POINCARÉ TYPE INEQUALITY FOR HYPERSURFACES AND RIGIDITY RESULTS

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ABSTRACT. In this paper, we deal with general divergence formulas involving symmetric endomorphisms. Using mild constraints in the sectional curvature and such divergence formulas we deduce a very general Poincaré type inequality. We apply such general inequality for higher-order mean curvature, in space forms and Einstein manifolds, to obtain several isoperimetric inequalities, as well as rigidity results for complete r-minimal hypersurfaces satisfying a suitable decay of the second fundamental form at infinity. Furthermore, using these techniques, we prove the flatness and non existence results for self-similar solutions of a large class of fully nonlinear curvature flows.

1. INTRODUCTION

In the last decades, many mathematicians are seeking for nice embeddings between spaces of functions and estimates providing regularity of solutions of some PDEs. For a domain Ω in $\mathbb{R}^n$, classical estimates that allow us to obtain interesting information on the space $W_0^{1,p}(\Omega)$ is the Poincaré inequality, for $1 \leq p < n$. The reader can learn more about the subject in [31], [43], [13], [24], [44] and references therein.

Many consequences of such types of inequalities are obtained in the literature, for instance, volume growth, spectral and regularity of solutions of elliptic equations, number of harmonic L^2 1-forms, number of endings of a manifold, and others. We point out that some rigidity is obtained from such type o inequality with additional constraints on the curvatures, see [18] and [46].

Results. In this work, we establish a very general Poincaré type inequality on manifolds. Using such estimate and additional mild conditions we obtain

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rigidity results for hypersurfaces of space forms and for suitable Einstein manifolds, as we briefly describe in the following. We prove

(i) Isoperimetric inequalities for domains of hypersurfaces of $\mathbb{R}^{m+1}$;
(ii) that $(r+1)$-minimal hypersurfaces of the space forms, which satisfy a suitable decay for the integral of the r-mean curvature over geodesic spheres, are foliated by totally geodesic submanifolds, becoming cylinders or totally geodesic hypersurfaces if its Ricci curvature is bounded from bellow;
(iii) that a hypersurface with a determined constant scalar curvature in a Einstein manifold is totally geodesic, provided the integral of its mean curvature over geodesic spheres satisfy a suitable decay;
(iv) Rigidity result for the hyperplane as the only homothetic self-similar solutions of a wide class of fully nonlinear curvature flows.

Organization of the paper. In section 2 we present the basic computations of this work. In section 3 we state our main general inequality and apply it for the setting of higher-order mean curvature and to derive isoperimetric inequalities. In section 4 we obtain rigidity results of item (ii) and (iii) above as a consequence of our Poincaré type inequality. We conclude the paper proving, in section 5, the rigidity results for self-similar solutions of some nonlinear curvature flows.

2. Notations and Preliminaries

Let M be a hypersurface of a Riemannian $(m+1)$-manifold $\overline{M}^{m+1}$. Denote by ∇ and $\overline{\nabla}$ the connections of M and $\overline{M}^{m+1}$, respectively. Given $\overline{X}: M\to T\overline{M}^{m+1}$ a vector field, write $\overline{X} = X^\top + X^\perp$, where $X^\top \in TM$ and $X^\perp \in TM^\perp$. Denoting by $\langle\cdot,\cdot\rangle$ the metric of $\overline{M}$, we have, for vector fields $Y, Z\in TM$,

$$\begin{aligned}\langle \overline{\nabla}_Y \overline{X}, Z\rangle &= \langle \overline{\nabla}_Y X^\top + \overline{\nabla}_Y X^\perp, Z\rangle\\ &= \langle \overline{\nabla}_Y X^\top, Z\rangle + \langle \overline{\nabla}_Y X^\perp, Z\rangle\\ &= \langle \overline{\nabla}_Y X^\top, Z\rangle - \langle X^\perp, \overline{\nabla}_Y Z\rangle\\ &= \langle \overline{\nabla}_Y X^\top, Z\rangle - \langle X^\perp, B(Y,Z)\rangle,\end{aligned}$$

where $B(Y,Z) = \overline{\nabla}_Y Z - \nabla_Y Z$ denotes the second fundamental form of M. If η is the normal vector field, then $X^\perp = \langle \overline{X}, \eta\rangle\eta$. It implies

$$\begin{aligned}\langle \overline{\nabla}_Y \overline{X}, Z\rangle &= \langle \overline{\nabla}_Y X^\top, Z\rangle - \langle \overline{X}, \eta\rangle\langle \eta, B(Y,Z)\rangle\\ &= \langle \overline{\nabla}_Y X^\top, Z\rangle - \langle \overline{X}, \eta\rangle\langle A(Y), Z\rangle,\end{aligned} \tag{1}$$

where $A : TM \to TM$ is the Weingarten operator which is given by

$$\langle A(V), W\rangle = \langle \eta, B(V,W)\rangle,\ V, W \in TM. \tag{2}$$

We first state a very general divergence formula which will be useful for us in the next section. A similar formula was obtained by the first and third named authors in [1].

Proposition 2.1. *If M is a hypersurface of a $(m+1)$-dimensional Riemannian manifold $\overline{M}^{m+1}$, $m \geq 2$, and $\overline{X} : M \to T\overline{M}$ is a vector field, then, for every symmetric linear operator $T : TM \to TM$, it holds*

$$\begin{aligned} \operatorname{div}_f(T(X^\top)) = &-\langle X^\top, T(\nabla f)\rangle + \operatorname{tr}\left(E \mapsto T\left(\left(\overline{\nabla}_E \overline{X}\right)^\top\right)\right) \\ &+ \langle \overline{X}, \eta\rangle \operatorname{tr}(AT) + (\operatorname{div} T)(X^\top). \end{aligned} \tag{3}$$

Here, $\operatorname{div}_f(Y) = e^f \operatorname{div}(e^{-f} Y)$ is the weighted divergent, $f : M \to \mathbb{R}$ is a smooth function, $(\operatorname{div} T)(Y) = \operatorname{tr}(E \mapsto (\nabla_E T)(Y))$, and tr denotes the trace of the operator.

Proof. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal frame in TM and $\overline{X} \in T\overline{M}$. First, since T is self-adjoint, we have

(4)

$$\operatorname{tr}\left(E \mapsto T\left(\left(\overline{\nabla}_E \overline{X}\right)^\top\right)\right) = \sum_{i=1}^m \left\langle T\left(\left(\overline{\nabla}_{e_i} \overline{X}\right)^\top\right), e_i\right\rangle = \sum_{i=1}^m \langle \overline{\nabla}_{e_i} \overline{X}, T(e_i)\rangle.$$

By using (1), p.2, and the self-adjointness of A, we obtain

$$\begin{aligned} \sum_{i=1}^m \langle \overline{\nabla}_{e_i} \overline{X}, T(e_i)\rangle &= \sum_{i=1}^m \langle \overline{\nabla}_{e_i} X^\top, T(e_i)\rangle - \left(\sum_{i=1}^m \langle A(e_i), T(e_i)\rangle\right) \langle \overline{X}, \eta\rangle \\ &= \sum_{i=1}^m \langle \overline{\nabla}_{e_i} X^\top, T(e_i)\rangle - \left(\sum_{i=1}^m \langle (AT)(e_i), e_i\rangle\right) \langle \overline{X}, \eta\rangle \\ &= \sum_{i=1}^m \langle \overline{\nabla}_{e_i} X^\top, T(e_i)\rangle - \operatorname{tr}(AT)\langle \overline{X}, \eta\rangle. \end{aligned}$$

Thus,

$$\sum_{i=1}^m \langle \overline{\nabla}_{e_i} X^\top, T(e_i)\rangle = \operatorname{tr}\left(E \mapsto T\left(\left(\overline{\nabla}_E \overline{X}\right)^\top\right)\right) + \operatorname{tr}(AT)\langle \overline{X}, \eta\rangle.$$

On the other hand, the self-adjointness of T implies

$$\begin{aligned}\sum_{i=1}^{m}\langle\overline{\nabla}_{e_i}X^\top, T(e_i)\rangle &= \sum_{i=1}^{m}\langle\nabla_{e_i}X^\top + B(e_i, X^\top), T(e_i)\rangle\\ &= \sum_{i=1}^{m}\langle T(\nabla_{e_i}X^\top), e_i\rangle\\ &= \sum_{i=1}^{m}\langle\nabla_{e_i}(T(X^\top)), e_i\rangle - \sum_{i=1}^{m}\langle(\nabla_{e_i}T)(X^\top), e_i\rangle\\ &= \operatorname{div}(T(X^\top)) - \operatorname{tr}(E\mapsto(\nabla_E T)(X^\top))\\ &= \operatorname{div}(T(X^\top)) - (\operatorname{div}T)(X^\top).\end{aligned}$$

Therefore,

$$\operatorname{div}(T(X^\top)) = \operatorname{tr}\left(E\mapsto T\left(\left(\overline{\nabla}_E\overline{X}\right)^\top\right)\right) + (\operatorname{div}T)(X^\top) + \operatorname{tr}(AT)\langle\overline{X},\eta\rangle.$$

Since

$$\operatorname{div}_f(Y) = e^f\operatorname{div}(e^{-f}Y) = \operatorname{div}(Y) - \langle\nabla f, Y\rangle,\ Y\in TM,$$

we conclude the result. □

In the next lemma we will estimate $\operatorname{tr}\left(E\mapsto T\left(\left(\overline{\nabla}_E\overline{X}\right)^\top\right)\right)$, for a special vector field $\overline{X}$, in terms of $\operatorname{tr}T$, the distance function of $\overline{M}$ and the bounds of the sectional curvatures of $\overline{M}$. This result is essentially in [12], Proposition 2.2, p.109, but by the difference of notations in both articles and by the sake of completeness, we include a (different) proof here.

Lemma 2.1. *Let $\overline{M}^{m+1}$, $m\geq 2$, be a Riemannian $(m+1)$-dimensional manifold whose sectional curvatures along the geodesics $\overline{\gamma}:[0,\ell]\to\overline{M}$ satisfy*

$$\operatorname{Sect}_{\overline{M}}(\overline{V},\overline{\gamma}') \leq -\frac{G''(t)}{G(t)},\ \forall\ \overline{V}\in T\overline{M},\ \overline{V}\perp\overline{\gamma}',$$

for a class $\mathcal{C}^2$ nondecreasing function $G:[0,b)\to\mathbb{R}$, positive for $t\neq 0$ and $b>l$. Let M be a hypersurface of $\overline{M}^{m+1}$, $T:TM\to TM$ be a non-negative symmetric linear operator, and $\rho(x)=\rho(p,x)$ be the geodesic distance of $\overline{M}^{m+1}$ starting at a point $x_0\in\overline{M}^{m+1}$. If $x\in M$ satisfies $\rho(x)<i(\overline{M})$, where $i(\overline{M})$ is the injectivity radius of $\overline{M}^{m+1}$, then the vector field $\overline{X}=G(\rho)\overline{\nabla}\rho$ satisfies

$$\operatorname{tr}\left(E\mapsto T\left(\left(\overline{\nabla}_E\overline{X}\right)^\top\right)\right)(x) \geq G'(\rho(x))(\operatorname{tr}T)(x). \tag{5}$$

Proof. Consider $\gamma:[0,\rho(x)]\to\overline{M}$ defined by $\gamma(t)=\exp_{x_0}(tu),\ u\in T_{x_0}\overline{M}$, the unit speed geodesic such that $\gamma(0)=x_0$ e $\gamma(\rho(x))=x$. Let $\{e_1,e_2,\dots,e_m\}$

be an orthonormal basis of T_xM composed by eigenvectors of T in $x \in M$, i.e.,

$$T(e_i(x)) = \theta_i(x)e_i(x),\ \ i = 1, 2, \dots, m.$$

Let Y_i, $i = 1, 2, \dots, m$, be the unitary projections of $e_i(x)$ over $\gamma'(\rho(x))^{\perp} \subset T_x\overline{M}$, namely,

$$Y_i = \frac{e_i(x) - \langle e_i(x), \gamma'(\rho(x))\rangle\gamma'(\rho(x))}{\|e_i(x) - \langle e_i(x), \gamma'(\rho(x))\rangle\gamma'(\rho(x))\|},\ \ i = 1, 2, \dots, m.$$

Thus,

$$e_i(x) = b_iY_i + c_i\gamma'(\rho(x)),$$

where $b_i = \|e_i(x) - \langle e_i(x), \gamma'(\rho(x))\rangle\gamma'(\rho(x))\|$ and $c_i = \langle e_i(x), \gamma'(\rho(x))\rangle$ satisfy $b_i^2 + c_i^2 = 1$ and $Y_i \perp \gamma'$ for all $i = 1, 2, \dots, m$. Since we are assuming that $\rho(x) < i(\overline{M})$, there are no conjugate points to x_0 along γ. Thus

$$\begin{aligned}
\operatorname{tr}\left(E \longmapsto T\left(\left(\overline{\nabla}_E\overline{X}\right)^{\top}\right)\right) &= \sum_{i=1}^{m}\langle\overline{\nabla}_{e_i}\overline{X}, T(e_i)\rangle = \sum_{i=1}^{m}\theta_i\langle\overline{\nabla}_{e_i}\overline{X}, e_i\rangle \\
&= \sum_{i=1}^{m}\theta_i\langle\overline{\nabla}_{b_iY_i+c_i\gamma'}\overline{X}, b_iY_i + c_i\gamma'\rangle \\
&= \sum_{i=1}^{m}\theta_ib_i^2\langle\overline{\nabla}_{Y_i}\overline{X}, Y_i\rangle + \sum_{i=1}^{m}\theta_ic_i^2\langle\overline{\nabla}_{\gamma'}\overline{X}, \gamma'\rangle \\
&\quad + \sum_{i=1}^{m}\theta_ib_ic_i\left[\langle\overline{\nabla}_{Y_i}\overline{X}, \gamma'\rangle + \langle\overline{\nabla}_{\gamma'}\overline{X}, Y_i\rangle\right].
\end{aligned}$$

Since $\overline{X}(t) = G(\rho(t))\overline{\nabla}\rho(t) = G(\rho(t))\gamma'(t)$ and $\overline{\nabla}_{\gamma'}\gamma' = 0$, we have

$$\begin{aligned}
\langle\overline{\nabla}_{\gamma'}\overline{X}, \gamma'\rangle &= \langle\overline{\nabla}_{\gamma'}(G(\rho)\gamma'), \gamma'\rangle = \langle G'(\rho)\langle\overline{\nabla}\rho, \gamma'\rangle\gamma' + G(\rho)\overline{\nabla}_{\gamma'}\gamma', \gamma'\rangle \\
&= G'(\rho)\langle\overline{\nabla}\rho, \gamma'\rangle\langle\gamma', \gamma'\rangle = G'(\rho), \\
\langle\overline{\nabla}_{Y_i}\overline{X}, \gamma'\rangle &= \langle\overline{\nabla}_{Y_i}(G(\rho)\gamma'), \gamma'\rangle = \langle G'(\rho)\langle Y_i, \overline{\nabla}\rho\rangle\gamma' + G(\rho)\overline{\nabla}_{Y_i}\gamma', \gamma'\rangle \\
&= G'(\rho)\langle Y_i, \gamma'\rangle + G(\rho)\langle\overline{\nabla}_{Y_i}\gamma', \gamma'\rangle \\
&= \frac{G(\rho)}{2}Y_i\langle\gamma', \gamma'\rangle = 0, \\
\langle\overline{\nabla}_{\gamma'}\overline{X}, Y_i\rangle &= \langle\overline{\nabla}_{\gamma'}(G(\rho)\gamma'), Y_i\rangle = \langle G'(\rho)\langle\gamma', \overline{\nabla}\rho\rangle\gamma' + G(\rho)\overline{\nabla}_{\gamma'}\gamma', Y_i\rangle = 0.
\end{aligned}$$

This gives

$$\operatorname{tr}\left(E \longmapsto T\left(\left(\overline{\nabla}_E\overline{X}\right)^{T}\right)\right) = \sum_{i=1}^{m}\theta_ib_i^2\langle\overline{\nabla}_{Y_i}\overline{X}, Y_i\rangle + G'(\rho)\sum_{i=1}^{m}\theta_ic_i^2.$$

On the other hand, let $\overline{N}^{m+1} = [0,b) \times \mathbb{S}^m$ with the metric $\langle \cdot, \cdot \rangle_{\overline{N}} = dt^2 + G(t)^2 d\omega^2$, where $d\omega^2$ is the metric of $\mathbb{S}^m$. It is well known that, if $\gamma_N : [0,\ell] \to \overline{N}$ is a geodesic and $\overline{V} \in T\overline{N}$ is such that $\overline{V} \perp \gamma_N'$, along γ_N, then

$$\mathrm{Sect}_{\overline{N}}(\overline{V}(t), \gamma_N'(t)) = -\frac{G''(t)}{G(t)}.$$

Thus, by using the hypothesis and the hessian comparison theorem, we have

$$\mathrm{Hess}_{\overline{M}}\, \rho(U,U) \geq \mathrm{Hess}_{\overline{N}}\, \rho(U,U) = \frac{G'(\rho)}{G(\rho)}[|U|^2 - \langle U, \overline{\nabla}\rho\rangle^2\rangle].$$

Since

$$\begin{aligned}
\langle \overline{\nabla}_{Y_i}\overline{X}, Y_i\rangle &= \langle \overline{\nabla}_{Y_i}(G(\rho)\overline{\nabla}\rho), Y_i\rangle \\
&= \langle G'(\rho)\langle Y_i, \overline{\nabla}\rho\rangle\overline{\nabla}\rho + G(\rho)\overline{\nabla}_{Y_i}\overline{\nabla}\rho, Y_i\rangle \\
&= G(\rho)\langle \overline{\nabla}_{Y_i}\overline{\nabla}\rho, Y_i\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
\langle \overline{\nabla}_{Y_i}\overline{X}, Y_i\rangle &= G(\rho)\langle \overline{\nabla}_{Y_i}\overline{\nabla}\rho, Y_i\rangle \\
&\geq G'(\rho)(|Y_i|^2 - \langle Y_i, \overline{\nabla}\rho\rangle^2) \\
&= G'(\rho).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{tr}\left(E \mapsto T\left(\left(\overline{\nabla}_E\overline{X}\right)^\top\right)\right) &= \sum_{i=1}^m \theta_i b_i^2 \langle \overline{\nabla}_{Y_i}\overline{X}, Y_i\rangle + G'(\rho)\sum_{i=1}^m \theta_i c_i^2 \\
&\geq G'(\rho)\sum_{i=1}^m \theta_i b_i^2 + G'(\rho)\sum_{i=1}^m \theta_i c_i^2 \\
&= G'(\rho)(\mathrm{tr}\, T).
\end{aligned}$$

□

Remark 1. Notice that, in $\overline{M}^{m+1} = [0,b) \times \mathbb{S}^m$, with the metric $\langle \cdot, \cdot \rangle_{\overline{M}} = dt^2 + G(t)^2 d\omega^2$, where $d\omega^2$ is the metric of $\mathbb{S}^m$, the inequality in the Lemma 2.1 becomes an equality and we do not need to assume that T is non negative definite, i.e., for hypersurfaces of $\overline{M}^{m+1}$ and for every symmetric linear operator $T : TM \to TM$ we have

$$\mathrm{tr}\left(E \mapsto T\left(\left(\overline{\nabla}_E\overline{X}\right)^\top\right)\right) = G'(\rho)(\mathrm{tr}\, T).$$

3. Poincaré type inequality

Here we state our Poincaré type inequality in full generality and next we present some consequences.

Theorem 3.1. *Let $\overline{M}^{m+1}$, $m \geq 2$, be a $(m+1)$-dimensional Riemannian manifold whose sectional curvatures along the geodesics $\overline{\gamma} : [0, \ell] \to \overline{M}$ satisfy*

$$\operatorname{Sect}_{\overline{M}}(\overline{V}, \overline{\gamma}') \leq -\frac{G''(t)}{G(t)}, \ \forall\, \overline{V} \in T\overline{M}, \ \overline{V} \perp \overline{\gamma}', \tag{6}$$

for a class $\mathcal{C}^2$ non decreasing function $G : [0, b) \to \mathbb{R}$, which is positive for $t \neq 0$ and $b > l$. Let M be a hypersurface of $\overline{M}^{m+1}$, $T : TM \to TM$ be a non negative symmetric linear operator and $\Omega \subset M$ be a connected and open domain with compact closure such that $\overline{\Omega} \cap \partial M = \emptyset$. If $\operatorname{diam} \Omega < 2i(\overline{M})$, where $i(\overline{M})$ is the injectivity radius of $\overline{M}^{m+1}$ and $\operatorname{diam} \Omega$ is the extrinsic diameter of Ω, then, for every class $\mathcal{C}^1$ functions $u, f \colon M \to \mathbb{R}$, with u non negative and compactly supported in Ω, we have

$$\begin{aligned}\int_\Omega G'(\rho)u(\operatorname{tr} T)e^{-f} d\mu \leq{}& G\left(\frac{\operatorname{diam}\Omega}{2}\right) \int_\Omega |T(\nabla u - u\nabla f)|e^{-f} d\mu \\ &+ G\left(\frac{\operatorname{diam}\Omega}{2}\right) \int_\Omega u \left[|| \operatorname{tr}(AT)| - (\operatorname{div} T)(\nabla\rho)|\right] e^{-f} d\mu,\end{aligned} \tag{7}$$

where ρ is the distance function of $\overline{M}^{m+1}$, restricted to M.

Moreover, if $\overline{M}^{m+1} = [0, b) \times \mathbb{S}^m$, with the metric $\langle \cdot, \cdot \rangle_{\overline{M}} = dt^2 + G(t)^2 d\omega^2$, where $d\omega^2$ is the metric of $\mathbb{S}^m$, then it is not necessary to assume that T is non-negative.

Proof of Theorem 3.1. Since $\operatorname{diam} \Omega < 2i(\overline{M})$, we can consider $B_R(x_0)$, $x_0 \in \overline{M}$, the smallest extrinsic ball containing $\overline{\Omega}$, and $\rho(x) = \rho(x_0, x)$ the extrinsic distance from x_0 to $x \in M$. Since $\Omega \subset B_R(x_0)$, then, for all $x \in \Omega$,

$$\rho(x) \leq R = \frac{\operatorname{diam}\Omega}{2}. \tag{8}$$

For every non negative class $\mathcal{C}^1$ function $u : M \to \mathbb{R}$, it holds

$$\begin{aligned}\operatorname{div}_f(uT(X^\top)) &= e^f \operatorname{div}(e^{-f} uT(X^\top)) \\ &= e^f \Big(u \operatorname{div}(e^{-f} T(X^\top)) + \langle \nabla u, e^{-f} T(X^\top)\rangle\Big) \\ &= u \operatorname{div}_f(T(X^\top)) + \langle \nabla u, T(X^\perp)\rangle,\end{aligned}$$

and so we have, using Proposition 2.1 and Lemma 2.1 for $\overline{X} = G(\rho)\overline{\nabla}\rho$,

$$\begin{aligned}\operatorname{div}_f(uT(X^\top)) &\geq G(\rho)\langle\nabla\rho, T(\nabla u - u\nabla f)\rangle + uG'(\rho)(\operatorname{tr} T)\\ &\quad + uG(\rho)\langle\overline{\nabla}\rho,\eta\rangle\operatorname{tr}(AT) + uG(\rho)(\operatorname{div} T)(\nabla\rho).\end{aligned}$$

On the other hand, by divergence theorem,

$$\int_\Omega \operatorname{div}_f(uT(X^\top))e^{-f}d\mu = \int_\Omega \operatorname{div}(e^{-f}uT(X^\top))d\mu = 0,$$

which implies, after integration and some rearrangement,

$$\begin{aligned}\int_\Omega uG'(\rho)(\operatorname{tr} T)e^{-f}d\mu &\leq \int_\Omega G(\rho)\langle-\nabla\rho, T(\nabla u - u\nabla f)\rangle e^{-f}d\mu\\ &\quad + \int_\Omega uG(\rho)\langle-\overline{\nabla}\rho,\eta\rangle\operatorname{tr}(AT)e^{-f}d\mu\\ &\quad + \int_\Omega uG(\rho)(\operatorname{div} T)(-\nabla\rho)e^{-f}d\mu.\end{aligned}\tag{9}$$

Now, since G is increasing and by using Cauchy-Schwartz inequality, we have

$$\begin{aligned}\int_\Omega uG'(\rho)(\operatorname{tr} T)e^{-f}d\mu &\leq G\left(\frac{\operatorname{diam}\Omega}{2}\right)\int_\Omega |T(\nabla u - u\nabla f)|e^{-f}d\mu\\ &\quad + G\left(\frac{\operatorname{diam}\Omega}{2}\right)\int_\Omega u\left|\left|\operatorname{tr}(AT)\right| - (\operatorname{div} T)(\nabla\rho)\right|e^{-f}d\mu.\end{aligned}$$

This gives (7). When $\overline{M}^{m+1} = [0,b)\times\mathbb{S}^m$, with the metric $\langle\cdot,\cdot\rangle_{\overline{N}} = dt^2 + G(t)^2d\omega^2$, the result follows from Remark 1, p.6. $\square$

3.1. **Space forms and the r-mean curvature.** For an oriented hypersurface M of $\overline{M}^{m+1}$, we recall that the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_m$, of A are called principal curvatures. The symmetric functions associated to the immersion are given by

$$S_r = \sum_{i_1<\ldots<i_r}\lambda_{i_1}\cdots\lambda_{i_r},$$

where $(i_1,\ldots,i_r)\in\{1,2,\ldots,m\}^r$. The r-mean curvature of M is defined by

$$H_r = \frac{1}{\binom{m}{r}}S_r.$$

When $r=1$, we have $H_1 = H = \frac{1}{m}\operatorname{tr} A$, the mean curvature of M. For $r=2$ and $\overline{M} = \mathbb{R}^{m+1}$, $H_2 = \frac{1}{m(m-1)}$Scal, where Scal is the non normalized scalar curvature of M, and for $r=m$, we have that $H_m = \det A$ is the Gauss-Kronecker curvature of M.

We recall that a hypersurface M of $\overline{M}^{m+1}$ is called r-minimal if H_r vanishes on M. Properties of hypersurfaces involving the r-mean curvaure, including the case of r-minimal hypersurfaces, has been object of research by many authors as, for example, [34], [41], [30], [5], [7], [36], [49], and [14].

Associated to the family of higher-order mean curvatures we have the Newton transformations $P_r : TM \to TM$, $r \in \{0, \ldots, m\}$, which are defined recursively as

$$P_0 = I, \ P_r = S_r I - AP_{r-1},$$

where $I : TM \to TM$ is the identity operator. Clearly P_r is a self-adjoint operator and $AP_r = P_r A$. This operator has nice properties related with the symmetric functions S_r. We first point out the following properties:

Lemma 3.1. *For each $0 \leq r \leq m-1$ it holds:*

(1) $\operatorname{tr} P_r = (m-r)S_r$;
(2) $\operatorname{tr} AP_r = (r+1)S_{r+1}$;
(3) $\operatorname{tr} A^2 P_r = S_1 S_{r+1} - (r+2)S_{r+2}$.

Proof. See [51] and [11]. □

Definition 3.1. Let $\mathcal{Q}_c^{m+1}$ be a $(m+1)$-dimensional, simply-connected, complete Riemannian manifold with constant sectional curvature c. If $c > 0$ consider $\mathcal{Q}_c^{m+1} = \mathbb{S}_+^{m+1}(c)$ be the open upper hemisphere. We call these manifolds space forms.

Before state the consequences of Theorem 3.1, we show sufficient conditions to the divergence of P_r vanish. Such result is well-known in literature, see [51] and [52], and we present a proof here for sake of completeness. The result is the following:

Lemma 3.2. *The divergence of the Newton transformations P_r vanishes, if the ambient manifold $\overline{M}$ is a space form.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal geodesic frame at $p \in M$. For X, Y vector fields on M, notice that

$$(\nabla_X P_r)Y = X(S_r)Y - (\nabla_X A)(P_{r-1}Y) - A((\nabla_X P_{r-1})Y),$$

and so

$$\operatorname{div} P_r = \nabla S_r - \sum_{i=1}^{n} (\nabla_{e_i} A)(P_{r-1}e_i) - A(\operatorname{div} P_{r-1}). \tag{10}$$

Using Codazzi equation we have for $Z \in TM$

$$\langle (\nabla_{e_i} A)(P_{r-1}e_i), Z \rangle = \langle P_{r-1}((\nabla_Z A)e_i), e_i \rangle + \langle \overline{\mathrm{R}}(N, P_{r-1}e_i)e_i, Z \rangle,$$

where R denotes the curvature tensor of the ambient manifold. Plugging the previous equality in (10), we obtain

$$\operatorname{div} P_r = dS_r - \operatorname{tr}(P_{r-1}\nabla_\star A) - A(\operatorname{div} P_{r-1}) - \sum_{i=1}^{n} \overline{\mathrm{R}}(N, P_{r-1}e_i)e_i, \tag{11}$$

where dS_r means the differential of S_r. To conclude the proof, we use equation (4.4) in [52], which states

$$dS_r = \operatorname{tr}(P_{r-1}\nabla_\star A),$$

and since the sectional curvature is constant, we deduce that

$$\operatorname{div} P_r = -A(\operatorname{div} P_{r-1}),$$

for $r \geq 1$ and thus it vanishes on M. $\square$

In order to state the next Poincaré type inequality, we need to define the special functions

$$\mathcal{S}_c(t) = \begin{cases} t, & \text{if } c = 0; \\ \frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}t), & \text{if } c < 0; \\ \frac{1}{\sqrt{c}}\sin(\sqrt{c}t), & \text{if } c > 0. \end{cases}$$

For space forms and Newton transformations we have the following result:

Theorem 3.2. *If M is a hypersurface of $\mathcal{Q}_c^{m+1}$ and $\Omega \subset M$, $\overline{\Omega} \cap \partial M = \emptyset$, is a connected and open domain with compact closure, then, for every class $\mathcal{C}^1$ functions $u, f\colon M \to \mathbb{R}$, with u non negative and compactly supported in Ω, we have*

$$\int_\Omega u S_r \mathcal{S}_c'(\rho) e^{-f} d\mu \leq C_0 \int_\Omega \left[|P_r(\nabla u - u\nabla f)| + (r+1)|S_{r+1}|u\right] e^{-f} d\mu, \tag{12}$$

for $C_0 = \frac{1}{(m-r)}\mathcal{S}_c\left(\frac{\operatorname{diam}\Omega}{2}\right)$. In particular, if $P_r : TM \to TM$ is non negative definite, then

$$\int_\Omega u H_r \mathcal{S}_c'(\rho) e^{-f} d\mu \leq C_1 \int_\Omega \left[|\nabla u - u\nabla f| H_r + |H_{r+1}|u\right] e^{-f} d\mu, \tag{13}$$

for $C_1 = (m-r)C_0$. Moreover, the equality holds if M is a geodesic sphere, $\Omega = M$, and f, u are constant functions.

Here, $\rho : M \to \mathbb{R}_+$ is distance function of $\mathcal{Q}_c^{m+1}$ restricted to M, S_r is the r-th symmetric function of the eigenvalues of M, $H_r = \binom{m}{r}^{-1} S_r$ is its r-mean curvature, and $\operatorname{diam}\Omega$ *denotes the extrinsic diameter of Ω.*

Proof. Indeed, in $\mathcal{Q}_c^{m+1}$ we have (6) for $G(t) = \mathcal{S}_c(t)$ and using Lemma 3.2 we have that $\operatorname{div} P_r = 0$ in space forms. From second item of Lemma 3.1, we have $\operatorname{tr}(AP_r) = (r+1)S_{r+1}$ and, by Theorem 3.1, we obtain (12). Moreover, if P_r is non-negative definite, then

$$|P_r(U)| \leq (\operatorname{tr} P_r)|U| = (m-r)S_r|U|,$$

which, together with $\binom{m}{r+1}\left(\frac{r+1}{m-r}\right)\binom{m}{r}^{-1} = 1$, gives (13), as desired. In order to verify the equality, just notice that, in geodesic spheres of radius R, it holds

$$\lambda_1 = \cdots = \lambda_m = \frac{\mathcal{S}_c'(R)}{\mathcal{S}_c(R)}.$$

The equality follows by direct substitution. □

Remark 2. There are some conditions to deduce that P_r is non-negative definite. We point out some of them below:

(a) If $S_{r+1} = 0$ and r odd, then we can choose an orientation such that P_r is non negative definite;
(b) $S_{r+1} = 0$, r even and $S_r \geq 0$;
(c) If r is odd, $S_{r+1} = 0$, and $S_{r+2} \neq 0$, then we can choose an orientation such that P_r is positive definite;
(d) If r is even, $S_{r+1} = 0$, $S_{r+2} \neq 0$, and $S_r \geq 0$, then P_r is positive definite;
(e) If $S_k > 0$ for some $1 \leq k \leq m-1$ and there exists a point where all the principal curvatures are non negative, then P_r is positive definite for every $1 \leq r \leq k-1$.

The proofs of these claims can be found in [17], Proposition 2.8., p.192, (for items (a) to (d)), [20], Proposition 3.2, p.188, (for item (e)).

In the following, we present some applications of the Poincaré inequalities of Theorem 3.2. Denote by $d\mu$ the m-Lebesgue measure of M and by dS_μ the $(m-1)$-dimensional Lebesgue measure of the boundaries of the m-dimensional subsets of M. We also denote the volume of a set Ω by $|\Omega|$.

Corollary 3.1. *Let M be a hypersurface of $\mathbb{R}^{m+1}$ such that $H_{r+1} > 0$, $r = 1, 2, \ldots, m-1$, and $\Omega \subset M$, $\Omega \cap \partial M = \emptyset$, be a connected and open domain with compact closure. If M has a point whose all the principal curvatures are non negative, then*

$$|\Omega| \leq \sum_{k=0}^{r} \left(\frac{\operatorname{diam}\Omega}{2}\right)^{k+1} \int_{\partial\Omega} H_k dS_\mu + \left(\frac{\operatorname{diam}\Omega}{2}\right)^{r+1} \int_{\Omega} H_{r+1} d\mu. \tag{14}$$

Proof. Taking $f \equiv 1$ and $u = u_\varepsilon$ in (13), where

$$u_\varepsilon(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, \partial\mathcal{B}_R) \geq \varepsilon; \\ \dfrac{1}{\varepsilon}\operatorname{dist}(x, \partial\mathcal{B}_R), & \text{if } \operatorname{dist}(x, \partial\mathcal{B}_R) < \varepsilon, \end{cases} \tag{15}$$

and dist stands for the distance function on M, we obtain

$$\int_\Omega H_r d\mu \leq \frac{\operatorname{diam}\Omega}{2}\left[\int_{\partial\Omega} H_r dS_\mu + \int_\Omega H_{r+1} d\mu\right]. \tag{16}$$

By applying successively (16) and using Remark 2, item (e) we obtain the result. □

Remark 3. In particular, for $r = 0$, we have

$$|\Omega| \leq \left(\frac{\operatorname{diam}\Omega}{2}\right)\left[|\partial\Omega| + \int_\Omega H d\mu\right], \tag{17}$$

and for $r = 1$, we obtain,

$$|\Omega| \leq \left(\frac{\operatorname{diam}\Omega}{2}\right)|\partial\Omega| + \left(\frac{\operatorname{diam}\Omega}{2}\right)^2\left[\int_{\partial\Omega} H dS_\mu + \frac{1}{m(m-1)}\int_\Omega \operatorname{Scal}\, d\mu\right], \tag{18}$$

where H is the mean curvature and Scal is the (non normalized) scalar curvature of M.

Remark 4. Isoperimetric inequalities in the spirit of (17) were obtained by the first and the third authors in [2] for immersions in warped product manifolds. We can also compare the previous results with Theorem 2 in [45], which states that

$$\int_M H_k \rho^p d\mu \leq \int_M H_r \rho^{p+r-k} d\mu$$

for every closed hypersurface M of $\mathbb{R}^{n+1}$ satisfying $H_r > 0$ and for every $p > 0$ and $0 \leq k < r$ (compare also with the results of [37]). Moreover, they prove that equality holds only for round spheres. On its turn, by the proof of our Poincaré type inquality (12), we obtain

$$\int_M H_{r-1} d\mu \leq \int_M \rho |H_r| d\mu \tag{19}$$

for every closed hypersurface M, by taking u and f constant functions, with the equality holding in the round spheres.

In the following results, let us denote by $\mathcal{B}_R \subset M$ the intrinsic ball (i.e., a geodesic ball of M) of radius R and center at a point $x_0 \in M$, and by $\partial\mathcal{B}_R$ its boundary, i.e., the geodesic sphere of radius R and center at x_0.

For a weakly locally convex hypersurfaces (i.e., M has non negative second fundamental form), P_r is non negative definite for every $r = 1, \dots, m-1$. Applying $m-1$ times (13) for $f \equiv 1$, and $u = u_\varepsilon$ we obtain, taking $\varepsilon \to 0$:

Corollary 3.2. *If M is a weakly locally convex hypersurface of $\mathbb{R}^{m+1}$, then the volume of any intrinsic ball $\mathcal{B}_R$ of radius R satisfies*

$$\frac{|\mathcal{B}_R|}{R^m} \leq \left[\frac{(R\max_{\partial\mathcal{B}_R}|A|)^m - 1}{(R\max_{\partial\mathcal{B}_R}|A|) - 1}\right]\frac{|\partial\mathcal{B}_R|}{R^{m-1}} + \int_{\mathcal{B}_R} H_m d\mu, \tag{20}$$

where H_m is the Gauss-Kronecker curvature of M and $|A|$ is the matrix norm its second fundamental form. In particular, if there exists $\alpha > 0$ such $\max_{\partial\mathcal{B}_R}|A| \leq \alpha/R$, then

$$\frac{|\mathcal{B}_R|}{R^m} \leq C(m,\alpha)\frac{|\partial\mathcal{B}_R|}{R^{m-1}} + \int_{\mathcal{B}_R} H_m d\mu, \tag{21}$$

where $C(m,\alpha) = \frac{\alpha^m - 1}{\alpha - 1}$. Moreover, if $0 < \alpha < 1$, then

$$\frac{|\mathcal{B}_R|}{R^m} \leq \frac{1}{1-\alpha}\frac{|\partial\mathcal{B}_R|}{R^{m-1}}. \tag{22}$$

Proof. Since $\lambda_i \leq |\lambda_i| \leq |A|$ we have $H_r \leq |A|^r$. Applying (14) to $\Omega = \mathcal{B}_R$ and $k = m-1$, we obtain

$$\begin{aligned} |\mathcal{B}_R| &\leq \sum_{r=0}^{m-1} R^{r+1}\int_{\partial\mathcal{B}_R} H_r dS_\mu + R^m \int_{\mathcal{B}_R} H_m d\mu \\ &\leq \sum_{r=0}^{m-1} R^{r+1}\max_{\partial\mathcal{B}_R}|A|^r|\partial\mathcal{B}_R| + R^m\int_{\mathcal{B}_R} H_m d\mu. \end{aligned}$$

This implies

$$\frac{|\mathcal{B}_R|}{R^m} \leq \left[\sum_{r=0}^{m-1}(R\max_{\partial\mathcal{B}_R}|A|)^r\right]\frac{|\partial\mathcal{B}_R|}{R^{m-1}} + \int_{\mathcal{B}_R} H_m d\mu,$$

which gives (20). Inequality (21) is an immediate consequence of (20) and the hypothesis $\max_{\partial\mathcal{B}_R}|A| \leq \alpha/R$. To conclude (22), just observe that $H_m \leq |A|^m \leq \alpha^m/R^m$, which implies

$$\frac{|\mathcal{B}_R|}{R^m} \leq \frac{\alpha^m - 1}{\alpha - 1}\frac{\partial\mathcal{B}_R}{R^{m-1}} + \frac{\alpha^m}{R^m}|\mathcal{B}_R|,$$

which gives the result. ∎

Remark 5. In fact, Corollary 3.2 holds for any hypersurface without any assumption of convexity, by applying successively (12). In this case, (20) becomes

$$\frac{|\mathcal{B}_R|}{R^m} \leq \mathcal{C}(m)\left[\frac{(R\max_{\partial\mathcal{B}_R}|A|)^m - 1}{(R\max_{\partial\mathcal{B}_R}|A|) - 1}\right]\frac{|\partial\mathcal{B}_R|}{R^{m-1}} + \int_{\mathcal{B}_R}|H_m|d\mu, \tag{23}$$

where $\mathcal{C}(m)$ is a constant, depending only on m. This constant exists and it holds $\mathcal{C}(m) \leq \frac{2^m-1}{m}$. In fact, since

$$P_r = \sum_{k=0}^{r}(-1)^k S_{r-k}A^k,$$

and $|S_k| \leq \binom{m}{k}|A|^k$, we obtain

$$\begin{aligned}
|P_r| &\leq \sum_{k=0}^{r}|S_{r-k}||A|^k \leq \left[\sum_{k=0}^{r}\binom{m}{r-k}\right]|A|^r\\
&= \left[\sum_{k=0}^{r}\binom{m}{k}\right]|A|^r \leq \left[\sum_{k=0}^{m-1}\binom{m}{k}\right]|A|^r\\
&= (2^m-1)|A|^r.
\end{aligned}$$

By (12) and reasoning as in the proof of Corollary 3.1, we obtain

$$|\Omega| \leq \sum_{r=0}^{m-1}\left(\frac{\operatorname{diam}\Omega}{2}\right)^{r+1}\int_{\partial\Omega}\left[\frac{|P_r|}{\binom{m}{r}(m-r)}\right]dS_\mu + \left(\frac{\operatorname{diam}\Omega}{2}\right)^m\int_\Omega |H_m|d\mu, \tag{24}$$

which gives, for $\Omega = \mathcal{B}_R$,

$$\begin{aligned}
\frac{|\mathcal{B}_R|}{R^m} &\leq \left[\sum_{r=0}^{m-1}\frac{2^m-1}{\binom{m}{r}(m-r)}(R\max_{\partial\mathcal{B}_R}|A|)^r\right]\frac{|\partial\mathcal{B}_R|}{R^{m-1}} + \int_{\mathcal{B}_R}|H_m|d\mu\\
&\leq \frac{2^m-1}{m}\left[\sum_{r=0}^{m-1}(R\max_{\partial\mathcal{B}_R}|A|)^r\right]\frac{|\partial\mathcal{B}_R|}{R^{m-1}} + \int_{\mathcal{B}_R}|H_m|d\mu,
\end{aligned}$$

since $\binom{m}{r}(m-r) = m\binom{m-1}{r} \geq m$.

3.2. **Einstein manifolds.** Recall that a Riemannian manifold $\overline{M}$ is Einstein if there is a real number λ, called Einstein constant, such that its Ricci tensor satisfies

$$\overline{\mathrm{Ric}}(X,Y) = \lambda\langle X,Y\rangle, X,Y \in T\overline{M}.$$

Such manifolds are very interesting in the point of view of physics and mathematics. In the first, the metric of Einstein manifolds are solutions of the vacuum Einstein field equations. In the latter because the metric in such manifolds is a critical point of the total scalar curvature with constraints, see for instance [15] for more details.

Example 1. The space forms $\mathcal{Q}_c^{m+1}$ are Einstein manifolds with Einstein constant $\lambda = mc$.

Next, we present spaces whose sectional curvature is not constant.

Example 2 (Product spaces)**.** Let $\overline{M} = \mathcal{Q}_{c_1}^{p_1} \times \mathcal{Q}_{c_2}^{p_2}$ be the product of two space forms. If $X, Y \in T\mathcal{Q}_{c_1}^{p_1}$ and $V, W \in T\mathcal{Q}_{c_2}^{p_2}$, then the sectional curvatures of $\overline{M}$ are

$$\mathrm{Sect}_{\overline{M}}(X,Y) = c_1,\ \mathrm{Sect}_{\overline{M}}(V,W) = c_2,\ \mathrm{Sect}_{\overline{M}}(X,V) = 0.$$

This gives $\mathrm{Ric}(X) = (p_1 - 1)c_1$ and $\mathrm{Ric}(V) = (p_2 - 1)c_2$. Thus, $\overline{M}$ is Einstein if and only if $(p_1 - 1)c_1 = (p_2 - 1)c_2$. The same reasoning holds for an arbitrary product $\mathcal{Q}_{c_1}^{p_1} \times \mathcal{Q}_{c_2}^{p_2} \times \cdots \times \mathcal{Q}_{c_k}^{p_k}$ or for an arbitrary product of Einstein manifolds.

Example 3 (Complex projective space)**.** The complex projective space $\mathbb{C}P^{m+1}$ is a compact Einstein manifold with sectional curvatures lying in the interval $[1/4, 1]$ and Einstein constant $m + 2$.

Example 4 (Schwarzschild metric)**.** Consider $\mathcal{S} = \mathbb{R}^2 \times \mathbb{S}^2$ furnished with a metric

$$ds^2 = dr^2 + \varphi^2(r)ds_1^2 + \psi^2(r)ds_2^2, \tag{25}$$

where we use polar coordinates in the plane $\mathbb{R}^2$, and ds_1^2 and ds_2^2 are the metrics on $\mathbb{S}^1$ and $\mathbb{S}^2$, respectively. It can be shown that the sectional curvatures of $\mathcal{S}$ satisfy

$$\mathrm{Sect}_{\mathcal{S}}(X, \partial_r) = -\frac{\varphi''(r)}{\varphi(r)}, X \in T\mathbb{S}^1,\ \mathrm{Sect}_{\mathcal{S}}(V, \partial_r) = -\frac{\psi''(r)}{\psi(r)}, V \in T\mathbb{S}^2,$$

where $-\varphi''/\varphi = 2\psi''/\psi$. Choose the functions φ and ψ verifying the following differential equations:

$$\begin{cases} \psi' &= 1 + C\psi^{-1}, \\ \psi' &= \alpha\varphi, \end{cases}$$

for α and C determined by the initial data. To obtain smoothness of the metric at the origin, we require that $\varphi(0) = 0$, $\varphi'(0) = 1$ and $\psi(0) = \beta$, for some $\beta > 0$. A simple computation gives $C = -\beta$ and $2\alpha = \beta^{-1}$. With this condition, we have $\psi'' = (\beta/2)\psi^{-2} > 0$. A straightforward computation show that the family of metrics (25) have Ricci curvature zero and so $(\mathcal{S}, ds^2)$ are Einstein manifolds, for more details see, for instance, [50].

We now bring our attention for a family of Einstein manifolds with a warped product metric. Such manifolds are very interesting and the reader can learn more about them in [27], [42], [19] and [39].

Example 5. Recall that given two Riemannian manifolds (M^n, g_M) and (F^m, g_F) and a positive smooth function w on M, the warped product metric on $M \times F$ is defined by

$$g = g_M + w^2 g_F.$$

We denote it as $M \times_w F$. In [19] the authors notice that $M \times_w F$ is an Einstein manifold if and only if

$$\mathrm{Ric}_M - \frac{m}{w}\mathrm{Hess}\, w = \lambda g_M,$$

where F^m is an m-dimensional Einstein manifold. If M has non-empty boundary, we assume that $w = 0$ on ∂M, see [39].

If M is a hypersurface of an Einstein manifold, then the first Newton transformation P_1 has divergence zero. This fact was proved in [33] and we include a proof here for the sake of completeness.

Lemma 3.3. *The divergence of the first Newton transformation P_1 vanishes if ambient manifold $\overline{M}$ is Einstein.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal geodesic frame at $p \in M$. For $r = 1$, equation (11) gives

$$\mathrm{div}\, P_1 = dS_1 - \mathrm{tr}(\nabla_\star A) - A(\mathrm{div}\, I) - \overline{\mathrm{Ric}}(N, \star).$$

Since $dS_1 = \mathrm{tr}(\nabla_\star A)$, and $\overline{M}$ is Einstein, we deduce that $\mathrm{div}\, P_1$ vanishes on M. □

Notice that, tracing the Gauss equation twice, for an adapted orthonormal frame $\{e_1, e_2, \ldots, e_m, \eta\}$, we have

$$\sum_{i=1}^{m} \overline{\mathrm{Ric}}(e_i, e_i) - \overline{\mathrm{Ric}}(\eta, \eta) = \mathrm{Scal} - 2S_2. \tag{26}$$

In particular, if $\overline{M}$ is an Einstein manifold with Einstein constant λ, i.e., $\overline{\mathrm{Ric}}(X, Y) = \lambda\langle X, Y\rangle$, then

$$(m-1)\lambda = \mathrm{Scal} - 2S_2, \tag{27}$$

where Scal denotes the scalar curvature of M.

Fix $p \in \overline{M}$. Let B_t be a ball of $\overline{M}$ with center at p and radius $t > 0$, and $\overline{\gamma}$ be a geodesic ray such that $\overline{\gamma}(0) = p$. Define $F : [0, i(\overline{M})) \to \mathbb{R}_+$ by

$$F(t) = \max_{B_t} \left\{ \mathrm{Sect}_{\overline{M}}(\overline{V}, \overline{\gamma}'), \forall\, \overline{V} \in T\overline{M}, \overline{V} \perp \overline{\gamma}' \right\},$$

where $i(\overline{M})$ is the injectivity radius of $\overline{M}$. Since F is a non decreasing function of t, it is differentiable almost everywhere. Let $G : [0, i(\overline{M})) \to \mathbb{R}_+$ be a weak solution of

$$G''(t) + F(t)G(t) \leq 0. \tag{28}$$

Example 6. For the examples we presented earlier, we have

(i) $F(t) = c := \max\{c_1, c_2\}$ for $\overline{M} = \mathcal{Q}^{p_1}_{c_1} \times \mathcal{Q}^{p_2}_{c_2}$, which gives $G(t) = \mathcal{S}_c(t)$;

(ii) $F(t) = 1$ for $\overline{M} = \mathbb{C}P^{m+1}$, which gives $G(t) = \mathcal{S}_1(t)$;

(iii) $F(t) = 2\psi''(t)/\psi(t) = -\varphi''(r)/\varphi(r)$ for $\overline{M} = \mathbb{R}^2 \times \mathbb{S}^2$ with the Schwarzschild metric, since $\psi'' > 0$, which gives $G(t) = \varphi(r)$.

For hypersurfaces of Einstein manifolds we have the following Poincaré type inequality:

Theorem 3.3. *If M is a hypersurface of an Einstein manifold $\overline{M}^{m+1}$, with Einstein constant λ, and $\Omega \subset M$, $\overline{\Omega} \cap \partial M = \emptyset$, is a connected and open domain with compact closure and such that* $\operatorname{diam}\Omega < 2i(\overline{M})$*, then, for every class $\mathcal{C}^1$ functions $u, f \colon M \to \mathbb{R}$, with u non negative and compactly supported in Ω, we have*

$$\int_\Omega u S_1 G'(\rho) e^{-f} d\mu \leq C_0 \int_\Omega \left[|P_1(\nabla u - u\nabla f)| + |\mathrm{Scal} - (m-1)\lambda| u \right] e^{-f} d\mu, \tag{29}$$

for $C_0 = \frac{1}{(m-1)} G\left(\frac{\operatorname{diam}\Omega}{2}\right)$. In particular, if $P_1 : TM \to TM$ is non negative definite, then

$$\int_\Omega u S_1 G'(\rho) e^{-f} d\mu \leq C_1 \int_\Omega \left[|\nabla u - u \nabla f| S_1 + \left| \frac{\mathrm{Scal}}{m-1} - \lambda \right| u \right] e^{-f} d\mu, \tag{30}$$

for $C_1 = (m-1)C_0$. Here, $\rho : M \to \mathbb{R}_+$ is distance function of $\overline{M}^{m+1}$ restricted to M and G is a solution of (28).

Proof. Indeed, using Lemma 3.3 we have that $\operatorname{div} P_1 = 0$ on an Einstein manifold. By the definition of G, we have that

$$\mathrm{Sect}_{\overline{M}}(\overline{V}, \overline{\gamma}') \leq -\frac{G''(t)}{G(t)}, \forall\, \overline{V} \in T\overline{M}, \overline{V} \perp \overline{\gamma}'.$$

From the second item of Lemma 3.1 we have $\operatorname{tr}(AP_1) = 2S_2$. Using (27) and Theorem 3.1 we obtain (29). Moreover, if P_1 is non-negative definite, then

$$|P_1(U)| \leq (\operatorname{tr} P_1)|U| = (m-1)S_1|U|,$$

which gives (30), as desired.

As an immediate consequence, we have

□

4. Rigidity results

In this section we state some rigidity results which are consequences of our Poincaré type inequality. Let

$$h_c(t) = \begin{cases} t, & \text{if } c = 0; \\ \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}t), & \text{if } c < 0; \\ 1, & \text{if } c > 0. \end{cases}$$

Notice that $\mathcal{S}_c(t) = h_c(t)$ for $c \leq 0$ and $\mathcal{S}_c(t) \leq \sqrt{c} h_c(t)$ for $c > 0$.

The first result reads as follows:

Theorem 4.1. *Let M be a complete $(r+1)$-minimal hypersurface, $1 \leq r \leq m-1$, of a space form $\mathcal{Q}_c^{m+1}$ of constant sectional curvature $c \in \mathbb{R}$ such that r is odd, or r is even and $H_r \geq 0$. If*

$$\limsup_{R\to\infty} h_c(R) \int_{\partial \mathcal{B}_R} H_r dS_\mu = 0, \tag{31}$$

then M is foliated by $(m-r+1)$-dimensional totally geodesic submanifolds of $\mathcal{Q}_c^{m+1}$. Moreover,

(i) *if $\mathcal{Q}_c^{m+1} = \mathbb{R}^{m+1}$ and M has non negative Ricci curvature, then $M = N^{r-1} \times \mathbb{R}^{m-r+1}$, where N^{r-1} is a $(r-1)$-dimensional Riemannian manifold;*

(ii) *if $\mathcal{Q}_c^{m+1} = \mathbb{S}_+^{m+1}(c)$ and M has Ricci curvature bounded from below by c, then M is totally geodesic.*

Proof. First, notice that, by the Proposition 2.8, p.192 of [17], since $S_{r+1} \equiv 0$, we have that P_r is semi-definite. If r is odd, we can choose an orientation such that P_r is positive semi-definite. This implies $\frac{1}{m-r} \operatorname{tr} P_r = S_r \geq 0$. If r is even it does not happen, but the assumption that $S_r \geq 0$ assures that P_r is positive semi-definite. This implies that $|P_r(U)| \leq (\operatorname{tr} P_r)|U| = (m-r)S_r|U|$,

$U \in TM$. Thus, by using inequality (13), we have that

$$\int_{\mathcal{B}_R} uS_r\mathcal{S}_c'(\rho)d\mu \le \mathcal{S}_c\left(\frac{\operatorname{diam}\mathcal{B}_R}{2}\right)\int_{\mathcal{B}_R}|\nabla u|S_r d\mu \le \mathcal{S}_c(R)\int_{\mathcal{B}_R}|\nabla u|S_r d\mu.$$

Plugging

$$u_\varepsilon(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x,\partial\mathcal{B}_R) \ge \varepsilon; \\ \frac{\operatorname{dist}(x,\partial\mathcal{B}_R)}{\varepsilon}, & \text{if } \operatorname{dist}(x,\partial\mathcal{B}_R) \le \varepsilon, \end{cases}$$

in the previous inequality and using co-area formula we obtain, after taking $\varepsilon \to 0$, that

$$\int_{\mathcal{B}_R} S_r\mathcal{S}_c'(\rho)d\mu \le \mathcal{S}_c(R)\int_{\partial\mathcal{B}_R} S_r dS_\mu \le \max\{1,\sqrt{c}\}h_c(R)\int_{\partial\mathcal{B}_R} S_r dS_\mu.$$

Making $R \to \infty$, we obtain

$$\int_M S_r d\mu \le \max\{1,\sqrt{c}\}\limsup_{R\to\infty} h_c(R)\int_{\partial\mathcal{B}_R} S_r dS_\mu = 0,$$

which implies that $S_r \equiv 0$. Since $S_{r+1} \equiv 0 \equiv S_r$, by Lemma 2.1, p.252, of [40], we obtain that A has rank at most $r-1$, i.e., M has index of relative nullity at least $m-r+1$. By using Proposition 1.18, p.24 of [29], we conclude that M is foliated by $(m-r+1)$-dimensional totally geodesic submanifolds of $\mathcal{Q}_c^{m+1}$. If $\mathcal{Q}_c^{m+1} = \mathbb{R}^{m+1}$ and M has non negative Ricci curvature, then by using Hartman splitting theorem (see [29], Theorem 7.15, p.196), $M = N^{r-1}\times\mathbb{R}^{m-r+1}$. If $\mathcal{Q}_c^{m+1} = \mathbb{S}_+^{m+1}(c)$ and the Ricci curvature of M is bounded from below by c, then by Corollary 7.12, of [29], M is totally geodesic. □

Remark 6. We do not need assume that $H_r \ge 0$ for r even if we replace the decay condition of H_r in the hypothesis of Theorem 4.1 by

$$\limsup_{R\to\infty} h_c(R)\int_{\partial\mathcal{B}_R}|A|^r dS_\mu = 0.$$

In fact, in this case we can use (12) and the discussion of Remark 5, p.13, to take estimates of $|P_r|$.

If the ambient manifold $\overline{M}$ is Einstein, define

$$\mathcal{G}(t) = \begin{cases} G(t), & \text{if } i(\overline{M}) = \infty; \\ 1, & \text{if } i(\overline{M}) < \infty, \end{cases}$$

where G is the solution of (28). For Einstein manifolds we have:

Theorem 4.2. *If M is a complete hypersurface with constant scalar curvature $(m-1)\lambda$ of an Einstein manifold $\overline{M}^{m+1}$, with Einstein constant λ, such that*

$$\limsup_{R\to\infty} \mathcal{G}(R)\int_{\partial\mathcal{B}_R} H dS_\mu = 0,$$

then M is totally geodesic.

Proof. The eigenvalues of P_1 are $S_1-\lambda_i$, where λ_i are the principal curvatures of M. Since $S_2\equiv 0$ and $S_1\geq 0$, we have

$$S_1-\lambda_i \leq S_1+|\lambda_i| \leq S_1+\sqrt{\lambda_1^2+\cdots+\lambda_m^2} = S_1+|A| = 2S_1,$$

i.e., $|P_1|\leq 2S_1$. Following the same reasoning of the proof of Theorem 4.1, we conclude that $S_1\equiv 0$. This gives $|A|=\sqrt{S_1^2-2S_2}=0$, i.e., M is totally geodesic. □

Since space forms are particular cases of Einstein manifolds for $\lambda = mc$, we have

Corollary 4.1. *If M is a complete hypersurface with constant scalar curvature $m(m-1)c$ of a space form $\mathcal{Q}_c^{m+1}$ of constant sectional curvature $c\in\mathbb{R}$, such that*

$$\limsup_{R\to\infty} h_c(R)\int_{\partial\mathcal{B}_R} H dS_\mu = 0,$$

then M is totally geodesic.

As a consequence of the proof of Theorem 4.1, we obtain,

Corollary 4.2. *There is no complete $(r+1)$-minimal hypersurface, $1\leq r\leq m-1$, a space form $\mathcal{Q}_c^{m+1}$ of constant sectional curvature $c\leq 0$, such that*

(i) *r is odd, or r is even and $H_r\geq 0$;*
(ii) *M is contained in a geodesic ball of $\mathcal{Q}_c^{m+1}$, and*
(iii) $\limsup_{R\to\infty}\int_{\partial\mathcal{B}_R} H_r dS_\mu = 0.$

5. Rigidity of self-similar solutions of curvature flows

Let $\psi: M^m\to\mathbb{R}^{m+1}$ be hypersurface. The evolution of $\psi(M)$ by the curvature is smooth a one-parameter family $\Psi: M\times I\to\mathbb{R}^{m+1}$ of immersions $\Psi_t := \Psi(\cdot,t): M\to\mathbb{R}^{m+1}$ solving the initial value problem

$$\left\{\begin{array}{lcl} \dfrac{\partial\Psi}{\partial t}(x,t) &=& (S_{r+1}(x,t))^\alpha \eta(x,t),\\ \Psi(x,0) &=& \psi(x), \end{array}\right. \tag{32}$$

for $\alpha \in \mathbb{R} - \{0\}$. The initial value problem (32) are also called a curvature flow. These flows have been studied by many authors in the last four decades, see, for example, [21], [53], [22], [54], [55], [8], [9], their citations, and references therein. We also can cite the recent book [10] for a extensive introduction of these flows.

A homothetic solution of the flow in (32) is a hypersurface satisfying the equation

$$S_{r+1}^{\alpha} = \delta\langle\psi, \eta\rangle, \tag{33}$$

for some non-zero real number δ. A hypersurface satisfying (33) evolves by the flow without changing their shapes, but only by dilation or contraction (that is, they remain the same after rescaling). If $\delta > 0$, then the hypersurface evolves by dilation and it is called a self-expander. If $\delta < 0$, then the hypersurface evolves by contraction and its is called a self-shrinker.

Remark 7. Homothetic solutions are examples of self-similar solutions, which are those solutions which evolves by flow without changing their shapes. Other examples are the translating solitons, which evolves translating the initial hypersurface in a fixed direction and those which evolves by a rotation of $\mathbb{R}^{m+1}$. For more details, see [10].

Homothetic solutions of curvature flows have received considerable attention in recent years, see, for example [47], [25], [32], [16],[35], [38], [23], [48], [3], and [4].

For homothetic solutions of the curvature flow (32) we can state:

Theorem 5.1. *Let M be a complete homothetic solution of the curvature flow* (32) *in $\mathbb{R}^{m+1}$, $1 \le r \le m-1$, such that*

(i) *$\alpha = p/q$, where p and q are odd integers;*
(ii) *or $\alpha \in \mathbb{R} - \{0\}$ and $S_{r+1} \ge 0$.*

If $\delta S_r \ge 0$ and

$$\limsup_{R\to\infty} R \int_{\partial\mathcal{B}_R} |A|^r dS_\mu = 0, \tag{34}$$

then

(i) *M is a hyperplane if $\alpha > 0$;*
(ii) *there is no such hypersurface if $\alpha < 0$.*

Proof. Using (9) and Remark 1, we have that

$$(m-r)\int_\Omega uS_r d\mu = \int_\Omega \langle -\overline{X}, P_r(\nabla u)\rangle d\mu + (r+1)\int_\Omega u\langle -\overline{X}, \eta\rangle S_{r+1} d\mu,$$

for $\overline{X} = \rho\overline{\nabla}\rho$. Since $S_{r+1}^{\alpha} = \delta\langle\psi, \eta\rangle$ and $\psi = \overline{X}$ in $\mathbb{R}^{m+1}$, we have

$$\begin{aligned}\int_\Omega u[(m-r)\delta S_r + (r+1)S_{r+1}^{\alpha+1}]d\mu &= \delta\int_\Omega \langle -\rho\nabla\rho, P_r(\nabla u)\rangle d\mu \\ &\leq \frac{|\delta|\operatorname{diam}\Omega}{2}\int_\Omega |P_r||\nabla u|d\mu \\ &\leq \frac{(2^m-1)|\delta|\operatorname{diam}\Omega}{2}\int_\Omega |A|^r|\nabla u|d\mu,\end{aligned}$$

since $|P_r| \leq (2^m-1)|A|^r$ by Remark 5, p.13. Taking $\Omega = \mathcal{B}_R$ a intrinsic ball of radius R, we have $\operatorname{diam}\mathcal{B}_R \leq 2R$, since the extrinsic distance is less than or equal to the intrinsic distance. Using

$$u_\varepsilon(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, \partial\mathcal{B}_R) \geq \varepsilon; \\ \frac{\operatorname{dist}(x,\partial\mathcal{B}_R)}{\varepsilon}, & \text{if } \operatorname{dist}(x, \partial\mathcal{B}_R) \leq \varepsilon, \end{cases}$$

and co-area formula we obtain, after taking $\varepsilon \to 0$, that

$$\int_{\mathcal{B}_R} u[(m-r)\delta S_r + (r+1)S_{r+1}^{\alpha+1}]d\mu \leq c(m,r)|\delta|R\int_{\partial\mathcal{B}_R}|A|^r d\mu.$$

Notice that, if $\alpha = \frac{2a+1}{2b+1}$, $a, b \in \mathbb{Z}$, then

$$S_{r+1}^{\alpha+1} = \left(S_{r+1}^{\frac{a+b}{2b+1}}\right)^2 \geq 0$$

no matter the signal of S_{r+1}. Taking $R \to \infty$ and using the hypothesis, we obtain

$$S_r \equiv S_{r+1} \equiv 0 \equiv \langle\psi, \eta\rangle,$$

which implies that M is a hyperplane, since it is smooth, for $\alpha > 0$, (see also [28]) and leads to a contradiction for $\alpha < 0$, since S_{r+1}^{α} is defined, in this case, only for $S_{r+1} > 0$. □

Remark 8. The proof of the Theorem 5.1 holds in the general setting of a Riemannian manifold $\overline{M}^{m+1}$ with bounded sectional curvatures by $G''(t)/G(t)$. So, we consider hypersurfaces satisfying the equation

$$S_{r+1}^{\alpha} = \delta\langle G(\rho)\overline{\nabla}\rho, \eta\rangle.$$

These surfaces have been object of research in recent years as self-similar solutions of curvature flows in ambient spaces other than $\mathbb{R}^{m+1}$, see, for example, [6] and [26]. If

$$\begin{cases} \limsup\limits_{R\to\infty} G(R)\displaystyle\int_{\partial\mathcal{B}_R}|A|^r dS_\mu = 0, & \text{if } G \text{ is unbounded;} \\ \limsup\limits_{R\to\infty}\displaystyle\int_{\partial\mathcal{B}_R}|A|^r dS_\mu = 0, & \text{if } G \text{ is bounded,} \end{cases}$$

then M satisfies

$$S_r \equiv S_{r+1} \equiv 0 \equiv \langle G(\rho)\overline{\nabla}\rho, \eta\rangle, \tag{35}$$

(assuming $c \neq 0$). The classification of which hypersurfaces satisfy (35) will depend on the ambient space we are considering. In the space form $\mathcal{Q}_c^{m+1}$, for example, by using the results in [28] and (35), we can conclude that M is totally geodesic if $\alpha > 0$ and that the hypersurface does not exist if $\alpha < 0$.

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