now ready to prove Proposition 4.1.1. First of all choosing an arbitrary parameter value $a \in \Omega^{(n)}$ for some fixed n, implied the existence of some interval $\omega^{(n)} \in \mathcal{P}^{(n)}$ such that $a \in \omega^{(n)}$. Let $\{r_i\}_{i=1}^{t+1}$ be a sequence of returns corresponded to $\Omega^{(n)}$ with $0 \leq r_1 < r_2 < ... < r_t \leq n < r_{t+1}$. Notice that if $\Omega^{(n)}$ does not have any return then $t = 0$. Let us introduce the shorthand:

$$|(f_a^{r_j})'(c_0(a))| \geq e^{\lambda r_j^\beta} \quad \text{and} \quad |(f_a^i)'(c_0(a))| \geq e^{\tilde{\lambda} i^\beta} \text{ for } i < r_j, \qquad (\tilde{\Phi})_{r_j}$$

In accordance of condition $(F1)$, we know $\lambda > \tilde{\lambda}$ and so $(\tilde{\Phi})_{r_j}$ coincides with $(\Phi)_{r_j}$ except at the return times for them we have a slightly stronger estimate. We shall prove $a \in \Omega^{(n)}$ implies $(\tilde{\Phi})_{r_{t+1}}$ which in particular implies $(\Phi)_{r_{t+1}}$ and so Proposition 4.1.1. The proof of this followed by the results of the next two subsections.

7.1.1 Derivative growth at the returns

This subsection is devoted to show that if we have $|(f_a^{r_i})'(c_0)| \geq e^{\lambda r_i^\beta}$ then at the subsequent return time r_{i+1} we have $|(f_a^{r_{i+1}})'(c_0)| \geq e^{\lambda r_{i+1}^\beta}$. This is the statement of Lemma 7.1.2, which comes after the following supplementary lemma.

Lemma 7.1.1. *Assume that $\omega_{r_i} \subseteq \hat{I}_{\mu,m}$ for some $I_{\mu,m} \in \mathcal{I}^+$ and $p = p(c_{r_i})$ is the binding period of c_{r_i}. Then for each $a \in \Omega^{(n)}$ we have*

$$|(f_a^{p+1})'(c_{r_i}(a))| \geq \frac{1}{\Gamma} e^{\lambda(r_i+p+1)^\beta - \lambda r_i^\beta} \tag{7.1.1}$$

Proof. First we prove that

$$|(f_a^{p+1})'(c_r(a))| \geq \frac{1}{\Gamma} e^{\lambda(\mu+p+1)^\beta - \lambda\mu^\beta} \tag{7.1.2}$$

For this applying Lemma 6.1.4 results that $|(f_a^{p+1})'(c_{r_i}(a))| \geq \frac{2}{\mathcal{B}^2 e^{2\alpha}} e^{(1-\gamma)\mu^\alpha}$ and so to obtain (7.1.2) it is enough to have

$$(1-\gamma)\mu^\alpha + \log\frac{2}{\mathcal{B}^2 e^{2\alpha}} - \lambda(\mu+p+1)^\beta + \lambda\mu^\beta - \ln(\Gamma) \geq 0, \tag{7.1.3}$$

However to get this, regarding $p \leq (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}}$, it is sufficient to show that

$$M_3(\mu) = (1-\gamma)\mu^\alpha + \log\frac{2}{\mathcal{B}^2 e^{2\alpha}} - \lambda(\mu + (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}} \mu^{\frac{\alpha}{\beta}} + 1)^\beta + \lambda\mu^\beta - \ln(\Gamma)$$

is a positive function. Condition $(F5)$ implies that $M_3(\mu_{\delta+})$ is positive and so we get the result if M_3 is monotone increasing with μ for $\mu \geq \mu_{\delta+}$. We have

$$M_3'(\mu) = \frac{\lambda\beta + \frac{\alpha(1-\gamma)}{\mu^{\beta-\alpha}}}{\mu^{1-\beta}} - \frac{\lambda\beta + \frac{2\alpha\tilde{\lambda}(\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}}}{\mu^{\frac{\beta-\alpha}{\beta}}}}{(\mu + (\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}}\mu^{\frac{\alpha}{\beta}} + 1)^{1-\beta}} \tag{7.1.4}$$

Now recalling condition $(F3)$ we obtain

$$0 \leq \alpha(1-\gamma)\mu_{\delta+}^{-\frac{\beta-\alpha}{\beta}}(\mu_{\delta+}^{(\beta-\alpha)(\frac{1}{\beta}-1)} - \frac{\lambda}{1-\gamma}(\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}}) = \frac{\alpha(1-\gamma)}{\mu_{\delta+}^{\beta-\alpha}} - \frac{\alpha\lambda(\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}}}{\mu_{\delta+}^{\frac{\beta-\alpha}{\beta}}}$$

which clearly implies $M_3'(\mu) \geq 0$ for $\mu \geq \mu_{\delta+}$ and so (7.1.2) is followed. We now show that (7.1.1) is true. For this we observe that the function $s(x) = \lambda(x+p+1)^\beta + \lambda x^\beta$ is decreasing for fixed p, since

$$s'(x) = \frac{\lambda\beta}{(x+p+1)^{1-\beta}} - \frac{\lambda\beta}{x^{1-\beta}} = \lambda\beta(\frac{1}{(x+p+1)^{1-\beta}} - \frac{1}{x^{1-\beta}}) \leq 0$$

and therefore because of the fact that $r_i \geq \mu$, the proof of Lemma is followed. ∎

Lemma 7.1.2. *Suppose that at a return time r_i we have $|(f_a^{r_i})'(c_0)| \geq e^{\lambda r_i^\beta}$, where a belongs to some $\omega^{(k)}$ with $r_i \leq k$. Then for the subsequent return time r_{i+1} we have*

$$|(f_a^{r_{i+1}})'(c_0)| \geq e^{\lambda r_{i+1}^\beta}.$$

Proof. Let $p = p(c_{r_i})$ be the corresponding binding period of $c_{r_i}(a)$. If $f_a^{r_i+p+1}(c_0) \notin \Delta$, then one can write

$$\begin{aligned} |(f_a^{r_{i+1}})'(c_0(a))| &= |(f_a^{r_{i+1}-(r_i+p+1)})'(f_a^{r_i+p+1}(c_0(a)))||(f_a^{r_i+p+1})'(c_0(a))| \\ &\geq \Gamma e^{\lambda(r_{i+1}-(r_i+p+1))}|(f_a^{r_i+p+1})'(c_0(a))| \end{aligned}$$

where the inequality comes from hypothesis $(H1)$ and using the fact that $f_a^s(c_0) \notin \Delta$ for $r_i + p + 1 < s < r_{i+1}$. Since $\beta < 1$ by $(F1)$ then

$$|(f_a^{r_{i+1}})'(c_0(a))| \geq \Gamma e^{\lambda(r_{i+1}-(r_i+p+1))^\beta}|(f_a^{r_i+p+1})'(c_0(a))| \tag{7.1.5}$$

On the other hand we have

$$\begin{aligned} |(f_a^{r_i+p+1})'(c_0(a))| &= |(f_a^{p+1})'(c_{r_i}(a))||(f_a^{r_i})'(c_0(a))| \\ &\geq e^{\lambda r_i^\beta}|(f_a^{p+1})'(c_{r_i}(a))| \end{aligned}$$

and then substituting the result of previous lemma gives us

$$|(f_a^{r_i+p+1})'(c_0(a))| \geq \frac{1}{\Gamma}e^{\lambda(r_i+p+1)^\beta} \geq e^{\lambda(r_i+p+1)^\beta} \tag{7.1.6}$$

where the second inequality used the fact that $\Gamma \leq 1$. Thus the result of Lemma, in this case, gained by combining (7.1.5) and (7.1.6). Otherwise if $f_a^{r_i+p+1}(c_0) \in \Delta$, i.e., $r_{i+1} = r_i + p + 1$ then

$$|(f_a^{r_{i+1}})'(c_0(a))| = |(f_a^{r_i+p+1})'(c_0(a))| \geq \frac{1}{\Gamma}e^{\lambda(r_i+p+1)^\beta} \geq e^{\lambda r_{i+1}^\beta}$$

where the third inequality followed by 7.1.6. This completes the proof. ∎

7.1.2 Derivative growth between the returns

Now we consider two consecutive returns $r_i < r_{i+1}$ and assume that $(\tilde{\Phi})_{r_i}$ holds for $a \in \Omega^{(k)}$ with $k \geq r_i$. In this subsection we shall consider various times s between r_i and r_{i+1} and show that $(\Phi)_s$ is valid. Notice that immediately by relation 7.1.6, $(\Phi)_{r_i+p+1}$ is true when $p = p(c_{r_i}(a))$ and so we give two separate arguments. We start by the following lemma for the times before $r_i + p + 1$

Lemma 7.1.3. *Let $(\tilde{\Phi})_{r_i}$ be valid and $0 \leq s \leq p-1$ where $p = p(c(r_i))$. Then for each $a \in \Omega^k$ with $k \geq r_i$ we have:*

$$|(f_a^{r_i+s+1})'(c_0)| \geq e^{\tilde{\lambda}(r_i+s+1)^\beta}. \tag{7.1.7}$$

Proof. By the chain rule and applying Lemma (6.1.2) we can write:

$$\begin{aligned}
|(f_a^{r_i+s+1})'(c_0(a))| &= |(f_a^{s+1})'(c_{r_i})||(f_a^{r_i})'(c_0)| \\
&\geq 2e^{\lambda r_i^\beta}|(f_a^s)'(f_a(c_{r_i}))||c_{r_i}(a)| \\
&\geq \frac{2}{\mathcal{B}} e^{\lambda r_i^\beta} e^{\tilde{\lambda} s^\beta} e^{-r_i^\alpha}
\end{aligned}$$

in which we used the assumption of $(\tilde{\Phi})_{r_i}$ and the fact that $a \in \Omega^{(k)} \subseteq \Omega^{(r_i)}$. Thus to have (7.1.7) it is enough to show that

$$\lambda r_i^\beta - r_i^\alpha + \log \frac{2}{\mathcal{B}} \geq \tilde{\lambda}(r_i+s+1)^\beta - \tilde{\lambda} s^\beta$$

for all $0 \leq s \leq p-1$. But we know that the expansion $\tilde{\lambda}(r_i+s+1)^\beta - \tilde{\lambda} s^\beta$ with respect to s is decreasing, so

$$\tilde{\lambda}(r_i+1)^\beta \geq \tilde{\lambda}(r_i+s+1)^\beta - \tilde{\lambda} s^\beta$$

and hence to get the result it is sufficient to conclude that the following function is positive for $r \geq \mu_{\delta^+}$.

$$r(r) = \lambda r^{\beta} - r^{\alpha} + \log \frac{2}{\mathcal{B}} - \tilde{\lambda}(r+1)^{\beta} \tag{7.1.8}$$

First we observe that condition $(F6)$ implies that $r(\mu_{\delta^+}) \geq 0$ Now differentiating $r(r)$ with respect to r we have

$$r'(r) = \beta\lambda r^{\beta-1} - \alpha r^{\alpha-1} - \tilde{\lambda}\beta(r+1)^{\beta-1} > r^{\alpha-1}(\beta(\lambda-\tilde{\lambda})r^{\beta-\alpha} - \alpha).$$

Again we recall condition $(F3)$ and for $r \geq \mu_{\delta^+}$ write:

$$\begin{aligned}
r^{\beta-\alpha} \geq (\frac{\lambda}{1-\gamma}(\frac{2}{\tilde{\lambda}})^{\frac{1}{\beta}})^{\frac{\beta}{1-\beta}} &= (\frac{\lambda}{1-\gamma})^{\frac{\beta}{1-\beta}}(\frac{2}{\tilde{\lambda}})^{\frac{\beta}{1-\beta}}(\frac{2}{\tilde{\lambda}}) \\
&= (\frac{2\lambda}{\tilde{\lambda}(1-\gamma)})^{\frac{\beta}{1-\beta}}(\frac{2}{\tilde{\lambda}}) \\
&> 4^{\frac{\beta}{1-\beta}}(\frac{2}{\tilde{\lambda}}) \quad \text{since } \tilde{\lambda} < \lambda \text{ and } \frac{1}{1-\gamma} > 2 \\
&\geq \frac{\alpha}{\beta(\lambda-\tilde{\lambda})} \quad \text{by } \tilde{\lambda} \leq \frac{2\beta 4^{\frac{\beta}{1-\beta}}}{\alpha + 2\beta 4^{\frac{\beta}{1-\beta}}}\lambda
\end{aligned}$$

This implies that $r'(r) > 0$ and so the proof of Lemma concluded. ∎

We notice that here if $r_{i+1} \leq r_i + p + 2$, then by Lemmas (7.1.3) and (7.1.2), $(\tilde{\Phi})_{r_{i+1}}$ is valid. Nevertheless if $r_{i+1} > r_i + p + 2$ then we let τ be the last time for which $c_{\tau}(a) \in \Delta^+$ with $r_i \leq \tau \leq r_i + p$ and give the following lemma.

Lemma 7.1.4. *Assume that $(\tilde{\Phi})_{r_i}$ holds for $a \in \Omega^{(k)}$ with $k > r_i$ and $p+1 \leq s \leq r_{i+1} - r_i - 2$ where $p = p(c_{r_i}(a))$. Then for each $a \in \Omega^{(k)}$ we have*

$$|(f_a^{r_i+s+1})'(c_0)| \geq e^{\tilde{\lambda}(r_i+s+1)^{\beta}} \tag{7.1.9}$$

Proof. We know that $f_a(c_\tau(a)) \in f_a(\Delta^+)$ and then hypothesize $(H1)$ and applying chain rule implies that

$$\begin{aligned}
|(f_a^{r_i+s+1})'(c_0)| &= |(f_a^{r_i+s-\tau})'(f_a^{\tau+1}(c_0))||(f_a^{\tau+1})'(c_0)| \\
&\geq e^{\lambda(r_i+s-\tau)} e^{\tilde{\lambda}(\tau+1)^\beta} \quad \text{by } (H1) \text{ and lemma } (7.1.3) \\
&\geq e^{\tilde{\lambda}(r_i+s-\tau)^\beta} e^{\tilde{\lambda}(\tau+1)^\beta} \quad (\text{since } \lambda \geq \tilde{\lambda} \text{ and } \beta < 1) \\
&\geq e^{\tilde{\lambda}(r_i+s+1)^\beta}
\end{aligned}$$

∎

7.1.3 Proof of Proposition (4.1.1)

We are turn now to the proof of Proposition (4.1.1).

Proof of Proposition (4.1.1). Suppose that n is given and $a \in \Omega^{(n)}$ arbitrary is chosen. Then there exists $\omega^{(n)} \in \mathcal{P}^n$ such that $a \in \omega^{(n)}$. If $\omega^{(n)}$ have not experienced any return then obviously hypothesize$(H1)$ results the Proposition. So we may assume that $\{r_i\}_{i=1}^{t+1}$ is the sequence of returns with $r_1 < ... < r_t \leq n \leq r_{t+1}$. First of all using hypothesize $(H1)$ we obtain $(\tilde{\Phi})_{r_1}$. Indeed, we have

$$|(f_a^{r_1})'(c_0(a))| \geq e^{\lambda r_1^\beta} \quad \text{and} \quad |(f_a^i)'(c_0(a))| \geq e^{\tilde{\lambda} i^\beta} \text{ for } i < r_1.$$

Now an application of Lemma 7.1.2 gives $|(f_a^{r_2})'(c_0)| \geq e^{\lambda r_2^\beta}$. Also we apply Lemmas 7.1.3, 7.1.4 and relation 7.1.6 to get $|(f_a^i)'(c_0)| \geq e^{\tilde{\lambda} i^\beta}$ for $r_1 < i < r_2$. Then $(\tilde{\Phi})_{r_2}$ is true and then similar arguments shows that $(\tilde{\Phi})_{r_{t+1}}$ is valid. This in particular implies $(\Phi)_{r_{t+1}}$ and therefore $(\Phi)_n$ i.e,

$$|(f_a^i)'(c_0(a))| \geq e^{\tilde{\lambda} i^\beta} \text{ for } i \leq n.$$

■

Chapter 8

Uniform Bounded Distortion

8.1 Uniform Bounded Distortion

We consider an arbitrary subset $\omega = \omega^{(n-1)} \in \mathcal{P}^{(n-1)}$ and let n be a return time for ω. Note that this assumption is not an issue that looses the generality here since the actual application of bounded distortion is at the time of returns. Associated to ω we have an increasing sequence $\{r_i\}_{i=1}^{t+1}$ of returns and a corresponding sequence $\{p_i\}_{i=1}^{t+1}$ of binding periods. In particular we assume that for each $1 \leq i \leq t$, $\omega_{r_i} \subseteq \hat{I}_{\mu_i, m_i}$ for some μ_i and m_i. The following proposition brings us the main objective of this section.

Proposition 8.1.1. *There exists a constant $\mathcal{D}$ such that for all $\omega \in \mathcal{P}^{(n-1)}$, each $a, b \in \omega$, we have*

$$\frac{|(c_k)'(a)|}{|(c_k)'(b)|} \leq \mathcal{D} \quad \text{for all } k \leq r_t + p_t + 1$$

where r_t is the last return less than n and p_t is its corresponding binding period. Moreover, the same estimate holds also for all $k \leq n$ restricted to the parameter interval $\tilde{\omega} \subseteq \omega$ where $\tilde{\omega}_n \subseteq \Delta$.

Notice that when $n = r_1$ is the first return time then we put $r_0 = 0$. The proof of this proposition is obtained after some abstract calculation which is giving in the following.

8.1.1 Abstract distortion calculation

We choose an arbitrary $\omega \in \mathcal{P}^{(n-1)}$ and so Proposition 4.1.1 implies that $(\Phi)_{n-1}$ holds for each $a \in \omega$. Beside of that since n is the subsequent return after r_t, then the arguments of the previous section and in particular Lemma 7.1.2 results $(\Phi)_n$ is true for each $a \in \omega$. So as we mentioned in Lemma (5.1.1) the rate of growth of $c_k'(a)$ and $(f_a^k)'(c_0)$ is the same for $k \leq n$. Indeed, for each $a, b \in \omega$ and all $k \leq n$ we have

$$\frac{|c_k'(a)|}{|c_k'(b)|} \leq \frac{\mathcal{P}_1}{\mathcal{P}_2} \frac{|(f_a^k)'(c_0)|}{|(f_b^k)'(c_0)|} \tag{8.1.1}$$

and therefore it is sufficient to find an upper bound for $\frac{|(f_a^k)'(c_0)|}{|(f_b^k)'(c_0)|}$. A standard argument, using $\log|1+x| \leq |x|$, leads to:

$$\log \frac{|(f_a^k)'(c_0)|}{|(f_b^k)'(c_0)|} = \log \prod_{j=0}^{k-1} \frac{|c_j(a)|}{|c_j(b)|} \leq \sum_{j=0}^{k-1} \log|\frac{c_j(a) - c_j(b)}{c_j(b)} + 1| \leq \sum_{j=0}^{k-1} \frac{|c_j(a) - c_j(b)|}{|c_j(b)|} \tag{8.1.2}$$

Now our strategy is to find an upper bound for the summation $S = \sum_{j=0}^{r_t+p_t} \frac{|\omega_j|}{|c_j(b)|}$ which in particular is an upper bound for the last term of 8.1.2 with $k \leq r_t + p_t + 1$. For this we shall subdivide the sum into

$$S = \sum_{i=1}^{t} \sum_{r_{i-1}+p_{i-1}+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} \tag{8.1.3}$$

which, for notational convenience, we put $r_0 + p_0 + 1 = 0$. Note that we have divided the sum S into a finite number of blocks corresponding to pieces of itinerary starting immediately after a binding period and going through to the end of the subsequent binding period. In the next lemma we find a bound for the sum over each individual block.

Lemma 8.1.2. *Let $\omega \in \mathcal{P}^{(n-1)}$. For each $a, b \in \Omega$ and all $s \le p_i - 1$ we have:*

$$\frac{|c_s(a)|}{|c_s(b)|} \le 2 \quad \forall a, b \in \omega.$$

Proof. Let $a, b \in \omega$ be chosen arbitrary and we write:

$$|c_s(a)| - |c_s(b)| \le |\omega_s| \le |\omega_{p_i - 1}| < |\omega_{\mu_i}| < |\omega_{r_t}|$$

where the second inequality comes from $(\Phi)_n$, the third inequality because of the fact that $p_i < \mu_i$ by Lemma 6.1.3 and in the last inequality we used the fact that r_t is the last return before n. So $|c_s(a)| < |c_s(b)| + |\omega_{r_t}|$, and then we have:

$$\frac{|c_s(a)|}{|c_s(b)|} < 1 + \frac{|\omega_{r_t}|}{|c_s(b)|} \le 1 + 3\frac{l_{\mu_t}}{e^{-\mu_t^\alpha}} = 1 + 3\frac{e^{\mu_t^\alpha - (\mu_t - 1)^\alpha} - 1}{\mu_t^2} < 2 \quad \text{for } \mu_t \ge \mu_{\delta^+} \ge 2$$

∎

Lemma 8.1.3. *Suppose that $b \in \omega$ is given and $(\Phi)_n$ holds for b. Then for all $i = 1, ..., t$ we have*

$$\sum_{r_{i-1}+p_{i-1}+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} \le \mathcal{D}_1 e^{\mu_{r_i}^\alpha} |\omega_{r_i}| \tag{8.1.4}$$

where $\mathcal{D}_1$ is defined in (3.1.5).

Proof. First of all we subdivide the left hand side of (8.1.4) into two summations as follows.

$$\sum_{r_{i-1}+p_{i-1}+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} = \sum_{r_{i-1}+p_{i-1}+1}^{r_i} \frac{|\omega_j|}{|c_j(b)|} + \sum_{r_i+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} \tag{8.1.5}$$

We shall estimate each of the two terms in separate arguments. In the first summation for $r_{i-1}+p_{i-1}+1 \leq j \leq r_i$ we have

$$\begin{aligned}
|\omega_{r_i}| = |(c_{r_i} \circ c_j^{-1})(\omega_j)| &= |(c_{r_i} \circ c_j^{-1})'(\zeta)||\omega_j| \quad \text{for some } \zeta \in \omega_j \\
&= \frac{|c'_{r_i}(\bar{a})|}{|c'_j(\bar{a})|}|\omega_j| \quad \text{for } \bar{a} = c_j^{-1}(\zeta) \\
&\geq \frac{\mathcal{P}_1}{\mathcal{P}_2}\frac{|(f_{\bar{a}}^{r_i})'(c_0)|}{|(f_{\bar{a}}^{j})'(c_0)|}|\omega_j| \quad \text{by Lemma 5.1.1} \\
&= \frac{\mathcal{P}_1}{\mathcal{P}_2}|(f_{\bar{a}}^{r_i-j})'(f_{\bar{a}}^{j}(c_0))||\omega_j|
\end{aligned}$$

and applying condition $(H1)$, implies that

$$|\omega_j| \leq \frac{\mathcal{P}_2}{\mathcal{P}_1}e^{-\lambda(r_i-j)}|\omega_{r_i}| \tag{8.1.6}$$

Also since $\omega_{r_i} \subseteq \hat{I}_{\mu_i,m_i}$ for some $I_{\mu_i,m_i} \in \mathcal{I}^+$ and so in particular $|c_j(b)| \geq |c_{r_i}(b)| \geq e^{-(\mu_i+1)^\alpha}$, we can write

$$\begin{aligned}
\sum_{r_{i-1}+p_{i-1}+1}^{r_i}\frac{|\omega_j|}{|c_j(b)|} \leq \sum_{r_{i-1}+p_{i-1}+1}^{r_i}\frac{\mathcal{P}_2}{\mathcal{P}_1}\frac{e^{-\lambda(r_i-j)}}{e^{-(\mu_i+1)^\alpha}}|\omega_{r_i}| &= \Big(\frac{\mathcal{P}_2}{\mathcal{P}_1}e\sum_{r_{i-1}+p_{i-1}+1}^{r_i}e^{-\lambda(r_i-j)}\Big)e^{\mu_i^\alpha}|\omega_{r_i}| \\
&\leq \Big(\frac{\mathcal{P}_2}{\mathcal{P}_1}e\sum_{j=0}^{\infty}e^{-\lambda j}\Big)e^{\mu_i^\alpha}|\omega_{r_i}|
\end{aligned}$$

and so the definition of $\mathcal{D}_2$ in (3.1.6) results that:

$$\sum_{r_{i-1}+p_{i-1}+1}^{r_i}\frac{|\omega_j|}{|c_j(b)|} \leq \mathcal{D}_2|\omega_{r_i}|e^{\mu_{r_i}^\alpha} \tag{8.1.7}$$

For the second summation of the right hand of (8.1.5), or in other words for $r_i+1 \leq j \leq$

$r_i + p_i$, using the auxiliary index $s = j - (r_i + 1)$, we can write

$$|\omega_j| = |\omega_{r_i+s+1}| = |(c_{r_i+s+1} \circ c_{r_i}^{-1})(\omega_{r_i})| = |(c_{r_i+s+1} \circ c_{r_i}^{-1})'(\zeta)||\omega_{r_i}| \quad \text{for some } \zeta \in \omega_j,$$

Thus letting $\bar{a} = c_{r_i}^{-1}(\zeta) \in \omega$, for all $0 \le s \le p_i - 1$, we get:

$$|\omega_j| = |\omega_{r_i+s+1}| \le \frac{\mathcal{P}_2}{\mathcal{P}_1}|(f_{\bar{a}}^{s+1})'(c_{r_i})||\omega_{r_i}| = \frac{2\mathcal{P}_2}{\mathcal{P}_1}|(f_{\bar{a}}^{s})'(f_{\bar{a}}(c_{r_i}))||(c_{r_i}(\bar{a}))||\omega_{r_i}| \tag{8.1.8}$$

To find a bound for $|(f_{\bar{a}}^s)'(f_{\bar{a}}(c_{r_i}))|$ we put $x = c_{r_i}(\bar{a})$ and apply the definition of binding period to obtain

$$|x_s(\bar{a}) - c_s(\bar{a})| = |(f_{\bar{a}}^s)'(\xi_0)||x_0(\bar{a}) - c_0(\bar{a})| \le \gamma_s|c_s(\bar{a})|$$

for some $\xi_0 \in (x_0(\bar{a}), c_0(\bar{a}))$ and now Lemma (6.1.2) implies that:

$$|(f_{\bar{a}}^s)'(f_{\bar{a}}(c_{r_i}))| \le \mathcal{B}^2|(f_{\bar{a}}^s)'(\xi_0)| \le \frac{\mathcal{B}^2\gamma_s|c_s(\bar{a})|}{|x_0(\bar{a}) - c_0(\bar{a})|} = \frac{\mathcal{B}^2\gamma_s|c_s(\bar{a})|}{|c_{r_i}(\bar{a})|^2}.$$

Then substituting this last inequality in (8.1.8) gives us

$$|\omega_{r_i+s+1}| \le \frac{2\mathcal{P}_2\mathcal{B}^2}{\mathcal{P}_1}\frac{\gamma_s|c_s(\bar{a})|}{|c_{r_i}(\bar{a})|}|\omega_{r_i}| \tag{8.1.9}$$

Also to estimate the dominator of the summation, i.e. $|c_j(b)|$ we proceed as following. First we let $y = c_{r_i}(b)$ and so one can write

$$|c_s(b)| - |y_s(b)| \le |c_s(b) - y_s(b)| \le \gamma_s|c_s(b)|$$

which results

$$|c_j(b)| = |c_{r_i+s+1}(b)| = f_b^{s+1}(c_{r_i}(b)) = |y_s(b)| \geq (1-\gamma_s)|c_s(b)|. \tag{8.1.10}$$

Now considering (8.1.9) and (8.1.10) we can write

$$\begin{aligned}
\sum_{s=0}^{p_i-1} \frac{|\omega_{r_i+s+1}|}{|c_{r_i+s+1}(b)|} &\leq (\sum_0^{p_i-1} \frac{2\mathcal{P}_2\mathcal{B}^2}{\mathcal{P}_1} \frac{\gamma_s}{1-\gamma_s} \frac{|c_s(\bar{a})|}{|c_{r_i}(\bar{a})||c_s(b)|})|\omega_{r_i}| \\
&\leq \frac{2\mathcal{P}_2\mathcal{B}^2}{\mathcal{P}_1}(\sum_0^{p_i-1} \frac{\gamma_s}{1-\gamma_s} \frac{|c_s(\bar{a})|}{|c_s(b)|})e^{(\mu_{r_i}+1)^\alpha}|\omega_{r_i}| \quad \text{since } |c_{r_i}(\bar{a})| \geq e^{-(\mu_{r_i}+1)^\alpha} \\
&\leq \frac{4\mathcal{P}_2\mathcal{B}^2}{\mathcal{P}_1}(\sum_0^{p_i-1} \frac{\gamma_s}{1-\gamma_s})e^{(\mu_{r_i}+1)^\alpha}|\omega_{r_i}| \quad \text{by Lemma 8.1.2} \\
&\leq \frac{4\mathcal{P}_2\mathcal{B}^2}{\mathcal{P}_1}(\sum_0^{N_0-1} \frac{\gamma_s}{1-\gamma_s} + \sum_{N_0}^{\infty} \frac{1}{s^2-1})e^{(\mu_{r_i}+1)^\alpha}|\omega_{r_i}| \quad \text{by } (H6)
\end{aligned}$$

and thus the definition of $\mathcal{D}_3$ in (3.1.6) gives us

$$\sum_{r_i+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} \leq \mathcal{D}_3 e^{\mu_{r_i}^\alpha}|\omega_{r_i}| \tag{8.1.11}$$

We recall (8.1.5) and write

$$\begin{aligned}
\sum_{r_{i-1}+p_{i-1}+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} &= \sum_{r_{i-1}+p_{i-1}+1}^{r_i} \frac{|\omega_j|}{|c_j(b)|} + \sum_{r_i+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} \\
&\leq \mathcal{D}_2 e^{\mu_i^\alpha}|\omega_{r_i}| + \mathcal{D}_3 e^{\mu_i^\alpha}|\omega_{r_i}| \quad \text{by (8.1.11) and (8.1.11)} \\
&= (\mathcal{D}_2+\mathcal{D}_3)e^{\mu_i^\alpha}|\omega_{r_i}| \\
&= \mathcal{D}_1 e^{\mu_i^\alpha}|\omega_{r_i}| \quad \text{by definition of } \mathcal{D}_1 \text{ in (3.1.5)}
\end{aligned}$$

$\blacksquare$

Substituting (8.1.4) in (8.1.3) we obtain

$$S = \sum_{i=1}^{t} \sum_{r_{i-1}+p_{i-1}+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} \leq \mathcal{D}_1 \sum_{i=1}^{t} |\omega_{r_i}| e^{\mu_{r_i}^{\alpha}} \tag{8.1.12}$$

Note that in the sum of the right-hand side of the last relation it may happen $\mu_{r_i} = \mu$ for a lot of i's and so we subdivide it as follows.

$$\sum_{i=1}^{t} |\omega_{r_i}| e^{\mu_{r_i}^{\alpha}} = \sum_{\mu \in \{\mu_{r_i}\}} e^{\mu^{\alpha}} \sum_{i:\mu_{r_i}=\mu} |\omega_{r_i}| \tag{8.1.13}$$

In the next lemma we estimate the total contribution of returns corresponding to a fixed μ.

Lemma 8.1.4. *For any* $\mu \geq (\log \frac{1}{\delta^+})^{\frac{1}{\alpha}}$,

$$\sum_{i:\mu_{r_i}=\mu} |\omega_{r_i}| \leq \mathcal{D}_4 \mathcal{D}_5 \frac{e^{-\mu^{\alpha}}}{\mu^2}$$

where $\mathcal{D}_4$ and $\mathcal{D}_5$ defined in (3.1.8).

Proof. Let $\{r_{i_j}\}_{j=1}^{s} \subseteq \{r_i\}$ be the increasing subsequence of returns for which $\omega_{r_{i_j}} \subseteq \hat{I}_{\mu,m}$ for some m. Using Proposition (6.1.1) implies that $|\omega_{r_{i_s}}| \geq (\mathcal{B}_2)^{k_j} |\omega_{r_{i_j}}|$, where $0 = k_s < k_{s-1} < ... < k_1$ and so one can write

$$\sum_{i:\mu_{r_i}=\mu} |\omega_{r_i}| = \sum_{j=1}^{s} |\omega_{r_{i_j}}| \leq \sum_{j=1}^{s} (\frac{1}{\mathcal{B}_2})^{k_j} |\omega_{r_{i_s}}| \leq (\sum_{j=0}^{\infty} (\frac{1}{\mathcal{B}_2})^{j}) |\omega_{r_{i_s}}|$$

and since $\mathcal{B}_2 > 1$ by $(F7)$, then the definition of $\mathcal{D}_4$ implies

$$\sum_{i:\mu_{r_i}=\mu} |\omega_{r_i}| \leq \mathcal{D}_4 |\omega_{r_{i_s}}| \tag{8.1.14}$$

But to find the length of $|\omega_{r_{i_s}}|$ we recall the procedure of construction $\mathcal{P}^{(k)}$ and $\omega^{(k)}$ which implies that $|\omega_{r_{i_s}}| \leq 2|I_{\mu,m}| + |I_{\mu-1,m}|$ and so

$$\begin{aligned}
|\omega_{r_{i_s}}| &\leq 2(\frac{e^{-(\mu-1)^\alpha} - e^{-\mu^\alpha}}{\mu^2}) + \frac{e^{-(\mu-2)^\alpha} - e^{-(\mu-1)^\alpha}}{(\mu-1)^2} \\
&= 2\frac{e^{-\mu^\alpha}}{\mu^2}(e^{\mu^\alpha-(\mu-1)^\alpha} - 1) + \frac{e^{-\mu^\alpha}}{(\mu-1)^2}(e^{\mu^\alpha-(\mu-2)^\alpha} - e^{\mu^\alpha-(\mu-1)^\alpha}) \\
&= \frac{e^{-\mu^\alpha}}{\mu^2}(2(e^{\mu^\alpha-(\mu-1)^\alpha} - 1) + (\frac{\mu}{\mu-1})^2(e^{\mu^\alpha-(\mu-2)^\alpha} - e^{\mu^\alpha-(\mu-1)^\alpha})) \\
&\leq \frac{e^{-\mu^\alpha}}{\mu^2}(2(e^{\mu^\alpha-(\mu-2)^\alpha} - 1)) \quad \text{since } (\frac{\mu}{\mu-1})^2 \leq 2 \quad \text{for } \mu \geq 4 \\
&\leq \frac{e^{-\mu^\alpha}}{\mu^2}(2(e^{2^\alpha} - 1)
\end{aligned}$$

and therefor the proof of Lemma follows by definition of $\mathcal{D}_5$ and (8.1.14). ∎

Now we summarize the results of this subsection in the following lemma which introduces an upper bound for the summation S.

Lemma 8.1.5. *Consider the summation* $S = \sum_{i=1}^{t} \sum_{r_{i-1}+p_{i-1}+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|}$, *then*

$$S \leq \mathcal{D}_7$$

in which $\mathcal{D}_7$ *was given by* (3.1.9)

Proof. As we have seen, by (8.1.12) and (8.1.13) we have

$$S = \sum_{i=1}^{t} \sum_{r_{i-1}+p_{i-1}+1}^{r_i+p_i} \frac{|\omega_j|}{|c_j(b)|} \le \mathcal{D}_1 \sum_{i=1}^{t} |\omega_{r_i}| e^{\mu_{r_i}^{\alpha}} \le \mathcal{D}_1 \sum_{\mu \in \{\mu_{r_i}\}} e^{\mu^{\alpha}} \sum_{i:\mu_{r_i}=\mu} |\omega_{r_i}|$$

We apply now Lemma 8.1.4 and get

$$S \le \mathcal{D}_1 \mathcal{D}_4 \mathcal{D}_5 \sum_{\mu \ge \mu_{\delta+}} \frac{1}{\mu^2} = \mathcal{D}_1 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6$$

where $\mathcal{D}_6$ is given by (3.1.9). Therefore the definition of $\mathcal{D}_7$ completes the proof of Lemma.

∎

8.1.2 Bounding the distortion

We turn now to the proof of the main result of this section.

Proof of Proposition 8.1.1. For $k \le r_t + p_t + 1$, recalling the definition of S and (8.1.1), 8.1.2 we can write

$$\frac{|c_k'(a)|}{|c_k'(b)|} \le \frac{\mathcal{P}_1}{\mathcal{P}_2} \frac{|(f_a^k)'(c_0)|}{|(f_b^k)'(c_0)|} \le \frac{\mathcal{P}_1}{\mathcal{P}_2} e^{S} \le \frac{\mathcal{P}_1}{\mathcal{P}_2} e^{\mathcal{D}_7} \le \mathcal{D}$$

where the last two inequalities come from Lemma 8.1.5 and the definition of $\mathcal{D}$. However if $k > r_t + p_t + 1$, then we restrict ourselves to some subinterval $\bar{\omega} \subseteq \omega$ with $\bar{\omega}_k \subseteq \Delta$. A similar argument as like as in (8.1.2), we write

$$\begin{aligned} \frac{|c_k'(a)|}{|c_k'(b)|} &\le \frac{\mathcal{P}_1}{\mathcal{P}_2} e^{\sum_{j=0}^{r_t+p_t} \frac{|\bar{\omega}_j|}{|c_j(b)|} + \sum_{r_t+p_t+1}^{k-1} \frac{|\bar{\omega}_j|}{|c_j(b)|}} \\ &\le \frac{\mathcal{P}_1}{\mathcal{P}_2} e^{S + \sum_{r_t+p_t+1}^{k-1} \frac{|\bar{\omega}_j|}{|c_j(b)|}} \qquad \text{since } |\bar{\omega}_j| \le |\omega_j| \end{aligned}$$

But to compute $\sum_{r_t+p_t+1}^{k-1} \frac{|\bar{\omega}_j|}{|c_j(b)|}$ we proceed as follows. First using $|\bar{\omega}_k| = |c_k \circ c_j^{-1})(\bar{\omega}_j)|$ and (D_1) we have:

$$|\bar{\omega}_j| \leq \Gamma^{-1} \frac{\mathcal{P}_2}{\mathcal{P}_1} e^{-\lambda(k-j)} |\bar{\omega}_k| \leq \Gamma^{-1} \frac{\mathcal{P}_2}{\mathcal{P}_1} e^{-\lambda(k-j)} \delta$$

and since $\omega_j \cap \Delta = \emptyset$ implies $|c_j(b)| \geq \delta$, then

$$\sum_{r_t+p_t+1}^{k-1} \frac{|\bar{\omega}_j|}{|c_j(b)|} \leq \Gamma^{-1} \frac{\mathcal{P}_2}{\mathcal{P}_1} \frac{\delta}{\delta} \sum_{r_t+p_t+1}^{k-1} e^{-\lambda(k-j)} \leq \frac{\mathcal{P}_2}{\mathcal{P}_1 \Gamma} \sum_{i=1}^{\infty} e^{-\lambda i}$$

and so the definition of $\mathcal{D}_2$ gives us

$$\sum_{r_t+p_t+1}^{k-1} \frac{|\bar{\omega}_j|}{|c_j(b)|} \leq \frac{e^{-\lambda-1} \mathcal{D}_2}{\Gamma}.$$

Therefore by the definition of $\mathcal{D}$ we get

$$\frac{|c_k'(a)|}{|c_k'(b)|} \leq \frac{\mathcal{P}_1}{\mathcal{P}_2} e^{S + (\frac{\mathcal{P}_1}{\mathcal{P}_2})^2 \frac{\mathcal{D}_2}{e\Gamma_1}} \leq \mathcal{D}.$$

∎

Chapter 9

Parameter exclusion estimates

9.1 Parameter exclusion estimates

In this section we prove proposition 4.1.2 which gives a bound the proportion of excluded parameters at each step n. Clearly it is sufficient to show that for every $\omega \in \mathcal{P}^{(n-1)}$ and letting

$$\omega' = \omega \setminus (\omega \cap \Omega^{(n)})$$

we have

$$|\omega'| \leq \alpha_n |\omega|.$$

Our strategy is to compare the sizes of the images ω'_n and ω_n of these intervals and then use the bounded distortion to show that the ratio between the size of the images is uniformly comparable to the ratio between the intervals themselves. Surely no exclusions are required if n is not a return time. We may therefore assume that n is a return time. We associate to ω a maximal sequence $r_1 < r_2 < ... < r_t < n = r_{t+1}$ of returns, and an associated sequence $p_1, p_2, ..., p_t$ of binding periods.

Lemma 9.1.1. *Let $\omega \in \Omega^{(n-1)}$ and n be the return time for ω. If $\omega' = \omega \setminus (\omega \cap \Omega^{(n)})$, then:*

$$|\omega'_n| \leq 2e^{-(n-1)^\alpha}.$$

Proof. We recall the construction of $\mathcal{P}^{(n)}$ and $\Omega^{(n)}$ from section 4.1. More precisely, ω'_n is exactly at most the union of interval $(-e^{-n^\alpha}, e^{-n^\alpha})$ and two intervals of length l_n. This gives the statement of Lemma. ∎

However to estimate $|\omega_n|$ we look to immediate parent interval $\tilde{\omega}$ which at some time $k < n$, ω was born. We need to distinguish two cases which requires distinct arguments. In the first case it might happen that, $I_{\mu,m} \subseteq \omega_{r_j} \subseteq \hat{I}_{\mu,m}$ for some $I_{\mu,m} \in \mathcal{I}^+$ and $r_j \leq r_t$. For the second case ω is a subset of it's parent interval $\tilde{\omega}$ such that at time k landed outside Δ^+ with $|\omega_k| \geq \frac{\mathcal{P}_2 e^{-\lambda}}{\Gamma \mathcal{P}_1} d$. Indeed in this case $\omega \in \{\omega^\pm\}$ and $k > r_t$.

Lemma 9.1.2. *Let n be a return time for $\omega \in \Omega^{(n-1)}$. Then:*

$$|\omega_n| \geq \min\{d, \frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}})^2 e^{(1-\gamma)(n-1)^\alpha} l_{n-1}\}$$

Proof. First we consider the case which $I_{\mu,m} \subseteq \omega_{r_j} \subseteq \hat{I}_{\mu,m}$ with $r_j \leq r_t$. Proposition 6.1.1 implies that

$$\begin{aligned}
|\omega_n| &\geq (\frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}})^2 e^{(1-\gamma)\mu^\alpha})^k |\omega_{r_j}| \quad \text{when} \quad k = t+1-j \\
&\geq \frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}})^2 e^{(1-\gamma)\mu^\alpha} |\omega_{r_j}| \quad \text{since } \mathcal{B}_2 > 1 \quad \text{by } (F7) \\
&\geq \frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}})^2 e^{(1-\gamma)\mu^\alpha} l_\mu
\end{aligned}$$

On the other hand the function $L(\mu) = e^{(1-\gamma)\mu^\alpha} l_\mu$ is a decreasing function with respect to

μ, and since $\mu \le r_t \le n-1$, then

$$|\omega_n| \ge \frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}})^2 e^{(1-\gamma)(n-1)^\alpha} l_{n-1} \tag{9.1.1}$$

For the remaining case which there exists some k with $r_t < k < n$ such that ω_k landed outside of Δ^+, we proceed as following:

$$\begin{aligned}
|\omega_n| &= |(c_n \circ c_k^{-1})'(\zeta_k)||\omega_k| \quad \text{for some } \zeta_k \in \omega_k \\
&\ge \frac{\mathcal{P}_1}{\mathcal{P}_2}\frac{|(f_{\bar{a}}^n)'(c_0)|}{|(f_{\bar{a}}^k)'(c_0)|}|\omega_k| \quad \text{with } \bar{a} = c_k^{-1}(\zeta_k) \in \omega \\
&= \frac{\mathcal{P}_1}{\mathcal{P}_2}|(f_{\bar{a}}^{n-k})'(c_k(\bar{a}))||\omega_k|
\end{aligned}$$

and if $\omega_n \subseteq \Delta^+$ then the fact that $|\omega_k| \ge \frac{\mathcal{P}_2}{\mathcal{P}_1\Gamma}e^{-\lambda}d$

$$|\omega_n| \ge \frac{\mathcal{P}_1}{\mathcal{P}_2}\Gamma e^{\lambda(n-k)}|\omega_k| \ge \frac{\mathcal{P}_1}{\mathcal{P}_2}\Gamma e^{\lambda}|\omega_k| \ge d$$

But if ω_n does not fully contained in Δ^+ and $\omega_n \cap (-e^{-n^\alpha}, e^{-n^\alpha}) \ne \emptyset$ then condition $(F9)$ gives us

$$|\omega_n| \ge \delta^+ - \delta \ge d,$$

We note that when $\omega_n \cap (-e^{-n^\alpha}, e^{-n^\alpha}) = \emptyset$ then $\alpha_n = 0$ and nothing remains to proof. This concludes the proof of Lemma. ∎

9.1.1 Proof of Proposition 4.1.2

Before turning to the detailed proof of the proposition, two supplementary lemmas are given as following.

Lemma 9.1.3. *For each $n \geq 2$ we have*

$$l_n \geq \frac{\alpha e^{-n^\alpha}}{n^{3-\alpha}}$$

Proof. By Mean Value Theorem there exists some $\zeta \in (n-1, n)$ such that

$$l_n = \frac{|e^{-(n-1)^\alpha} - e^{-n^\alpha}|}{n^2} = \frac{|-\alpha\zeta^{\alpha-1}e^{-\zeta^\alpha}((n-1)-n)|}{n^2} \geq \frac{\alpha n^{\alpha-1}e^{-n^\alpha}}{n^2} = \frac{\alpha e^{-n^\alpha}}{n^{3-\alpha}}$$

∎

Lemma 9.1.4. *For each $n \geq 1 + \mu_\delta$, we have*

$$\frac{\mathcal{D}e^{2\alpha}}{\alpha}(\frac{\mathcal{P}_2\mathcal{B}}{\mathcal{P}_1})^2(n-1)^{3-\alpha}e^{(\gamma-1)(n-1)^\alpha} \leq \alpha_n$$

Proof. We recall the definition of α_n and for $n \geq 1 + \mu_\delta$ let:

$$N(n) = \ln(\frac{\mathcal{D}e^{2\alpha}}{\alpha}(\frac{\mathcal{P}_2\mathcal{B}}{\mathcal{P}_1})^2) + (3-\alpha)\ln(n-1) + (\gamma-1)(n-1)^\alpha - (\gamma-1)(n-1)^\nu$$

To prove the Lemma it is sufficient to show that $N(n)$ is negative for $n \geq 1 + \mu_\delta$. Differentiating of $N(n)$ with respect to n we have

$$N'(n) = \frac{1}{n-1}(3 - \alpha - \alpha(1-\gamma)(n-1)^\alpha + \nu(1-\gamma)(n-1)^\nu) \leq 0$$

where in the last inequality we used condition $(F8)$ and the fact that its left hand term is decreasing with n and $\mu_\delta \leq n-1$. To achieve that $N(n)$ is negative and then complete the proof we recall condition $(F0)$ which implies $N(\mu_\delta)$ and so $N(n)$ is negative. ∎

Proof of Proposition 4.1.2. It is enough to show that for each $\omega \in \mathcal{P}^{n-1}$ we have $\frac{|\omega'|}{|\omega|} \leq \alpha_n$ where $|\omega'|$ is the portion of $|\omega|$ which excluded at time n. A straightforward application of Mean Value Theorem gives us

$$\frac{|\omega_n'|}{|\omega_n|} = \frac{|c_n'(a)|}{|c_n'(b)|}\frac{|\omega'|}{|\omega|} \qquad \text{for some } a \in \omega' \text{ and } b \in \omega,$$

and so using Proposition 8.1.1 we get $\frac{|\omega'|}{|\omega|} \leq \mathcal{D}\frac{|\omega_n'|}{|\omega_n|}$. By Lemma 9.1.1 we know that $|\omega_n'| \leq 2e^{-(n-1)^\alpha}$. Observing Lemma 9.1.2 one may consider two cases. In the first case it happens that $|\omega_n| \geq \frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}})^2 e^{(1-\gamma)(n-1)^\alpha} l_{n-1}$ and then for $n-1 \geq \mu_\delta$ we can write:

$$\begin{aligned}
\frac{|\omega'|}{|\omega|} \leq \mathcal{D}\frac{|\omega_n'|}{|\omega_n|} &\leq \mathcal{D}\frac{2e^{-(n-1)^\alpha}}{\frac{2}{e^{2\alpha}}(\frac{\mathcal{P}_1}{\mathcal{P}_2\mathcal{B}})^2 e^{(1-\gamma)(n-1)^\alpha} l_{n-1}} \\
&\leq \frac{\mathcal{D}e^{2\alpha}}{\alpha}(\frac{\mathcal{P}_2\mathcal{B}}{\mathcal{P}_1})^2 \frac{e^{-(n-1)^\alpha}(n-1)^{3-\alpha}}{e^{(1-\gamma)(n-1)^\alpha} e^{-(n-1)^\alpha}} \qquad \text{by Lemma 9.1.3} \\
&\leq \frac{\mathcal{D}e^{2\alpha}}{\alpha}(\frac{\mathcal{P}_2\mathcal{B}}{\mathcal{P}_1})^2 e^{(\gamma-1)(n-1)^\alpha}(n-1)^{3-\alpha} \\
&\leq \alpha_n \quad \text{by Lemma 9.1.4}
\end{aligned}$$

For the second case where $|\omega_n| \geq d$, applying definition of d one can write:

$$\frac{|\omega'|}{|\omega|} \leq \mathcal{D}\frac{|\omega_n'|}{|\omega_n|} \leq \frac{2\mathcal{D}e^{-(n-1)^\alpha}}{d} \leq \frac{2\mathcal{D}e^{-(n-1)^\alpha}}{2\mathcal{D}e^{-\mu_\delta^\alpha+(1-\gamma)\mu_\delta^\nu}} \leq \frac{e^{-(n-1)^\alpha}}{e^{-(n-1)^\alpha+(1-\gamma)(n-1)^\nu}} \leq \alpha_n$$

where the third inequality uses the fact that the function $N_3(n) = -n^\alpha + (1-\gamma)n^\nu$ is monotone decreasing with n and $n-1 \geq \mu_\delta$. ∎

Chapter 10

Computation of number

10.1 Computation of number

In this section we find an explicit estimate of the set of stochastic parameters for the quadratic family $f_a(x) = -x^2+a$. Indeed, we take advantage of being closed to parameter value $a = 2$ in parameter space to be able to obtain a lower bound for Ω^*.

10.1.1 Preliminary dynamical facts

This part is devoted to give some preliminary facts about the dynamic of the quadratic family which helps us to choice the constants explicitly in such a way that dynamical hypotheses hold. We start by a supplementary lemma that is applied in the subsequent lemma.

Lemma 10.1.1. *Let $a \in [\sqrt{A} - \varepsilon, \sqrt{A}]$ in which $\sqrt{A} = 2 - \sigma$ and σ satisfies*

$$\sigma(2 - \sigma - \varepsilon) \geq \delta^2 \tag{10.1.1}$$

then

(a) $2a \geq a^2 + \delta^2$.

(b) If $|x| \geq \delta$, *then* $|f_a(x)| \leq \sqrt{A} - \delta^2$.

Proof. We can write

$$2a - a^2 = a(2-a) \geq (\sqrt{A} - \varepsilon)(2 - \sqrt{A}) = \sigma(2 - \sigma - \varepsilon)$$

and then part (a) follows from (10.1.1). For the proof of part (b), first since $|x| \geq \delta$, then we have $f_a(x) = -x^2 + a \leq \sqrt{A} - \delta^2$. On the other hand using the fact that $|x| \leq a$ implies that $-x^2 + a \geq -a^2 + a$, and so by part (a) we get

$$f_a(x) = -x^2 + a \geq -a + \delta^2 \geq -(\sqrt{A} - \delta^2)$$

Therefor the proof is completed. ∎

Lemma 10.1.2. *Suppose that* $\sqrt{A} = 2 - \sigma$ *where* $\sigma = 0$ *or satisfies in* (10.1.1). *Assume that for any* $a \in [\sqrt{A} - \varepsilon, \sqrt{A}]$:

$$|f_a^j(x)| \geq \delta, \quad j = 0, ..., n-1$$

Then, we have

$$|(f_a^n)'(x)| \geq \sqrt{\frac{A - (f_a^n(x))^2}{A - \delta^2}} \left(\frac{2\delta}{\varepsilon^{\frac{1}{2}} + \delta} \sqrt{\frac{\delta^2(2\sqrt{A} - \delta^2)}{\sigma(2-\sigma) + \delta^2(2\sqrt{A} - \delta^2)}}\right)^n \qquad (10.1.2)$$

Proof. We let $y = f_a^i(x)$ for some i with $0 \le i \le n-1$ and then by using expression

$$\frac{1}{\sqrt{A-y^2}}\sqrt{\frac{A-y^2}{A-(f_a(y))^2}}\sqrt{\frac{A-(f_a(y))^2}{A-(f_a^2(y))^2}}\cdots\sqrt{\frac{A-(f_a^{k-1}(y))^2}{A-(f_a^k(y))^2}}\sqrt{A-(f_a^k(y))^2} = 1$$

write

$$\begin{aligned} |(f_a^k)'(y)| &= |2y||2f_a(y)|...|2f_a^{k-1}(y)| \\ &= \frac{1}{\sqrt{A-y^2}}|2y|\sqrt{\frac{A-y^2}{A-(f_a(y))^2}}...|2f_a^{k-1}(y)|\sqrt{\frac{A-(f_a^{k-1}(y))^2}{A-(f_a^k(y))^2}}\sqrt{A-(f_a^k(y))^2} \end{aligned}$$

To simplify the argument we put $B(z) = 2|z|\sqrt{\frac{A-z^2}{A-(f_a(z))^2}}$ where $z = f_a^j(y), j = 0, ..., k$, and so we have

$$|(f_a^k)'(y)| = \sqrt{\frac{A-(f_a^k(y))^2}{A-y^2}}B(y)B(f_a(y))...B(f_a^{k-1}(y)) \tag{10.1.3}$$

In order to find a lower bound for $B(z)$, we substitute $f_a(z) = -z^2 + a$ in the definition of $B(z)$. So we get Also substituting $f_a(z) = -z^2 + a$ in $B(z)$, implies that

$$\begin{aligned} B(z) = \frac{2|z|\sqrt{A-z^2}}{\sqrt{\sqrt{A}-f_a(z)}\sqrt{\sqrt{A}+f_a(z)}} &= \frac{2|z|}{\sqrt{\sqrt{A}-a+z^2}}\frac{\sqrt{A-z^2}}{\sqrt{\sqrt{A}+a-z^2}} \\ &\ge \frac{2|z|}{\sqrt{\sqrt{A}-a+z^2}}\frac{\sqrt{A-z^2}}{\sqrt{2\sqrt{A}-z^2}} \end{aligned}$$

where in the second inequality we used the assumption $a \le \sqrt{A}$. Now first we observe that if $\sigma = 0$ then for z with $|z| \ge \delta$ one may write

$$B(z) \ge \frac{2|z|}{\sqrt{2-a+z^2}}\frac{\sqrt{4-z^2}}{\sqrt{4-z^2}} \ge \frac{2|z|}{\varepsilon^{\frac{1}{2}}+|z|} \ge \frac{2\delta}{\varepsilon^{\frac{1}{2}}+\delta} \tag{10.1.4}$$

in which we used the facts that $|2-a| < \varepsilon$ and $\frac{2|z|}{\varepsilon^{\frac{1}{2}}+|z|}$ is monotone increasing in$|z|$. Therefore putting $y = x, k = n$ in (10.1.3) and using (10.1.4) we have

$$|(f_a^n)'(x)| = \sqrt{\frac{A-(f_a^n(x))^2}{A-\delta^2}}(\frac{2\delta}{\varepsilon^{\frac{1}{2}}+\delta})^n$$

This concludes the proof of Lemma for $\sigma = 0$. In the rest of proof we assume $\sigma \neq 0$ and satisfies in (10.1.1), i.e.; $\sigma(2-\sigma-\varepsilon) \geq \delta^2$. For this, depending on the location of x we need to consider two distinct cases which requires different arguments as follows.

Case 1. $\delta < |x| \leq -\delta^2 + a$.

Let $z = f_a^i(x), i = 0, ..., n-1$. Part (b) of Lemma 10.1.1 implies that $|z| \leq \sqrt{A} - \delta^2$ and since $\sqrt{1-\frac{2\sqrt{A}-A}{2\sqrt{A}-z^2}}$ is decreasing in $|z|$, we can write: we get

$$\frac{\sqrt{A-z^2}}{\sqrt{2\sqrt{A}-z^2}} = \sqrt{1-\frac{2\sqrt{A}-A}{2\sqrt{A}-z^2}} \geq \sqrt{1-\frac{2\sqrt{A}-A}{2\sqrt{A}-(\sqrt{A}-\delta^2)^2}}\sqrt{\frac{\delta^2(2\sqrt{A}-\delta^2)}{2\sqrt{A}-A+\delta^2(2\sqrt{A}-\delta^2)}}.$$

Then, recalling the above argument for $B(z)$ and using the facts that $\sqrt{A}-a < \varepsilon$ and $\frac{2|z|}{\sqrt{\sqrt{A}-a+z^2}}$ is increasing in $|z|$ we get

$$B(z) \geq \frac{2|z|}{\sqrt{\sqrt{A}-a+z^2}}\frac{\sqrt{A-z^2}}{\sqrt{2\sqrt{A}-z^2}} \geq \frac{2\delta}{\varepsilon^{1/2}+\delta}\sqrt{\frac{\delta^2(2\sqrt{A}-\delta^2)}{\sigma(2-\sigma)+\delta^2(2\sqrt{A}-\delta^2)}} \tag{10.1.5}$$

Therefor, letting $y = x, z = f_a^i(x)$ for $i = 0, ..., n-1$ and $k = n$, applying (10.1.3) and (10.1.4) lead to:

$$|(f_a^n)'(x)| \geq \sqrt{\frac{A-(f_a^n(x))^2}{A-\delta^2}}(\frac{2\delta}{\varepsilon^{1/2}+\delta}\sqrt{\frac{\delta^2(2\sqrt{A}-\delta^2)}{\sigma(2-\sigma)+\delta^2(2\sqrt{A}-\delta^2)}})^n$$

Case 2. $-\delta^2 + a < |x| \leq a$.

We let $y = f_a(x)$ in (10.1.3) and then a similar argument, like the previous case, for $z = f_a^i(y)$ with $i = 0, , n-2$ results that

$$|(f_a^{n-1})'(f_a(x))| \geq \sqrt{\frac{A - (f_a^n(x))^2}{A - \delta^2}}(\frac{2\delta}{\varepsilon^{1/2} + \delta}\sqrt{\frac{\delta^2(2\sqrt{A} - \delta^2)}{\sigma(2-\sigma) + \delta^2(2\sqrt{A} - \delta^2)}})^{n-1}$$

On the other hand due to the fact that $|x| > -\delta^2 + a \geq 1$, it follows that

$$2|x| \geq 2 \geq \frac{2\delta}{\varepsilon^{1/2} + \delta}\sqrt{\frac{\delta^2(2\sqrt{A} - \delta^2)}{\sigma(2-\sigma) + \delta^2(2\sqrt{A} - \delta^2)}}$$

and therefore we get

$$|(f_a^n)'(x)| = 2|x||f_a^{n-1})'(f_a(x))| \geq \sqrt{\frac{A - (f_a^n(x))^2}{A - \delta^2}}(\frac{2\delta}{\varepsilon^{1/2} + \delta}\sqrt{\frac{\delta^2(2\sqrt{A} - \delta^2)}{\sigma(2-\sigma) + \delta^2(2\sqrt{A} - \delta^2)}})^n$$

This concludes the proof of Lemma. ∎

Note that observing dynamical hypothesize $(H1)$, the statement of previous lemma allows us to assume that

$$\Gamma = \sqrt{\frac{A - (f_a^n(x))^2}{A - \delta^2}}, \quad \lambda = \ln(\frac{2\delta}{\varepsilon^{1/2} + \delta}\sqrt{\frac{\delta^2(2\sqrt{A} - \delta^2)}{\sigma(2-\sigma) + \delta^2(2\sqrt{A} - \delta^2)}})$$

It is also necessary, at least, to require $\varepsilon^{1/2} < \delta$ in order to get positive values of λ. So we fix here the length of Ω as

$$\varepsilon = \delta^{2+\frac{1}{10}}$$

In our knowledge in the literature related to the quadratic family the previous lemma is stated based on the conjugacy of tent map and the quadratic family for $a = 2$. But, we don't use it which permits us to study the parameter intervals not necessarily of the form $[a_*, 2]$. In fact we can consider $\Omega = [a_*, a^*]$, in which $a^* = \sqrt{A} = 2 - \sigma$. However, since the calculations all here are analytic , we should study the values of A for which $\sqrt{\frac{\delta^2(2\sqrt{A}-\delta^2)}{\sigma(2-\sigma)+\delta^2(2\sqrt{A}-\delta^2)}} > \frac{1}{2}$. A simple calculation then shows that we may consider those values of $A = 2 - \sigma$ in which $\delta^2 \leq \sigma \leq 5\delta^2$.

We emphasize that the calculations are all completely analytic in this work. It should be noted, however, that the constants Γ and λ, using a rigorous computational method, can be estimated for all parameter values.

Lemma 10.1.3. *Let $\Omega = [a_*, a^*]$ be an interval of parameters. For each $a \in \Omega$, we have*

$$f_a^n(\Delta) \cap \Delta = \emptyset \quad \textit{for all} \quad 1 \leq n \leq N_0 = 2 + [\frac{1}{\log 4} \log \frac{2 - \sqrt{2+\sqrt{2}}}{|2 + a_* - a_*^2 + \delta^2(2a_* - \delta^2)|}]$$

Indeed, $f_a^n(\Delta)$ located in the left of the point $x = -\sqrt{a+\sqrt{a}}$.

Proof. Let $p = \frac{-1-\sqrt{1+4a}}{2}$ be the fixed point of f_a located in $[-2, 0]$. We let also $J_1 = [p, f_a(f_a(\delta))] = [p, f_a(\delta_0)]$ and $J_{m+1} = f_a^m(J_1)$. Since f_a is increasing on $[-2, 0]$, then considering those values of m for them J_m is still remained in $[-2, 0]$, implies that

$$J_{m+1} = [p, f_a^{m=1}(\delta_0)] = [p, \delta_{m+1}] \supseteq f_a^{m+1}(f_a(\Delta)).$$

Thus we get the result by finding the values of m which $J_{m+1} \subseteq [p, -\sqrt{a+\sqrt{a}}]$, or in particular satisfies

$$|J_{m+1}| \leq |-2 + \sqrt{a+\sqrt{a}}| \tag{10.1.6}$$

In fact, this shows that J_{m+1} and subsequently f_a^{m+2} remains in the left of $x = -\sqrt{a+\sqrt{a}}$. On the other hand a straightforward application of Mean Value Theorem, using the facts

$$|(f_a)'(x)| \leq 4, \forall x \in [-2,2] \quad \text{and} \quad |J_1| \leq |[-2, f_a(\delta_0)]| = |2+a-a^2+\delta^2(2a-\delta^2)|,$$

gives

$$|J_{m+1}| = |f_a^m(J_1)| \leq 4^m|J_1| \leq 4^m|2+a-a^2+\delta^2(2a-\delta^2)| \tag{10.1.7}$$

So, to conclude the proof, in according of relations 10.1.6, 10.1.7 and the fact that $|2 - \sqrt{a+\sqrt{a}}| \geq 2-\sqrt{2+\sqrt{2}}$ for $a \leq 2$ it is enough to choose m in such a way that satisfies

$$4^m|2+a-a^2+\delta^2(2a-\delta^2)| \leq |2-\sqrt{2+\sqrt{2}}|$$

We now observe that the left hand of the previous inequality is decreasing in a and therefore $f_a^{m+2}(\Delta)$ for $a \in [a_*, a^*]$ is suited in the left of $x = -\sqrt{a+\sqrt{a}}$, if

$$m \leq \frac{1}{\ln 4} \ln \frac{2-\sqrt{2+\sqrt{2}}}{|2+a_*-a_*^2+\delta^2(2a_*-\delta^2)|}$$

which concludes the statement of Lemma. ∎

10.1.2 Primitive constants and dynamical hypotheses

First we fix

$$\Omega = [2-\varepsilon, 2], \alpha = 0.31, \beta = 0.65, \delta = e^{-96}, \delta^+ = e^{-58}, \gamma = \frac{1}{2} \text{ and } \nu = \frac{14\alpha}{15}.$$

The result of the previous subsection and considering the formal conditions lead us to choose the primitive constants as are stated in the statement of the next proposition.

Proposition 10.1.4. *hypotheses* $(H1)-(H7)$ *are satisfied with constants*

$$\Gamma=\sqrt{\tfrac{4-(\delta^+)^2}{4-\delta^2}}\simeq 1, \quad ,\lambda=\ln(\tfrac{2}{1+\delta^{\frac{1}{20}}})\simeq 0.685, \quad \tilde{\lambda}=0.6,$$
$$N_0=[\tfrac{1}{\log 4}\log\tfrac{2-\sqrt{2+\sqrt{2}}}{|2+a_*-a_*^2+\delta^2(2a_*-\delta^2)|}]\simeq 137, \quad N=120$$
$$\mathcal{P}_1=1-\tfrac{1}{3.8}(\tfrac{1-(\frac{1}{2\sqrt{1.9+\sqrt{1.9}}})^N}{1-\frac{1}{2\sqrt{1.9+\sqrt{1.9}}}})-\tfrac{((e^{-\tilde{\lambda}})^m(2m+1-e^{-\tilde{\lambda}}(2m-1)))}{(1-e^{-\tilde{\lambda}})^2}\simeq 0.63$$
$$\mathcal{P}_2=\tfrac{3}{4}+\tfrac{((e^{-\tilde{\lambda}})^m(2m+1-e^{-\tilde{\lambda}}(2m-1)))}{(1-e^{-\tilde{\lambda}})^2}\simeq 0.76, \quad \mathcal{B}=???,$$
$$\begin{cases}\gamma_i\geq \sup_{a\in\Omega}\frac{|\Delta_i^+(a)|}{|c_i(a)|} & \textit{if} \quad 0\leq i<N_0\\ \gamma_i=\frac{1}{i^2}, & \textit{if} \quad i\geq N_0\end{cases}$$

Proof. Letting $A=4,\sigma=0$ and $\varepsilon=\delta^{2+\frac{1}{10}}$ in Lemma 10.1.2, we get

$$|(f_a^n)'(x)|\geq\sqrt{\frac{4-(f_a^n(x))^2}{4-\delta^2}}(\frac{2\delta}{\delta^{1+\frac{1}{20}}+\delta})^n=\Gamma e^{\lambda n}$$

where $\Gamma=\sqrt{\frac{4-(f_a^n(x))^2}{4-\delta^2}}$ and $\lambda=\ln(\frac{2\delta}{\delta^{1+\frac{1}{20}}+\delta})$. If $|f_a^n(x)|\leq|x|$, e.g. if $x\in f_a(\Delta^+)$, or $f_a^n(x)\in\Delta$ then we have $\Gamma\simeq 1$. Also if $f_a^n(x)\in\delta^+$, then substituting δ and δ^+, we get $\Gamma\geq\sqrt{\frac{4-(\delta^+)2}{4-\delta^2}}\simeq 1$. Furthermore, we can write

$$\lambda=\ln(\frac{2\delta}{\delta^{1+\frac{1}{20}}+\delta})=\ln(\frac{2}{\delta^{\frac{1}{20}}+1})\geq\ln 2-\ln(1+\delta^{\frac{1}{20}})\geq\ln 2-\delta^{\frac{1}{20}}\simeq 0.686.$$

Hence, we have $(H1)$. The first part of $(H2)$ is satisfied by

$$N_0 = 144 \leq 2 + \frac{1}{\ln 4} \ln \frac{2 - \sqrt{2+\sqrt{2}}}{|2 + a_* - a_*^2 + \delta^2(2a_* - \delta^2)|}, \tag{10.1.8}$$

which comes from Lemma 10.1.3 and considering $a_* = 2 - \varepsilon = 2 - \delta^{2+\frac{1}{10}}$. To verify the second part of $(H2)$ we look for a natural number $N_1 \leq N_0$ such that $|c_{N_1}(\Omega)| \geq \delta^+$. For this since $N_1 \leq N_0$ and so $(\Phi)_{N_1}$ is true for all $a \in \Omega$, then by Mean Value Theorem there exists a parameter $\bar{a} \in \Omega$ which we can write

$$|c_{N_1}(\Omega)| = |c'_{N_1}(\bar{a})||\Omega| \geq \mathcal{P}_1 |(f_a^{N_1})'(c_0)||\Omega| \tag{10.1.9}$$

Because $N_1 \leq N_0$, Lemma 10.1.2 and relation 10.1.8 implies that for all i with $i \leq N_1$, we have $|f_a^i(c_0)| \geq \sqrt{a + \sqrt{a}} \geq \sqrt{a_* + \sqrt{a_*}} \geq \sqrt{1.9 + \sqrt{1.9}} \simeq 1.8$. In the other words, by chain rule $|(f_a^{N_1})'(c_0)| \geq (3.6)^{N_1}$ and so using (10.1.9), we get $|c_{N_1}(\Omega)| \geq \mathcal{P}_1 (3.6)^{N_1} \delta^{2+\frac{1}{10}}$. Therefore to find N_1 it is enough to solve

$$\mathcal{P}_1 (3.6)^{N_1} \delta^{2+\frac{1}{10}} \geq \delta^+$$

Now, since the length $|c_n(\Omega)|$ is increasing at least for $N_1 \leq n \leq N_0$, we have $(H2)$. Now we let $b = 2$, $m = 11$ and then we write

$$\begin{aligned}
1 + \sum_{i=1}^{N} \frac{1}{(f_a^i)'(c_0)} &= 1 + \frac{1}{f'_a(c_0)} + \frac{1}{f'_a(c_0) f'_a(c_1)} + ... + \frac{1}{f'_a(c_0) ... f'_a(c_{N-1})} \\
&= 1 + \frac{1}{f'_a(c_0)} (1 + \frac{1}{f'_a(c_1)} + ... + \frac{1}{f'_a(c_1) ... f'_a(c_{N-1})}) \\
&= 1 + \frac{1}{-2a} (1 + \frac{1}{f'_a(c_1)} + ... + \frac{1}{f'_a(c_1) ... f'_a(c_{N-1})})
\end{aligned}$$

Since $N < N_0$, then by lemma 10.1.3 we have $-2 < c_i \leq -\sqrt{a_* + \sqrt{a_*}}$ for all $a \in \Omega$ and $i \leq N$. A straightforward computation for $a \in [a_*, a^*] \subseteq [1.9, 2]$ shows that:

$$0.636 \simeq 1 - \frac{1}{3.8}\left(\frac{1-(\frac{1}{2\sqrt{1.9+\sqrt{1.9}}})^{120}}{1-2\sqrt{1.9+\sqrt{1.9}}}\right) \leq 1 + \sum_{i=1}^{120} \frac{1}{(f_a^i)'(c_0)} \leq \frac{3}{4}$$

On the other hand to estimate the infinite summation appeared in $(H3)$, using a simple calculation gives us

$$\sum_m^{\infty}((j+1)^b - j^b)(\frac{1}{e^{\tilde{\lambda}}})^j = \sum_m^{\infty}(2j+1)(e^{-\tilde{\lambda}})^j = \frac{(e^{-\tilde{\lambda}})^m(2m+1-e^{-\tilde{\lambda}}(2m-1))}{(1-e^{-\tilde{\lambda}})^2} \simeq 0.007$$

Clearly $(H3)$ is satisfied and moreover

$$\inf_{a\in\Omega} \min\{\min_{1\leq k\leq N}\{|1+\sum_{i=1}^{k}\frac{1}{(f_a^i)'(c_0)}|\}, |1+\sum_{i=1}^{N}\frac{1}{(f_a^i)'(c_0)}| - \sum_m^{\infty}((j+1)^b - j^b)(\frac{1}{e^{\tilde{\lambda}}})^j\}$$
$$= |1+\sum_{i=1}^{120}\frac{1}{(f_a^i)'(c_0)}| - \sum_{11}^{\infty}(2j+1)(\frac{1}{e^{\tilde{\lambda}}})^j \geq 0.636 - 0.007 \geq 0.62 = \mathcal{P}_1,$$

$$\sup_{a\in\Omega} \max\{\max_{1\leq k\leq N}\{|1+\sum_{i=1}^{k}\frac{1}{(f_a^i)'(c_0)}|\}, |1+\sum_{i=1}^{N}\frac{1}{(f_a^i)'(c_0)}| + \sum_m^{\infty}((j+1)^b - j^b)(\frac{1}{e^{\tilde{\lambda}}})^j\}$$
$$= ???? \leq \frac{3}{4} + 0.007 \leq 0.76 = \mathcal{P}_2$$

So we have $(H4)$ and $(H5)$. Automatically $(H6)$ is satisfied. Finally to verify the validation of $(H7)$ we can write

$$|\Delta_0^+(a)| = |f_a(\Delta^+)| = (\delta^+)^2 \quad \text{and} \quad |\Delta_i^+(a)| = |f_a(\Delta_{i-1}^+)| \leq 4^i(\delta^+)^2$$

So using the fact that $|c_i(a)| \geq \sqrt{a+\sqrt{a}} \geq \sqrt{a_* + \sqrt{a_*}}$, for $i \leq N_0$ we get

$$\ln(\frac{|\Delta_i^+(a)| + |c_i(a)|}{|c_i(a)|}) \leq \ln(1 + \frac{4^i(\delta^+)^2}{\sqrt{a_* + \sqrt{a_*}}}) \leq \frac{4^i(\delta^+)^2}{\sqrt{a_* + \sqrt{a_*}}}$$

and therefore we obtain:

$$\begin{aligned} & \exp(\sup_{a\in\Omega} \sum_{i=0}^{N_0-1} \log(\frac{|\Delta_i^+(a)| + |c_i(a)|}{|c_i(a)|}) + \frac{1}{2}(\frac{1}{N_0 - 1} + \frac{1}{N_0})) \\ \leq\ & \exp(\frac{(\delta^+)^2}{\sqrt{a_* + \sqrt{a_*}}} \sum_{0}^{143} 4^i + \frac{1}{2}(\frac{1}{143} - \frac{1}{144})) \\ \leq\ & 1.57 = \mathcal{B} \end{aligned}$$

This ends the proof of Proposition. ∎

Auxiliary constants and formal conditions

By explicit computations we then get

$$\mathcal{B}_1 = 1.953269467 * 10^{21} \quad \text{and} \quad \mathcal{B}_2 = 1.300050448 * 10^{21}$$

We now recall the constants which used to find the bounded distortion. First

$$\mathcal{D}_2 = \frac{e\mathcal{P}_2}{\mathcal{P}_1} \sum_{j=0}^{\infty} e^{-\lambda j} = \frac{e\mathcal{P}_2}{\mathcal{P}_1(1 - e^{-\lambda})} \simeq 8.1$$

and to find $\mathcal{D}_3$ we first observe that

$$\begin{aligned}
\sum_0^\infty \frac{\gamma_s}{1-\gamma_s} &= \sum_0^{N_0-1} \frac{\gamma_s}{1-\gamma_s} + \sum_{N_0}^\infty \frac{1}{s^2-1} \\
&\leq \sum_0^{N_0-1} (\frac{1}{1-\gamma_s} - 1) + \frac{1}{2}(\frac{1}{N_0-1} + \frac{1}{N_0}) \\
&\leq \sum_0^{N_0-1} (\frac{1}{1-\frac{4^s(\delta^+)^2}{\sqrt{a_*+\sqrt{a_*}}}} - 1) + \frac{1}{2}(\frac{1}{N_0-1} + \frac{1}{N_0})
\end{aligned}$$

and so

$$\mathcal{D}_3 \simeq 2.9$$

Easily we find

$$\mathcal{D}_1 \simeq 11, \quad \mathcal{D}_4 \simeq 1 \quad \mathcal{D}_5 \simeq 4.57 \quad \mathcal{D}_6 \simeq 1 * 10^{-16} \quad \mathcal{D}_7 \simeq 5 * 10^{-15},$$

which all of these constants gives us

$$\mathcal{D} \simeq 2.48$$

Chapter 11

Persistence of Periodic Solutions at Resonance of a Third Order Oscillator

11.1 Introduction

Oscillations play an important role in many physical and biological systems. The equation

$$\dddot{x} + \omega^2 \dot{x} = 0 \tag{11.1.1}$$

is a linear third-order equation of an oscillator which discusses the motion of an object on an ellipse in a three dimensional manifold. One of the most important problems is to determine if any of the periodic solutions of the system (11.1.1) persist when subjected to a small external periodic perturbation. These kinds of equations arise in problems related to energy, acceleration and fluid mechanics [56], electric circuits theory [55], nonlinear vibrations [46] and mathematical biology [45, 47, 56]. So existence of the periodic solution for these kind of systems is of great importance. Mehri and Niksirat in [49] considered the nonlinear autonomous equation

$$\dddot{x} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}) \tag{11.1.2}$$

and obtained some conditions under them the system (11.1.2) has periodic solution. Then Rabiei and Afsharnejad in [51] studied the equation (11.1.2) and let $f(x, \dot{x})$ be C^r $(r \geq 3)$ and

$$F(x, \dot{x}, \ddot{x}) = f_x(x, \dot{x})\dot{x} + f_{\dot{x}}(x, \dot{x})\ddot{x}.$$

They showed that if $f_{x\dot{x}}(0,0) \neq 0$, then (11.1.2) has many periodic solutions. Later in [52] they extended the concept of [51] for the following non-autonomous third-order differential equation

$$\dddot{x} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}, t, \varepsilon) \tag{11.1.3}$$

and showed that if

$$F(x, \dot{x}, \ddot{x}, t, \varepsilon) = H_x(x, \dot{x}, t, \varepsilon)\dot{x} + H_{\dot{x}}(x, \dot{x}, t, \varepsilon)\ddot{x} + H_\varepsilon(x, \dot{x}, t, \varepsilon)$$

where, $H(x, \dot{x}, t, \varepsilon) = f(x, \dot{x}) + \varepsilon g(x, \dot{x}, t)$ then (11.1.3) has many periodic solutions. Also Afsharnejad and Golmakani in [54] studied the equation

$$\dddot{x} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}, \varepsilon) \tag{11.1.4}$$

where ε is a real parameter characterizing the strength of the external driving force and obtained some conditions for the existence of periodic solutions of the system (11.1.4).

They assumed that if there exists a C^r - function $f(x, \dot{x}, \varepsilon)$ such that

$$F(x, \dot{x}, \ddot{x}, \varepsilon) = \dot{x} f_x(x, \dot{x}, \varepsilon) + \ddot{x} f_{\dot{x}}(x, \dot{x}, \varepsilon)$$

and $f_{\dot{x}}(x_0, \dot{x}_0, \varepsilon_0) \neq 0$, then (11.1.4) has many periodic solutions. On the other hand Chicone [41, 42, 43] considered the family of differential equations

$$\dot{y} = f(y) + \varepsilon g(y, t, \varepsilon) \quad y \in \mathbb{R}^n, \quad \varepsilon \geq 0 \tag{11.1.5}$$

and proved a continuation theorem for periodic orbits at resonance under some conditions for the system (11.1.5). Also Z.Zhang and Z.Wang in [57] and P.Amster, P.De Napoli and M.C.Mariani in [40] have been worked on the existence of periodic solutions for third-order differential equations by using continuation theorem of the coincidence degree.

The concept of dimension reduction for ordinary differential equations is recently the matter of much researches. In what follows we consider the three dimensional perturbed linear oscillator

$$\dot{\ddot{x}} + \omega^2 \dot{x} = \mu F(x, \dot{x}, \ddot{x}, t, \mu) \quad (x, \dot{x}, \ddot{x}) \in \mathbb{R}^3, \quad \mu \in \mathbb{R} \tag{11.1.6}$$

where F is periodic map in t with period $T > 0$. We moreover assume that T is at resonance with the period of oscillation.

11.2 Preliminaries

We here in the first subsection remind some preliminaries and facts in order to get the displacement function of desired equation. Second subsedtion is devoted to explain about the Lyapunov- Schmidt Reduction method that will being used as our main technic to resolve our question.

11.2.1 Resonance Condition

Rewriting the system (11.1.6) as

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -\omega^2 y + \mu F(x, y, z, t, \mu) \tag{11.2.1}$$

that $X = (x, y, z) \in \mathbb{R}^3$, then we get

$$\dot{X} = f(X) + \mu g(X, t, \mu) \tag{11.2.2}$$

in which

$$f(X) = \begin{pmatrix} y \\ z \\ -\omega^2 y \end{pmatrix} \quad , \quad g(X, t, \mu) = \begin{pmatrix} 0 \\ 0 \\ F(x, y, z, t, \mu) \end{pmatrix}$$

We assume that the unperturbed system

$$\dot{X} = f(X) \tag{11.2.3}$$

has $\mathbb{R}^3$ as a *periodic manifold* with period $2\pi/\omega$. Indeed all of the solutions of the unpertebed system are periodic with period $2\pi/\omega$. We moreover suppose that the map

$g(t, X, \mu)$ is periodic with period

$$T := T(\mu) = \frac{M}{N}\frac{2\pi}{\omega} + k\mu + O(\mu^2), \quad (M, N) = 1. \tag{11.2.4}$$

This last assumption on the map $g(X, t, \mu)$ implise that the periodic solutions of the unperturbed system (11.2.3) are in $(M : N)$ *resonance* with the force term $g(X, t, \mu)$ for which $NT(0) = M2\pi/\omega$ is the *resonance relation.*

11.2.2 Fundamental Matrix

For $\upsilon \in \mathbb{R}^3$ suppose that, function $t \to X(\upsilon, t, \mu)$ denote the solution of the unperturbed system (11.2.3) such that $X(\upsilon, 0, \mu) = \upsilon$ and let $\Phi(\upsilon, t)$ be the *principal fundamental matrix* at time $t = 0$ of the *first variational equation*

$$\dot{W} = Df(X(\upsilon, t, 0))W. \tag{11.2.5}$$

we remind that, the *principal fundamental matrix*solution is just the partial derivative of the solution of the original unperturbed differential equation with respect to υ, i.e

$$\Phi(\upsilon, t) = X_\upsilon(\upsilon, t, 0).$$

A simple calculation shows that the first variational equation (11.2.5) has a fudamental matrix of the following form

$$\Phi(\upsilon, t) = \begin{pmatrix} 1 & \frac{1}{\omega}\sin\omega t & \frac{1}{\omega^2}(1-\cos\omega t) \\ 0 & \cos\omega t & \frac{1}{\omega}\sin\omega t \\ 0 & -\omega\sin\omega t & \cos\omega t \end{pmatrix}. \tag{11.2.6}$$

We have the following theorem.

Theorem 11.2.1. *(Variation of Constants Formula) Consider the initial value problem $\dot{x} = A(t)x + g(t,x)$, $x(t_0) = x_0$ and $t \to \Phi(t)$ be a fundamental matrix solution for the homogeneous system $\dot{x} = A(t)x$ that is defined on some interval J_0 containing t_0. If $t \to \phi(t)$ is the solution of the initial value problem defined on some subinterval of J_0, then*

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s,\phi(s))ds.$$

Proof. see[42, Sec.2.3] ∎

11.2.3 Continuation of Initial Condition

For $\upsilon \in \mathbb{R}^3$ we say that the unperturbed periodic orbit $X(\upsilon, t, 0)$ of unperturbed system (11.2.3) *persists* if there is some $\mu_0 > 0$ and a continuous function $\beta : (-\mu_0, \mu_0) \to \mathbb{R}^3$ such that $\beta(0) = \upsilon$ and $X(\beta(\mu), NT, \mu) - \beta(\mu) = 0$. In this case the curve $\mu \to \beta(\mu)$ is called a *continuation of initial condition* for the periodic solution. We note that the resonance condition (11.2.4) and in particular the fact that $NT(0) = M2\pi/\omega$ implies that the periodic solutions $t \to X(\upsilon, t, 0)$ of the system (11.2.3) are also periodic with period NT for each $\upsilon \in \mathbb{R}^3$.

11.2.4 Displacement Function

It is obvious that the periodic solutions of (11.2.2) with period NT corresponds to zeros of $X(\upsilon, NT, \mu) - \upsilon$. The function

$$\sigma(\upsilon, t, \mu) = X(\upsilon, t, \mu) - \upsilon \tag{11.2.7}$$

is called the *displacement function* of (11.2.2). Let $\sigma(\upsilon_0, t, \mu_0) = 0$, we call σ *nondegenerate* at (υ_0, t, μ_0) if rank $(d\sigma)_{(\upsilon_0,t,\mu_0)} = 3$, otherwise σ is called *degenerate* at (υ_0, t, μ_0).

11.2.5 Lyapunov-Schmidt Reduction

Here we indicate the Lyapunov - Schmidt procedure from [44] which be used for reducing of the displacement function σ. This method is explained in detail in [48]. Putting $L = (d\sigma)_{(t,\upsilon_0,0)}$ by decomposition theorems in linear algebra we have

$$\mathbb{R}^n = K \oplus K^{\perp}, \quad \mathbb{R}^n = R^{\perp} \oplus R \tag{11.2.8}$$

where $K = kerL$, and $K^{\perp}$ denotes a complement of K in $\mathbb{R}^n$ with $dimK = k > 0$, $dimK^{\perp} = n - k$. By linear algebra linear transformation L has $rangeR$, $dimR = n - k$ and suppose that $R^{\perp}$ is a complement of R in $\mathbb{R}^n$ with $dimR^{\perp} = k$. We let $\pi : \mathbb{R}^n \to R$ denote the linear projection onto the range and $\pi^{\perp} : \mathbb{R}^n \to R^{\perp}$ the complementary projection namely $\pi^{\perp} = I - \pi$. We define now

$$\Psi : K \times K^{\perp} \times \mathbb{R} \to R$$

$$\Psi(k, k', \mu) = \pi\sigma(k, k', t, \mu)$$

The fact that $\dim K^{\perp} = \dim R = n - k$ result that its partial derivative

$$\Psi_{k'}(k_0, k'_0, 0)|_{K^{\perp}} : K^{\perp} \to R$$

at $(k_0, k_0') = v_0$ is an isomorphism. Using the Implicit Function Theorem there exists a function $\alpha : K \times \mathbb{R} \to K^\perp$ such that

$$\Psi(k, \alpha(k, \mu), \mu) = 0, \quad \alpha(k_0, 0) = k_0'$$

for sufficiently small $|k - k_0|$ and $|\mu|$. The Lyapunov - Schmidt reduced function is defined as the complementary function

$$\varphi : K \times R \to R^\perp$$

$$\varphi(k, \mu) = \pi^\perp \sigma(k, \alpha(k, \mu), t, \mu), \quad \varphi(k_0, 0) = 0$$

If there is a curve $\mu \to \beta(\mu)$ in K such that $\beta(0) = 0$ and $\alpha(\beta(\mu), \mu) = 0$ then (k_0, k_0') is a zero of σ with the required curve of zero $\mu \to \gamma(\mu)$ in $K \times K^\perp$ given by $\gamma(\mu) := (\beta(\mu), \alpha(\beta(\mu), \mu))$. Of course, we will not be able to determine the existence of the curve β by a direct application of the Implicit Function Theorem because L has a nontrivial kernel but we may be able to apply it after a further reduction. In [44] is indicated a reduction can be made when the displacement function has maximal rank. In the following theorem, we suppose maximal rank of the unperturbed function is $n - k$, we obtain of the curve β by using of the Implicit Function Theorem directly.

Theorem 11.2.2. *we consider the displacement function σ and let rank $(d\sigma)_{(v_0,t,0)}$ be $n - k$, then the zeros of the equation $\sigma(v, t, \mu) = 0$ are in one to one correspondence with the zeros of the function*

$$B : K \to R^\perp$$

$$B(k_0) = \pi^\perp \sigma_\mu(k_0, k_0', t, 0).$$

Proof. We consider the corresponding above the kernel K and range R of L together with their complements $K^{\perp}$ and $R^{\perp}$, the mappings $\pi : \mathbb{R}^n \to R$, $\pi^{\perp} : \mathbb{R}^n \to R^{\perp}$. we define

$$\Psi : K \times K^{\perp} \times \mathbb{R} \to R$$

$$\Psi(k, k', \mu) = \pi\sigma(k, k', t, \mu)$$

then, for each $k_0 \in K$ can result that $\Psi_{k'}(k_0, k'_0, 0)|_{K^{\perp}} : K^{\perp} \to R$ at $(k_0, k'_0) = \upsilon_0$ is an isomorphism. Now by using the Implicit Function Theorem there exists a function $\alpha : K \times \mathbb{R} \to K^{\perp}$ such that

$$\Psi(k, \alpha(k, \mu), \mu) = 0, \quad \alpha(k_0, 0) = k'_0$$

for sufficiently small $|k - k_0|$ and $|\mu|$. Now we define

$$\varphi : K \times \mathbb{R} \to R^{\perp}$$

$$\varphi(k, \mu) = \pi^{\perp}\sigma(k, \alpha(k, \mu), t, \mu), \quad \varphi(k_0, 0) = 0.$$

Thus, by Taylor's Theorem, we have

$$\varphi(k, \mu) = \mu(\varphi_{\mu}(k, 0) + O(\mu))$$

where

$$\varphi_{\mu}(k, 0) = \pi^{\perp}\sigma_{k'}(k_0, k'_0, t, 0)\alpha_{\mu}(k_0, 0) + \pi^{\perp}\sigma_{\mu}(k_0, k'_0, t, 0)$$

But, the range of $\sigma_{k'}(k_0, k'_0, t, 0)$ is R, so the first term of the last formula vanishes and we define the reduced function

$$B : K \to R^{\perp}$$

$$B(k_0) = \pi^{\perp}\sigma_{\mu}(k_0, k_0', t, 0).$$

By the above reduction, if there is a curve $\mu \to \beta(\mu)$ in K such that $\varphi(\beta(\mu), \mu) = 0$ and $\mu \to \gamma(\mu)$ is the curve in $K \times K^{\perp}$ defined by $\gamma(\mu) := (\beta(\mu), \varphi(\beta(\mu), \mu))$, then $\sigma(\gamma(\mu), \mu) = 0$ and $\gamma(0) = (\beta(0), 0)$ in $K \times K^{\perp}$ is a zero of σ. Therefore we can result if $k_0 \in K$ is a zero of the function B such that $B(k_0) = 0$ and $DB(k_0) : K \to R^{\perp}$ is an isomorphism, then $(k_0, k_0') \in K \times K^{\perp}$ is a zero of σ. ∎

11.3 Nondegenerate Case

In order to desire continuation in the Nondegenerate Case, we obtain a general formula for the displacement function by using Tylor series expansion in powers of μ and find the bifurcation function then we use the Implicit Function Theorem to reduce the problem of the persistence of resonant unperturbed periodic solutions to finding simple zeros of the bifurcation function. We recall that a zero of the equation

$$\sigma(\upsilon, NT, \mu) = X(\upsilon, NT, \mu) - \upsilon \tag{11.3.1}$$

corresponds to a periodic solution of the system (11.2.2) with period NT.

Lemma 11.3.1. *Suppose that $\sigma(\upsilon, NT, \mu)$ is the displacement function for system* (11.2.2). *Then*

$$\sigma(\upsilon, NT, \mu) = \mu\Phi(\upsilon, NT) \int_0^{NT} \Phi^{-1}(\upsilon, s) g(X(\upsilon, s, 0), s, 0) ds + O(\mu^2)$$

Proof. We write the Taylor expansion (MC Lourant), at $\mu = 0$, of the right hand expression of (11.3.1) as following

$$X(\upsilon, NT, \mu) - \upsilon = (X(\upsilon, NT, 0) - \upsilon) + \mu X_\mu(\upsilon, NT, 0) + O(\mu^2)$$

where, X_μ is the derivative of X with respect to μ. So the fact $X(\upsilon, NT, 0) = \upsilon$ implies that

$$\sigma(\upsilon, NT, \mu) = \mu X_\mu(\upsilon, NT, 0) + O(\mu^2) \tag{11.3.2}$$

On the other hand differentiating (11.2.2) with respect to μ, gives us

$$\dot{X}_\mu = Df(X(\upsilon, t, 0))X_\mu + g(\upsilon, t, \mu) + \mu(D_X g(\upsilon, t, \mu).X_\mu + \frac{\partial g}{\partial \mu}(\upsilon, t, \mu))$$

and so by putting $\mu = 0$ in the above formula we get

$$\dot{X}_\mu(\upsilon, t, 0) = Df(X(\upsilon, t, 0))X_\mu(\upsilon, t, 0) + g(X(\upsilon, t, 0), t, 0)$$

so the function $t \to X_\mu(\upsilon, t, 0)$ is the solution of the variation equation

$$\dot{w} = Df(X(\upsilon, t, 0))w + g(X(\upsilon, t, 0), t, 0) \quad , \quad w(0) = 0$$

By theorem (11.2.1), the solution of this equation is

$$X_\mu(\upsilon, t, 0) = \Phi(\upsilon, t) \int_0^t \Phi^{-1}(\upsilon, s) g(X(\upsilon, s, 0), s, 0) ds \tag{11.3.3}$$

and now substituting (11.3.3) in (11.3.2) and putting $t = NT$ we get the statement. ∎

Corollary 11.3.2. *By definition of displacement function* (11.2.7) *we know that the zeros of* $\sigma(\upsilon, NT, \mu)$ *are correspond to the periodic solutions of* (11.1.5). *So a straightforward* *application of Lemma 11.3.1 and plugging* $\Phi(\upsilon,t)$ *and* $\Phi^{-1}(\upsilon,t)$ *from* (11.2.6) *we have*

$$\sigma(\upsilon,NT,\mu)=\mu\begin{pmatrix}\frac{1}{\omega^2}\int_0^{NT}(1-\cos(\omega t))F(X(\upsilon,t,0),t,0)dt\\ \frac{1}{\omega}\int_0^{NT}\sin(\omega t)F(X(\upsilon,t,0),t,0)dt\\ \int_0^{NT}\cos(\omega t)F(X(\upsilon,t,0),t,0)dt\end{pmatrix}+O(\mu^2)$$

Theorem 11.3.3. *Suppose that in the system* (11.1.6)*, the term* $F(x,\dot{x},\ddot{x},t,\mu)$ *is periodic in* t *and satisfies the resonance condition* (11.2.4)*. If the bifurcation function*

$$\upsilon\to X_\mu(\upsilon,NT,0) \tag{11.3.4}$$

has a simple zero at $\upsilon_0\in\mathbb{R}^3$*, then* $X(\upsilon_0,t,0)$ *persists. In particular this implies that* $X(\upsilon_0,t,\mu)$ *is a periodic solution of* (11.1.6) *for small* $\mu\neq 0$*.*

Proof. we define the function

$$\Psi:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3$$

$$(\upsilon,\mu)\to X_\mu(\upsilon,NT,0)$$

let υ_0 be a simple zero of $X_\mu(\upsilon,NT,0)$, then $\Psi(\upsilon_0,0)=0$ and $\det(\frac{\partial\Psi}{\partial\upsilon}|_{\upsilon_0})\neq 0$. Therefore by the Implicit Function Theorem there is a function $\mu\to\upsilon(\mu)$ defined for μ in a neighborhood of $\mu=0$ such that $\Psi(\upsilon(\mu),\mu)=0$ and $\upsilon(0)=\upsilon_0$ and completes the proof. ∎

Example 11.3.4. *We consider the equation*

$$\dddot{x}+\dot{x}=\mu(x^2+x\sin t) \tag{11.3.5}$$

A simple computation shows that all the orbits of (11.3.5) are periodic with period 2π, and the solution of the unperturbed system at an initial point $\xi = (x, y, z)$ is as following:

$$X(\xi, t, 0) = (-z\cos t + y\sin t + x + z, z\sin t + y\cos t, z\cos t - y\sin t)$$

We observe that the function $F(x, \dot{x}, \ddot{x}, t, \mu) = (x^2 + x\sin t)$ is also periodic with period 2π. This in particular implies that it is at resonance with the unperturbed system and $M = N = 1$. A simple calculation, using the definition of $X_\mu(\upsilon, NT, 0)$ in (11.4.3), gives us

$$X_\mu(\xi, 2\pi, 0) = \pi(2x^2 + y + y^2 + 6xz + 5z^2, -x - 2xy - z - 2yz, -2(xz + z^2))$$

So, differentiating it we get

$$DX_\mu(\xi, 2\pi, 0) = \begin{pmatrix} \pi(4x + 6z) & \pi(1 + 2y) & \pi(6x + 10z) \\ -\pi(1 + 2y) & -\pi(2x + 2z) & -\pi(1 + 2y) \\ -2\pi z & 0 & -2\pi(x + 2z) \end{pmatrix}$$

and then a straightforward calculation results that $\det DX_\mu(\xi, 2\pi, 0) = -2\pi^3 x + 16\pi^3 x^3 - 8\pi^3 xy - 8\pi^3 xy^2 - 2\pi^3 z + 48\pi^3 x^2 z - 8\pi^3 yz - 8\pi^3 y^2 z + 40\pi^3 xz^2 + 8\pi^3 z^3$. Letting $\xi_0 = (x_0, y_0, z_0) = (\sqrt{\frac{1}{8}}, -\frac{1}{2}, 0)$ we have

$$X_\mu(\xi_0, 2\pi, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad DX_\mu(\xi_0, 2\pi, 0) = \pi\sqrt{\frac{1}{8}} \begin{pmatrix} 4 & 0 & 6 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Thus ξ_0 is a simple zero of $X_\mu(\upsilon, 2\pi, 0)$ since $DX_\mu(\xi_0, 2\pi, 0) = \pi^3/\sqrt{2} \neq 0$. Therefore by applying Theorem 11.3.3 the equation (11.3.5), for $|\mu|$ small enough, has a one-parameter family of periodic solutions around ξ_0.

11.4 Degenerate case

We suppose here that $rank(d\sigma)_{(t,\upsilon_0,0)} = 2$ and so as like as previous section we can not use the Implicit Function Theorem directly. By [41] we can obtain the bifurcation equation for degenerate case by helping an extension of the bifurcation equations for nondegenerate case and use the Lyapanov-Schmidt procedure to finding the bifurcation function, thus the first we will find a new formula for the displacement function. To simplify the notation we let

$$Y(\upsilon,s) := \Phi^{-1}(\upsilon,s)[D^2 f(X(\upsilon,t,0))X_\mu^2(\upsilon,t,0)$$
$$+2(D_X g(X(\upsilon,s,0),s,0).X_\mu(\upsilon,s,0) + \frac{\partial g}{\partial\mu}(X(\upsilon,s,0),t,0))] \tag{11.4.1}$$

Lemma 11.4.1. *Suppose that we have the displacement function σ defined in section 2, then*

$$\sigma(\upsilon,NT,\mu) = \mu\Phi(\upsilon,NT)\int_0^{NT}\Phi^{-1}(s)g(X(\upsilon,s,0),s,0)ds$$

$$+\frac{\mu^2}{2}\Phi(\upsilon,NT)\int_0^{NT}Y(s)ds + O(\mu^3)$$

that

Proof. The Taylor series of $\sigma(\upsilon,NT,\mu)$ at $\mu = 0$ is

$$\sigma(\upsilon,NT,\mu) = (X(\upsilon,NT,0)-\upsilon) + \mu X_\mu(\upsilon,NT,0) \tag{11.4.2}$$
$$+\frac{\mu^2}{2}X_{\mu\mu}(\upsilon,NT,0) + O(\mu^3)$$

By differentiating from the equation (11.2.2) with respect to the parameter μ, we have

$$\dot{X}_\mu(\upsilon, t, \mu) = Df(X(\upsilon, t, \mu))X_\mu(\upsilon, t, \mu) + g(X(\upsilon, t, \mu), t, \mu)$$

$$+\mu(D_X g(X(\upsilon, t, \mu), t, \mu).X_\mu(\upsilon, t, \mu) + \frac{\partial g}{\partial \mu}(X(\upsilon, t, \mu), t, \mu))$$

and twice differentiating from the equation (11.2.2) with respect to the parameter μ, gives us

$$\dot{X}_{\mu\mu}(\upsilon, t, \mu) = Df(X(\upsilon, t, \mu))X_{\mu\mu}(\upsilon, t, \mu) + D^2 f(X(\upsilon, t, \mu))X_\mu^2(\upsilon, t, \mu)$$

$$+D_X g(X(\upsilon, t, \mu), t, \mu).X_\mu(\upsilon, t, \mu) + \frac{\partial g}{\partial \mu}(X(\upsilon, t, \mu), t, \mu)$$

$$+D_X g(X(\upsilon, t, \mu), t, \mu).X_\mu(\upsilon, t, \mu) + \frac{\partial g}{\partial \mu}(X(\upsilon, t, \mu), t, \mu) + \mu H_\mu$$

in which

$$H_\mu = \frac{\partial}{\partial \mu}(D_X g(X(\upsilon, t, \mu), t, \mu).X_\mu(\upsilon, t, \mu) + \frac{\partial g}{\partial \mu}(X(\upsilon, t, \mu), t, \mu))$$

Now putting $\mu = 0$,

$$\dot{X}_{\mu\mu}(\upsilon, t, 0) = Df(X(\upsilon, t, 0))X_{\mu\mu}(\upsilon, t, 0) + D^2 f(X(\upsilon, t, 0))X_\mu^2(\upsilon, t, 0)$$

$$+2(D_X g(X(\upsilon, t, 0), t, 0).X_\mu(\upsilon, t, 0) + \frac{\partial g}{\partial \mu}(X(\upsilon, t, 0), t, 0))$$

so the function $t \to X_{\mu\mu}(\upsilon, t, 0)$ is the solution of the variation equation

$$\dot{w} = Df(X(\upsilon,t,0))w + D^2f(X(\upsilon,t,0))X_\mu^2(\upsilon,t,0)$$

$$+2(D_Xg(X(\upsilon,t,0),t,0).X_\mu(\upsilon,t,0) + \frac{\partial g}{\partial \mu}(X(\upsilon,t,0),t,0))$$

and $w(0) = 0$, hence we can write

$$X_{\mu\mu}(\upsilon,t,0) = \Phi(\upsilon,t)(\int_0^t \Phi^{-1}(\upsilon,s)[D^2f(X(\upsilon,t,0))X_\mu^2(\upsilon,t,0)$$

$$+2(D_Xg(X(\upsilon,s,0),s,0).X_\mu(\upsilon,s,0) + \frac{\partial g}{\partial \mu}(X(\upsilon,s,0),t,0))]ds)$$

Therefore by replacing X_μ and $X_{\mu\mu}$ in (11.4.2) the proof is completed.

∎

Theorem 11.4.2. *Suppose that in the system* (11.1.6)*, the term* $F(x,\dot{x},\ddot{x},t,\mu)$ *is periodic in* t *and satisfies the resonance condition* (11.2.4)*. If the bifurcation function*

$$\upsilon \to \pi^\perp(X_\mu(\upsilon,NT,0) + \frac{\mu}{2}X_{\mu\mu}(\upsilon,NT,0)) \tag{11.4.3}$$

has a simple zero at $\upsilon_0 \in \mathbb{R}^3$*, then* $X(\upsilon_0,t,0)$ *persists.*

Proof. we define the function

$$\Psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^\perp$$

$$(\upsilon,\mu) \to \pi^\perp(X_\mu(\upsilon,NT,0) + \frac{\mu}{2}X_{\mu\mu}(\upsilon,NT,0))$$

let υ_0 be a simple zero of $X_\mu(\upsilon,NT,0)+\frac{\mu}{2}X_{\mu\mu}(\upsilon,NT,0)$, then $\Psi(\upsilon_0,0) = 0$ and $\det(\frac{\partial\Psi}{\partial\upsilon}|_{\upsilon_0}) \neq 0$. Therefore by the implicit function theorem there is a function $\mu \to \upsilon(\mu)$ defined for μ in a neighborhood of $\mu = 0$ such that $\Psi(\upsilon(\mu),\mu) = 0$ and $\upsilon(0) = \upsilon_0$ and completes the proof. ∎

Example 11.4.3. : *Consider the system*

$$\dddot{x} + \dot{x} = \mu(-x^2 + \dot{x}\cos t) \tag{11.4.4}$$

Here $F(t, x, \dot{x}, \ddot{x}, \mu) = (-x^2 + \dot{x}\cos t)$ is periodic with period 2π, and the solution of the unperturbed system is

$$\begin{pmatrix} -z\cos t + y\sin t + x + z \\ z\sin t + y\cos t \\ z\cos t - y\sin t \end{pmatrix}$$

which is periodic with period 2π, so $M = N = 1$. By direct computation we have

$$X_\mu(2\pi, x, y, z, 0) = \begin{pmatrix} -\pi(2x^2 - y + y^2 + 6xz + 5z^2) \\ 2\pi(xy + yz) \\ 2\pi(xz + z^2) \end{pmatrix}$$

So

$$DX_\mu(2\pi, x, y, z, 0) = \begin{pmatrix} -\pi(4x + 6z) & -\pi(-1 + 2y) & -\pi(6x + 10z) \\ 2\pi y & 2\pi(x + z) & 2\pi y \\ 2\pi z & 0 & 2\pi(x + 2z) \end{pmatrix}$$

We see that

$$X_\mu(2\pi, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$DX_\mu(2\pi, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0) = \begin{pmatrix} \pi & 0 & 2\pi \\ \pi & 0 & \pi \\ -\pi & 0 & -\pi \end{pmatrix}$$

Then $rank(DX_\mu(2\pi, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0)) = 2$, and

$$X_{\mu\mu} = \begin{pmatrix} 4\pi^2(4x^3 + 18x^2z + 3y^2z + 15z^3 - 3xy + 2xy^2 + 28xz^2 - 4yz) \\ 4\pi^2(-2x^2y + y^2 - y^3 - 6xyz - 5yz^2) \\ 4\pi^2(-2x^2z + yz - zy^2 - 6xz^2 - 5z^3 + xy + yz) \end{pmatrix}$$

So

$$DX_{\mu\mu}(2\pi, x, y, z, 0) = \begin{pmatrix} \frac{\partial X^1_{\mu\mu}}{x} & \frac{\partial X^2_{\mu\mu}}{y} & \frac{\partial X^3_{\mu\mu}}{z} \end{pmatrix}$$

,

$$\frac{\partial X^1_{\mu\mu}}{x} = \begin{pmatrix} 4\pi^2(12x^2 - 3y + 2y^2 + 36xz + 28z^2) \\ 4\pi^2(-4xy - 6yz) \\ 4\pi^2(-4xz - 6z^2 + y) \end{pmatrix}$$

$$\frac{\partial X^2_{\mu\mu}}{y} = \begin{pmatrix} 4\pi^2(-3x + 4xy - 4z + 6zy) \\ 4\pi^2(-2x^2 + 2y - 3y^2 - 6xz - 5z^2) \\ 4\pi^2(z - 2zy + x + z) \end{pmatrix}$$

$$\frac{\partial X^3_{\mu\mu}}{z} = \begin{pmatrix} 4\pi^2(18x^2 + 56xz - 4y + 3y^2 + 45z^2) \\ 4\pi^2(-6xy - 10yz) \\ 4\pi^2(-2x^2 + y - y^2 - 12xz - 15z^2 + y) \end{pmatrix}$$

thus

$$DX_{\mu\mu}(2\pi, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0) = \begin{pmatrix} 0 & 0 & 2\pi^2 \\ 2\pi^2 & 0 & 4\pi^2 \\ 0 & 0 & -2\pi^2 \end{pmatrix}$$

then $rank(DX_{\mu\mu}(2\pi, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0)) = 2$ and we have

$$rank(D\sigma(2\pi, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0)) = 2$$

Since $\frac{\partial\sigma}{\partial x}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 2\pi, 0) \neq 0$, $\frac{\partial\sigma}{\partial z}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 2\pi, 0) \neq 0$ and $\frac{\partial\sigma}{\partial y}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 2\pi, 0) = 0$, then by using the Implicit Function theorem we have $x = h_1(y, \mu)$ and $z = h_2(y, \mu)$ in a sufficiently small neighborhood of the point $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 2\pi, 0)$ such that $\frac{1}{2} = h_1(\frac{1}{2}, 0)$, $-\frac{1}{2} = h_2(\frac{1}{2}, 0)$ and $\sigma(h_1(y, \mu), y, h_2(y, \mu)2\pi, , \mu) = 0$.

Bibliography

[1] Arbieto, Alexander, and Carlos Matheus. 2004. ÒDecidability of Chaos for Some Families of Dynamical Systems.Ó Foundations of Computational Mathematics. 4 (3): 269Ð275.

[2] Avila, Artur, Mikhail Lyubich, and Welington de Melo. 2003. ÒRegular or Stochastic Dynamics in Real Analytic Families of Unimodal Maps.Ó Inventiones Mathematicae 154 (3): 451Ð550.

[3] Avila, Artur, and Carlos Gustavo Moreira. 2005. ÒStatistical Properties of Unimodal Maps: the Quadratic Family.Ó Annals of Mathematics 161 (2): 831Ð881.

[4] Avila, Artur, Mikhail Lyubich, and Weixiao Shen. 2011. ÒParapuzzle of the Multibrot Set and Typical Dynamics of Unimodal Maps.Ó Journal of the European Mathematical Society (JEMS) 13

[5] Ali Golmakani, Steffano Luzzatto and Pawel Pilarczyk. 2015. Uniform Expansivity Outside The Critical Neighborhood in The Quadratic Family.

[6] Benedicks M and Carleson L 1985 On iterations of 1 - ax2 on (-1, 1) *Ann. Math.* 122: 1-25

[7] Borwein, Jonathan M, and O-Yeat Chan. 2009. ÒUniform Bounds for the Complementary Incomplete Gamma Function.Ó Mathematical Inequalities & Applications 12 (1): 115Ð121.

[8] Collet P and Eckmann J-P 1983 Positive Lyapunov exponents and absolute continuity for maps of the interval *Ergod. Theory Dynam. Syst.* 3: 13-46

[9] Day S, Kokubu H, Luzzatto S, Mischaikow K, Oka H and Pilarczyk P 2008 Quantitative hyperbolicity estimates in one-dimensional dynamics *Nonlinearity* 21: 1967-87

[10] de Melo, Welington, and Sebastian van Strien. 1993. One-Dimensional Dynamics. Vol. 25. Berlin: Springer-Verlag.

[11] Devaney, Robert L. 2003. An Introduction to Chaotic Dynamical Systems. Westview Pr.

[12] Gao, Bing, and Weixiao Shen. 2011. ÒSummability Implies Collet-Eckmann Almost Surely.Ó arXiv.org.

[13] J. Graczyk and G. Swiatek 1997 Generic hyperbolicity in the logistic family. *Ann. Math.* 146:1-52

[14] Huang, Yu-Ru. 2011, Measure of paremeters with acim nonadjacent to the Chebyshev value in the quadratic family. PhD Thesis. University of Maryland.

[15] Jakobson M V 2001 Piecewise smooth maps with absolutely continuous invariant measures and uniformly scaled Markov partitions *Proc. Symp. Pure Math.* 69: 825-81

[16] Jakobson M V 2004 Parameter choice for families of maps with many critical points *Modern Dynamical Systems and Applications* (Cambridge: Cambridge University Press)

[17] Kozlovski O 2003 Axiom A maps are dense in the space of unimodal maps in the C^k topology. *Ann. Math.* 157: 1-43

[18] Kozlovski O, Shen W and van Strien S 2007 Density of hyperbolicity in dimension one. *Ann. Math.* 166: 145-82

[19] Kozlovski, Oleg, Weixiao Shen, and Sebastian van Strien. 2007. ÒRigidity for Real Polynomials.Ó The Annals of Mathematics 165 (3): 749Ð841.

[20] Day S, Kokubo H, Luzzatto S, Mischaikow K, Oka H and Pilarczyke P 2008 Quantitaive hyperbolicity estimates in one-dimensional dynamics. *Nonlinearity*, 21:1967-1987.

[21] Luzzatto, stefano and Marcelo Viana. Positive Lyapunov exponents for Lorenz-like families with criticalities Astérisque, tome 261 (2000), p. 201-237

[22] Luzzatto S and Takahashi K 2006 Computable conditions for the occurrence of non-uniform hyperbolicity in families of onedimensional maps. em Nonlinearity 19: 1657-1695

[23] Lyubich M 1997 Dynamics of quadratic polynomials. I *Acta Math.* 178: 185-247. Lyubich M 1997 Dynamics of quadratic polynomials. II *Acta Math.* 178: 247-97.

[24] Lyubich, Mikhail. 2002. Almost Every Real Quadratic Map Is Either Regular or Stochastic. The Annals of Mathematics 156 (1).

[25] de Melo W and van Strien S 1993 One-dimensional dynamics Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)) vol 25 (Berlin: Springer)

[26] Mañé, Ricardo. 1985. ÒHyperbolicity, Sinks and Measure in One-Dimensional Dynamics.Ó Communications in Mathematical Physics 100 (4): 495Ð524.

[27] Natalini, Pierpaolo, and Biagio Palumbo. 2000. ÒInequalities for the Incomplete Gamma Function.Ó Mathematical Inequalities & Applications 3 (1): 69Ð77.

[28] Nowicki, Tomasz, and Duncan Sands. 1998. ÒNon-Uniform Hyperbolicity and Universal Bounds for S-Unimodal Maps.Ó Inventiones Mathematicae 132 (3)

[29] Pacifico, M J, A Rovella, M Viana, M J Pacifico, A Rovella, and M Viana. 1998. ÒInfinite-Modal Maps with Global Chaotic BehaviorÓ 148 (2): 441Ð484.

[30] G. Panti. 2013. Personal communication.

[31] Rovella, Alvaro. 1993. ÒThe Dynamics of Perturbations of the Contracting Lorenz Attractor.Ó Boletim Da Sociedade Brasileira De Matemática. Nova Série 24 (2): 233Ð259.

[32] Rychlik M R 1988 Another proof of Jakobson's theorem and related results *Ergod. Theory Dyn. Syst.* 8: 93-109

[33] Simó, C, and J C Tatjer. 1991. ÒWindows of Attraction of the Logistic Map.Ó In European Conference on Iteration Theory (Batschuns, 1989), 335Ð342. World Sci. Publ., River Edge, NJ.

[34] M. Shishikura 2012. A Proof of JakobsonÕs Theorem via Yoccoz puzzles and the measure of stochastic parameters. Conference slides.

[35] Thieullen Ph, Tresser C and Young L-S 1994 Positive Lyapunov exponent for generic one-parameter families of unimodal maps *J. Anal. Math.* 64: 121-72

[36] Thunberg H 1999 Positive exponent in families with flat critical point *Ergod. Theory Dyn. Syst.* 19: 767-807

[37] Tsujii M 1993 Positive Lyapunov exponents in families of one-dimensional dynamical systems *Invent. Math.* 111: 113-37

[38] Tucker, Warwick. 2011. Validated Numerics. Princeton, NJ: Princeton University Press.

[39] Tucker, Warwick, and Daniel Wilczak. 2009. A Rigorous Lower Bound for the Stability Regions of the Quadratic Map. Physica D 238 (18). 1923Ð1936.

[40] P.Amster, P.De Napoli and M.C.Mariani, *Periodic Solutions of a Resonant Third-Order Equation*, Nonlinear Analysis, Theory, Methods and Applications 60, 3(2005) pp 399 - 410.

[41] M.B.H.Rhouma and C.Chicone, *On The Continuation of Periodic Orbits*, Methods and Applications of Analysis, vol. 7, (2000) pp 85 - 104.

[42] C.Chicone, *Ordinary Differential Equations with Applications*, Texts in Applied Mathematics, vol. 34, New York: Springer-Verlage, 1999.

[43] C.Chicone, *Periodic Solutions of a System of Coupled Oscillators Near Resonance*, SIAM Journal on Mathematical Analysis, vol.26, (1995) pp 1257 - 1283.

[44] C.Chicone, *Lyapunov - Schmidt Reduction and Melnikov Integrals for Bifurcation of Periodic Solutions in Coupled Oscillators*, Journal of differential equations, vol.112, (1994) pp 407 - 447.

[45] J.Cronin, *Some mathematics of biological oscillations*, SIAM Rev. 19 (1977) pp 100 - 137.

[46] K.O. Friedrichs, *On nonlinear vibrations of the third-order, in:Studies in Nonlinear Vibration Theory*, Inst.Math.Mech., New York University, 1946, pp. 65-103.

[47] R.S.Holland, M.S. Keener, *Analysis of linear compartment models for ecosystems*, J. Theoret. Biol. 44 (1974) pp 106 - 116.

[48] M.Golubitsky and D.G.Schaeffer, *Singularities and Groups in Bifurcation Theory*, Texts in Applied Mathematics, vol. 1, Springer-Verlage, New York,1985.

[49] B.Mehri and M. Niksirat, *The existence of periodic solutions for the non-linear third-order autonomous O.D.Es*, Nonlinear Anal. Forum 5 (2000) pp 163 - 171.

[50] L.Perko, *Differential Equations and Dynamical Systems*, Springer-Verlage, 1991.

[51] O. Rabiei Motlagh and Z.Afsharnejad, *Existence of periodic solutions and homoclinic orbits for third-order nonlinear differential equations* , International Journal of Mathematics and Mathematical Science (IJMMS)., (2003-4) pp 209 - 228.

[52] O. Rabiei Motlagh and Z.Afsharnejad, *Existence of periodic solutions and Bonded Invariant Sets for Non-Autonomous Third-Order Differential Equations*, Vietnam Journal of Mathematics.31 : 3 (2003) pp 295 - 308.

[53] O. Rabiei Motlagh, J.A. Esfahani, and Z.Afsharnejad, *Existence of turbulent behavior for non-chaotic two dimensional jets*, J. Math. Anal. Appl. 293 (2004) pp 329 - 344.

[54] z.Afsharnejad and A.Golmakani, *Continuation of Periodic Orbits of a Third Order Oscillator under Autonamous Perturbation*,Jornal of Dynamical System and Geometric Theories, vol. 4 , No. 2 ,(2006) pp 191 - 197

[55] L.L.Rauch, *Oscillations of a third order nonlinear autonomous system,in: Contributions to the Theory of Nonlinear Oscillations*, in: Ann. of Math. Stud., vol. 20, 1950 pp. 39-88.

[56] B. Li, *Uniqueness and stability of a limit cycle for a third-order dynamical system arising in neuron modeling*, Nonlinear Anal. 5 (1981) pp 13-19.

[57] Z.Zhang and Z.Wang, *Periodic solutions of the third order functional equations* , J. Math. Anal. Appl. 292 (2004) pp 115 - 134.