# ON THE SPECTRUM AND INDEX OF EXPANDING AND TRANSLATING SOLITONS OF THE MEAN CURVATURE FLOW IN $\mathbb{R}^3$

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ABSTRACT. In this paper we prove that two-dimensional translating solitons in $\mathbb{R}^3$ with finite $L$-index are homeomorphic to a plane or a cylinder and that a two-dimensional self-expander with finite $L$-index and sub exponential weighted volume growth has finite topology. We also prove that translating solitons and self-expanders have finite topology, provided the bottom of the spectrum of the $L$-stability operator is bounded from below and their weighted volume have subexponential growth.

## 1. INTRODUCTION

1.1. **Translating solitons.** We say a hypersurface $\Sigma$ of $\mathbb{R}^{n+1}$ is translating soliton (or, shortly, a translator) of the mean curvature flow if

$$H = \langle \nu, V \rangle,$$

where $H$ is its mean curvature, $V$ is a parallel unitary vector field of $\mathbb{R}^{n+1}$, and $\nu$ is the outward normal vector field of $\Sigma$. Throughout this work, we will use the convention that $H = \operatorname{trace} A$, where $A(Y) = \overline{\nabla}_Y \nu$ and $\overline{\nabla}$ is the connection of $\mathbb{R}^{n+1}$. Under this convention, the mean curvature of the round sphere $\mathbb{S}^n(R)$ of radius $R$ is $n/R$ and the mean curvature of the right circular cylinder $\mathbb{S}^k(R) \times \mathbb{R}^{n-k}$ of radius $R$ is $k/R$.

Translating solitons are known as type II singularities of the mean curvature flow. Huisken and Sinestrari showed in [22] that if the initial hypersurface is mean convex and the singularity is of type II, then any limit hypersurface is a convex translating soliton.

Hamilton [18] proved that any strictly convex eternal solution to the mean curvature flow where the mean curvature assumes its maximum value at a point in space-time must be a translating soliton. These hypersurfaces are also known as self-similar solutions of the mean curvature flow moving by translation, i.e., if $\Sigma$ is a translating soliton, then $\Sigma_t = \Sigma + tV$ is a solution of the mean curvature flow for all times $t \in \mathbb{R}$.

On the other hand, translating solitons can also be seen as critical points of the weighted area functional

$$\int_\Sigma e^{\langle x, V\rangle} d\Sigma,$$

under every compactly supported normal normal variation of $\Sigma$. Taking the second derivative, we obtain

$$\frac{d^2}{dt^2}\left(\int_\Sigma e^{\langle x,V\rangle}d\Sigma\right)\Bigg|_{t=0} = -\int_\Sigma \xi\left[\Delta\xi + \langle V,\nabla\xi\rangle + |A|^2\right]e^{\langle V,x\rangle}d\Sigma$$

$$=: -\int_\Sigma \xi L\xi e^{\langle x,V\rangle}d\Sigma$$

for variations of the form $\xi\nu$, where $\xi$ is a smooth function with compact support in $\Sigma$, where $L\xi = \Delta\xi + \langle V,\nabla\xi\rangle + |A|^2$. Since $L$ is an elliptic operator, we can consider its spectrum. Given a bounded domain $\Omega \subset \Sigma$, define the $L$-index of $\Omega$ by

$$\mathrm{Ind}^L(\Omega) = \#\{\text{negative eigenvalues of } L \text{ on } C_0^\infty(\Omega)\}$$

and the $L$-index of $\Sigma$ as

$$\mathrm{Ind}_L(\Sigma) := \sup_{\Omega\subset\Sigma}\mathrm{Ind}^L(\Omega).$$

The $L$-index is the maximal dimension of the subspace in $C_0^\infty(\Sigma)$ such that the quadratic form

$$Q_L(\xi,\xi) = -\int_\Sigma \xi L\xi e^{\langle x,V\rangle}d\Sigma$$

is negative. Intuitively, this is the maximal dimension of the subspaces in $C_0^\infty(\Sigma)$ such that the compact variations decrease the weighted area.

In this subject, Impera and Rimoldi, see Theorem D of [27], proved that a $n$-dimensional translating soliton in $\mathbb{R}^{n+1}$ with finite $L$-index has finitely many ends. We prove

**Theorem 1.1.** *If $\Sigma \subset \mathbb{R}^3$ is a two-dimensional complete translating soliton with finite $L$-index, then $\Sigma$ is homeomorphic to $\mathbb{C}$ or $\mathbb{C}\backslash\{0\}$. In particular, $\Sigma$ has at most two ends.*

*Moreover, for every $\varepsilon \in (0,1)$,*

$$\lim_{Q\to\infty} \frac{1}{e^Q} \int_{B(\varepsilon Q)} |A|^2 e^{\langle x,V\rangle} d\Sigma < \infty \ and \ \lim_{Q\to\infty} \frac{1}{Q^2 e^Q} \int_{B(\varepsilon Q)} e^{\langle x,V\rangle} d\Sigma < \infty.$$

*Remark* 1.1. We say that translating solitons with $L$-index equal to zero are $L$-stable. Equivalently, a translating soliton is said $L$-stable if and only if

$$\frac{d^2}{dt^2} \left( \int_\Sigma e^{\langle x,V\rangle} d\Sigma \right) \Big|_{t=0} \geq 0$$

for all the compactly supported normal variations. In Theorem C of [27], Impera and Rimoldi proved that a complete, $L$-stable, translating hypersurface of $\mathbb{R}^{n+1}$ has at most one end. For dimension two, the topological classification of complete $L$-stable translating solitons was completed by Kunikawa and Saito, see [24], proving that any complete, two-dimensional, $L$-stable translating soliton has genus zero. Theorem 1.1 generalizes these two results for the two-dimensional situation.

*Remark* 1.2. There are many interesting classification results for two-dimensional translating solitons in $\mathbb{R}^3$. For example, Spruck and Xiao proved in [31] that complete, immersed, translating solitons with nonnegative mean curvature are convex. Shahriyari in [28] proved that complete translating graphs in $\mathbb{R}^3$ are $L$-stable and there is no complete translating graph in $\mathbb{R}^3$ over a bounded connected domain with smooth boundary. In [21], Hoffman, Ilmanen, Martín, and White classified all complete translating graphs in $\mathbb{R}^3$ : they are the grim reaper surface, the tilted grim reaper surfaces, the bowl soliton and a family of graphs $u^b : \mathbb{R} \times (-b,b) \to \mathbb{R}$, for $b > \pi/2$. We can also mention that Tasayco and Zhou, see [33], proved that the grim reaper hypersurface is only nonplanar translating soliton of $\mathbb{R}^3$ and $\mathbb{R}^4$ whose weighted integral of $|A|^2$ over a geodesic ball has at most quadratic growth for large radius $R$.

Inspired by Colding and Minicozzi (see the section 9 of [10]) we can define the bottom of the spectrum of the elliptic operator $L$ by

$$\mu_1 = \inf_\xi \frac{-\int_\Sigma \xi L\xi e^{\langle x,V\rangle} d\Sigma}{\int_\Sigma \xi^2 e^{\langle x,V\rangle} d\Sigma} = \inf_\xi \frac{\int_\Sigma \left[|\nabla\xi|^2 - |A|^2\xi^2\right] e^{\langle x,V\rangle} d\Sigma}{\int_\Sigma \xi^2 e^{\langle x,V\rangle} d\Sigma},$$

where the infimum is taken over all smooth functions $\xi$ with compact support in $\Sigma$. We prove

**Theorem 1.2.** *Let $\Sigma \subset \mathbb{R}^3$ be a translating soliton. If $\mu_1 \geq -\delta$, for some $\delta > 0$ and*

$$\lim_{Q \to \infty} \frac{1}{e^Q} \int_{B(Q)} e^{\langle x, V \rangle} d\Sigma < \infty,$$

*then $\Sigma$ has finite topology. Moreover, for every $\varepsilon \in (0,1)$, it holds*

$$\lim_{Q \to \infty} \frac{1}{e^Q} \int_{B(\varepsilon Q)} |A|^2 e^{\langle x, V \rangle} d\mu < \infty.$$

*Here, $B(Q)$ is the geodesic ball of $\Sigma$ with center in a reference point $x_0 \in \Sigma$ and radius $Q > 0$.*

A immediate consequence of the result above is the following

**Corollary 1.1.** *If a two-dimensional complete translating soliton $\Sigma \subset \mathbb{R}^3$ has infinite topology, then $\mu_1 = -\infty$ or*

$$\lim_{Q \to \infty} \frac{1}{e^Q} \int_{B(Q)} e^{\langle x, V \rangle} d\Sigma = \infty.$$

*Remark* 1.3. There are examples of two-dimensional translating solitons with infinite genus constructed by Nguyen, see [25] and [26].

1.2. **Self-expanders.** A self-expander of the mean curvature flow is a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ which satisfies

$$H(x) = -\frac{1}{2}\langle x, \nu \rangle,$$

where $H$ is its mean curvature and $\nu$ is the outward unitary normal vector field of $\Sigma$. Self-expanders play an important role in the mean curvature flow. They describe the asymptotic long time behavior for the flow and its local structure after the singularities in the very short time. In fact, Ecker and Huisken proved in [13] that the mean curvature flow exists for all times $t > 0$ if the initial surface is an entire graph which is "straight" at infinity in the sense that $|\langle x, \nu \rangle| \leq C(1 + \|x\|)^{1-\delta}$ for some $\delta > 0$ and $C < \infty$. Moreover, the flow converges to a self-expander. Later, Stavrou [32] proved the same result under the weaker hypothesis that the graphical function have an unique tangent cone at infinity.

Self-expanders are also known as self-similar solutions which expands homothetically under the mean curvature flow in the sense that, if $\Sigma$ is a self-expander, then $\Sigma_t = \sqrt{t}\Sigma$ is a solution of the flow for every $t > 0$.

Examples of asymptotically conical self-expanders were obtained by Ecker and Huisken in [13], Angenent, Ilmanen and Chopp, see [3], and by Helmensdorfer in [20]. Recent

results about self-expanders were obtained, for example, by Cheng and Zhou, see [9], by Bersntein and Wang, see [5] and [6], Ding, see [12], Deruelle and Schulze, see [11], and Ancari and Cheng, see [2].

Self-expanders are also critical points of the weighted area functional

$$\int_\Sigma e^{\frac{1}{4}\|x\|^2} d\Sigma.$$

Taking the second derivative of this functional, we have

$$\frac{d^2}{dt^2}\left(\int_\Sigma e^{\frac{1}{4}\|x\|^2} d\Sigma\right)\bigg|_{t=0} = -\int_\Sigma \xi\left[\Delta\xi + \frac{1}{2}\langle x,\nu\rangle + \left(|A|^2 - \frac{1}{2}\right)\xi\right]e^{\frac{1}{4}\|x\|^2}d\Sigma$$

$$= -\int_\Sigma \xi L\xi e^{\frac{1}{4}\|x\|^2}d\Sigma,$$

where $L\xi = \Delta\xi + \frac{1}{2}\langle x,\nu\rangle + \left(|A|^2 - \frac{1}{2}\right)\xi$. Since $L$ is an elliptic operator, we can consider its spectrum.

Analogously it was done for translating solitons, we can also define the $L$-index for self-expander. For self-expanders with finite $L$-index, we have

**Theorem 1.3.** *Let $\Sigma \subset \mathbb{R}^3$ be a two-dimensional complete self-expander of the mean curvature flow. If $\Sigma$ has finite L-index and there exist $\beta > 1$ such that*

$$\lim_{Q\to\infty} \frac{1}{e^{\frac{\beta}{4}Q^2}}\int_{B(Q)} e^{\frac{1}{4}\|x\|^2}d\Sigma < \infty,$$

*then $\Sigma$ has finite topology. Moreover, for every $\varepsilon \in (0,1)$ it holds*

$$\lim_{Q\to\infty} \frac{1}{e^{\frac{\beta}{4}Q^2}}\int_{B(\varepsilon Q)} |A|^2 e^{\frac{1}{4}\|x\|^2}d\mu < \infty.$$

*Here, $B(Q)$ is the geodesic ball of $\Sigma$ with center in a reference point $x_0 \in \Sigma$ and radius $Q > 0$.*

As an immediate consequence of the result above we have

**Corollary 1.2.** *If a complete two-dimensional self-expander $\Sigma \subset \mathbb{R}^3$ has infinite topology, then it has infinite index or*

$$\lim_{Q\to\infty} \frac{1}{e^{\frac{\beta}{4}Q^2}}\int_{B(Q)} e^{\frac{1}{4}\|x\|^2}d\Sigma = \infty,$$

*for every $\beta > 1$.*

We can define the bottom of the $L$ operator for self-expanders by

$$\mu_1 = \inf_{\xi} \frac{-\int_\Sigma \xi L \xi e^{\frac{1}{4}\|x\|^2} d\Sigma}{\int_\Sigma \xi^2 e^{\frac{1}{4}\|x\|^2} d\Sigma} = \inf_{\xi} \frac{\int_\Sigma \left[|\nabla \xi|^2 - \left(|A|^2 - \frac{1}{2}\right)\xi^2\right] e^{\frac{1}{4}\|x\|^2} d\Sigma}{\int_\Sigma \xi^2 e^{\frac{1}{4}\|x\|^2} d\Sigma},$$

where the infimum is taken over all smooth functions $\xi$ with compact support in $\Sigma$. We have, for complete self-expanders, the following result:

**Theorem 1.4.** *Let $\Sigma \subset \mathbb{R}^3$ be a two-dimensional complete self-expander of the mean curvature flow.*

(i) *If $\mu_1 \geq 1/2$, then $\Sigma$ is homeomorphic to $\mathbb{C}$ or $\mathbb{C}\backslash\{0\}$ and, moreover, if $\mu_1 > 1/2$, then, for every $\varepsilon \in (0,1)$ and $\beta > 1$, it holds*

$$\lim_{Q\to\infty} \frac{1}{e^{\frac{\beta}{4}Q^2}} \int_{B(\varepsilon Q)} e^{\frac{1}{4}\|x\|^2} d\mu < \infty;$$

(ii) *If $\mu_1 \in (-\infty, 1/2)$, and there exist $\beta > 1$ such that*

$$\lim_{Q\to\infty} \frac{1}{e^{\frac{\beta}{4}Q^2}} \int_{B(Q)} e^{\frac{1}{4}\|x\|^2} d\mu < \infty;$$

*then $\Sigma$ has finite topology;*

(iii) *In both situations of items (i) and (ii) it holds*

$$\lim_{Q\to\infty} \frac{1}{e^{\frac{\beta}{4}Q^2}} \int_{B(\varepsilon Q)} |A|^2 e^{\frac{1}{4}\|x\|^2} d\mu < \infty.$$

*Here, $B(Q)$ is the geodesic ball of $\Sigma$ with center in a reference point $x_0 \in \Sigma$ and radius $Q > 0$.*

*Remark* 1.4. If $\Sigma$ is homeomorphic to $\mathbb{C}\backslash\{0\}$ then both limits in items (i) and (iii) are equal to zero. We can also prove that, if $\mu_1 = 1/2$, then

$$\lim_{Q\to\infty} \frac{1}{Q^2 e^{\frac{\beta}{4}Q^2}} \int_{B(\varepsilon Q)} e^{\frac{1}{4}\|x\|^2} d\mu < \infty;$$

for every $\varepsilon \in (0,1)$ and $\beta > 1$.

As an immediate consequence of the result above we have

**Corollary 1.3.** *If a complete two-dimensional self-expander $\Sigma \subset \mathbb{R}^3$ has infinite topology, then $\mu_1 = -\infty$ or*

$$\lim_{Q\to\infty} \frac{1}{e^{\frac{\beta}{4}Q^2}} \int_{B(Q)} e^{\frac{1}{4}\|x\|^2} d\Sigma = \infty,$$

*for every $\beta > 1$.*

*Remark* 1.5. Recently, Ancari and Cheng, see [2], proved some upper bounds for the bottom $\mu_1$ of the $L$-stability operator of self-expanders. They also proved that the cylinders $\Gamma \times \mathbb{R}^{n-1}$, where $\Gamma$ is a self-expanding curve (which were classified by Ishimura in [23] and Halldorson in [17]), are self-expanders with $\mu_1 = \frac{n+1}{2}$. We observe that, for $n = 2$, these surfaces are a class of self-expanders which satisfies the hypothesis of item (i) of Theorem 1.4.

*Remark* 1.6. The results we present here will be proven in the more general setting for $f$-minimal surfaces $\Sigma$ in three-dimensional weighted Riemannian manifolds $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$ satisfying

$$\overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) \geq k,$$

for some $k \in \mathbb{R}$. Here, $\overline{\mathrm{Scal}}$ is the scalar curvature of $M^3$ and $\overline{\mathrm{Hess}} f(\nu, \nu)$ is the hessian tensor of $f$ in $M^3$ applied to the unitary normal vector field $\nu$ of $\Sigma$ in $M^3$.

## 2. Preliminaries

Let $\Sigma$ be a Riemannian surface with Gaussian curvature $K$. Fixed a point $x_0 \in \Sigma$, let $r(x)$ be the Riemannian (intrinsic) distance in $\Sigma$ from $x_0$ to $x \in \Sigma$. Let $B(s)$ be the open geodesic ball in $\Sigma$ of center $x_0$ and radius $s$. Denote by $L(s)$ the length of the boundary of $B(s)$. This length function is a priori only defined for $s \in \mathbb{R}_+ \backslash E$, where the set of exceptional values $E$ is closed, and has Lebesgue measure zero. For $t < s$, denote for $C(t,s) = B(s) \backslash \overline{B}(t)$, where $\overline{B}(t)$ is the closure of $B(t)$.

**Definition 2.1.** Let $\chi(B(t))$ be the Euler characteristic of the open ball $B(t)$. We set

$$\widehat{\chi}(s) = \sup\{\chi(B(t)) | t \in [s, \infty)\}.$$

The following result is basic and well known. For a more details, we refer to the work [4] of Bérard and Castillon.

**Lemma 2.1.** *The function $\widehat{\chi}(s)$ is continuous on the left, nonincreasing from $[0, \infty)$ to $\mathbb{Z}$, and with at most countably many discontinuities. Moreover, if $\{t_j\}_{j=1}^{\overline{N}} = \{0 < t_1 < t_2 < \cdots < t_n < \cdots\}$ is the set of discontinuities, where $\overline{N} \in N \cup \{\infty\}$, $\overline{N} = 0$, when the sequence is empty, and $\overline{N} = \infty$, when the sequence is infinite, then*

   (i) *at each discontinuity $t_n$, $n \geq 1$, the function $\widehat{\chi}$ has a jump*

$$\omega_n = \widehat{\chi}(t_n^-) - \widehat{\chi}(t_n^+), \ \omega_n \in \mathbb{N}, \omega_n \geq 1;$$

(ii) *it holds $\widehat{\chi}(s) = 1$, for $s \in [0, t_1]$, and*

$$\widehat{\chi}(s) = 1 - (\omega_1 + \cdots + \omega_n) \leq -(n-1),$$

*for $s \in (t_n, t_{n+1}]$.*

*Remark* 2.1. Notice that this sequence depends on the reference point $x_0$.

In the proof of our results we will use following inequalities, which were proved first by Fiala [14] for the set $\mathbb{R}_+ \backslash E$ and were extended to $\mathbb{R}_+$ by the work Hartman [19], Shiohama and Tanaka [29] and [30].

**Lemma 2.2** ((Fiala's inequality))**.** *On the set $\mathbb{R}_+ \backslash E$, the function $L$ is of class $C^1$ and its extension to $\mathbb{R}_+$ satisfies*

(i) $L'(t) \leq 2\pi \chi(B(t)) - \displaystyle\int_{B(t)} K d\Sigma;$

(ii) $L(b) - L(a) \leq L(b^-) - L(a) \leq \displaystyle\int_a^b L'(t) dt,$

*whenever $0 \leq a < b$, where $\chi(B(t))$ is the Euler characteristic of $B(t)$.*

The proof of the following Lemma can also be found in [4].

**Lemma 2.3.** *Let $\Sigma$ be a complete Riemannian surface. Let $\{t_n\}_{n=1}^{\overline{N}}$ be the set of discontinuities of the function $\overline{\chi}$, with jumps $\omega_n$, relative to some reference point $x_0 \in \Sigma$. Let $\chi(\Sigma)$ be the Euler characteristic of $\Sigma$, with $\chi(\Sigma) = -\infty$ if $\Sigma$ does not have finite topology. Then*

$$1 - \sum_{n=1}^{\overline{N}} \omega_n \leq \chi(\Sigma).$$

We will also need the following consequence of the coarea formula.

**Lemma 2.4.** *For every $g : \Sigma \to \mathbb{R}$ locally integrable,*

$$\int_{B(t)} g d\Sigma = \int_{-\infty}^t \left[ \int_{\partial B(u)} g ds \right] du,$$

*where $ds$ is the length element of $\partial B(u)$. In particular,*

$$\frac{d}{dt} \left[ \int_{B(t)} g d\Sigma \right] = \int_{\partial B(u)} g ds.$$

**Definition 2.2.** Let $0 \leq R < S$. We say that a function $\xi : [R, S] \to \mathbb{R}$ is admissible in the interval $[R, S]$, if

(i) $\xi$ is of class $C^1$ and piecewise $C^2$ in $[R, S]$;

(ii) $\xi \geq 0$, $\xi' \leq 0$ and $\xi'' \geq 0$.

The next lemma uses the ideas of the proof of Theorem 3.4, p. 223 of Gulliver and Lawson, see [16], see also Lemma 2.2, p.1245 of [4] and Lemma 1.8 p.276, of the work [7] of Castillon.

**Lemma 2.5.** *Fix $x_0 \in \Sigma$ and let $r(x)$ be the distance to $x_0$ in $\Sigma$. Given $f : \Sigma \to \mathbb{R}$ be a locally integrable function, let $F : [0, \infty) \to \mathbb{R}$ be a function such that $F(Q) \leq \inf_{B(Q)} f(x)$. Then, for every $0 \leq R < Q$, and for any admissible function $\xi$ on $[R, Q]$,*

(2.1)
$$\int_{C(R,Q)} K\xi(r)^2 e^{-f} d\Sigma \leq e^{-F(Q)} \left[ \xi^2 G + 2\xi\xi' L - 2\pi\widehat{\chi}(R)\xi^2 \right]\Big|_R^Q - \int_{C(R,Q)} (\xi^2)''(r) e^{-f} d\Sigma.$$

*Proof.* Let

$$G(t) = \int_{B(t)} K d\Sigma \quad \text{and} \quad H(t) = \int_R^t G(u) du.$$

Since $f \geq F(Q)$ in $C(R, Q)$, we have $e^{-f} \leq e^{-F(Q)}$. This gives

$$\int_{C(R,Q)} K\xi(r)^2 e^{-f} d\Sigma \leq e^{-F(Q)} \int_{C(R,Q)} K\xi(r)^2 d\Sigma.$$

On the other hand, by using the coarea formula (see Lemma 2.4), we have

$$\int_{C(R,Q)} K\xi(r)^2 d\Sigma = \int_R^Q \xi(t)^2 \int_{S(t)} K ds dt$$

$$= \int_R^Q \xi(t)^2 G'(t) dt = \xi^2 G\Big|_R^Q - \int_R^Q (\xi^2)' G dt$$

$$= \xi^2 e^{-F} G\Big|_R^Q - \int_R^Q (\xi^2)' H' dt$$

$$= \xi^2 e^{-F} G\Big|_R^Q - (\xi)' H\Big|_R^Q + \int_R^Q (\xi^2)'' H dt.$$

By using the Fiala's inequality, see Lemma 2.2, we obtain

$$H(t) = \int_R^t G(u) du \leq \int_R^t [2\pi\chi(B(u)) - L'(u)] du$$

$$\leq 2\pi\widehat{\chi}(R)(t - R) - L(t) + L(R).$$

Since $\xi$ is admissible, then $(\xi^2)' = 2\xi\xi' \leq 0$ and $(\xi^2)'' = 2(\xi')^2 + 2\xi\xi'' \geq 0$. Thus, using that $H(R) = 0$,

$$\int_{C(R,Q)} K\xi(r)^2 d\Sigma \leq \xi^2 G\big|_R^Q - (\xi^2)'(Q)[2\pi\widehat{\chi}(R)(Q-R) - L(Q) + L(R)]$$

$$+ 2\pi\widehat{\chi}(R)\int_R^Q (\xi^2)''(t)(t-R)dt + L(R)\int_R^Q (\xi^2)''(t)dt$$

$$- \int_R^Q (\xi^2)''(t)L(t)dt$$

$$= \xi^2 G\big|_R^Q - 2\pi\widehat{\chi}(R)(\xi^2)'(Q)(Q-R)$$

$$+ L(Q)(\xi^2)'(Q) - L(R)(\xi^2)'(Q)$$

$$+ 2\pi\widehat{\chi}(R)\left[(\xi^2)'(Q)(Q-R) - \int_R^Q (\xi^2)'(t)dt\right]$$

$$+ L(R)(\xi^2)'(Q) - L(R)(\xi^2)'(R) - \int_R^Q (\xi^2)''(t)L(t)dt$$

$$= \xi^2 G\big|_R^Q + (\xi^2)'L\big|_R^Q - 2\pi\widehat{\chi}(R)(\xi^2)\big|_R^Q - \int_R^Q (\xi^2)''(t)L(t)dt$$

$$= [\xi^2 G + (2\xi\xi')L - 2\pi\widehat{\chi}(R)\xi^2]\big|_R^Q - \int_R^Q (\xi^2)''(t)L(t)dt.$$

Thus,

$$\int_{C(R,Q)} K\xi(r)^2 e^{-f}d\Sigma \leq e^{-F(Q)}[\xi^2 G + (2\xi\xi')L - 2\pi\widehat{\chi}(R)\xi^2]\big|_R^Q - e^{-F(Q)}\int_R^Q (\xi^2)''(t)L(t)dt.$$

By using the coarea formula again and the fact that $(\xi^2)''(t) \geq 0$, we have

$$e^{-F(Q)}\int_R^Q (\xi^2)''(t)L(t)dt = e^{-F(Q)}\int_R^Q (\xi^2)''(t)\int_{S(t)} dsdt$$

$$= e^{-F(Q)}\int_{C(R,Q)} (\xi^2)''(r)|\nabla_\Sigma r|d\Sigma$$

$$= e^{-F(Q)}\int_{C(R,Q)} (\xi^2)''(r)d\Sigma$$

$$\geq \int_{C(R,Q)} (\xi^2)''(r)e^{-f}d\Sigma.$$

The result then follows.                                                                   $\square$

**Lemma 2.6.** *Let $\{t_n\}_{n=1}^{\overline{N}}$ be the discontinuities of the function $\widehat{\chi}$. Define the index $N(R)$ to be the largest integer $n$ such that $t_n \leq R$. Let $\xi$ be an admissible function in the interval*

$[R, Q]$. If $f : \Sigma \to \mathbb{R}$ is a locally integrable function and $F(Q) \leq \inf_{B(Q)} f(x)$, then

$$e^{F(Q)} \int_{C(R,Q)} K\xi(r)^2 e^{-f} d\Sigma \leq [\xi^2 G + (\xi^2)' L]\big|_R^Q + 2\pi \widehat{\chi}(t_{N(R)}) \xi(R)^2$$

(2.2)
$$- \sum_{n=N(R)+1}^{N(Q)} 2\pi \omega_n \xi(t_n)^2 - 2\pi \widehat{\chi}(t_{N(Q)}) \xi(Q)^2$$

$$- e^{F(Q)} \int_{C(R,Q)} (\xi^2)''(r) e^{-f} d\Sigma.$$

In particular, if $R = 0$ and assuming that $\xi(Q) = 0$, then

(2.3) $\quad \int_{B(Q)} K\xi(r)^2 e^{-f} d\Sigma \leq 2\pi e^{-F(Q)} \left[ \xi(0)^2 - \sum_{n=1}^{N(Q)} \omega_n \xi(t_n)^2 \right] - \int_{B(Q)} (\xi^2)''(r) e^{-f} d\Sigma.$

*Proof.* Applying Lemma 2.5, we have

$$e^{F(Q)} \int_{C(R,Q)} K\xi(r)^2 e^{-f} d\Sigma = e^{F(Q)} \int_{C(R,t_{N(R)+1})} K\xi(r)^2 e^{-f} d\Sigma$$

$$+ \sum_{n=N(R)+1}^{N(Q)-1} e^{F(Q)} \int_{C(t_n,t_{n+1})} K\xi(r)^2 e^{-f} d\Sigma$$

$$+ e^{F(Q)} \int_{C(t_{N(Q)},Q)} K\xi(r)^2 e^{-f} d\Sigma$$

$$\leq [\xi^2 G + 2\xi\xi' L]\big|_R^Q - 2\pi \widehat{\chi}(t_{N(R)})[\xi(t_{N(R)+1})^2 - \xi(R)^2]$$

$$- 2\pi \sum_{n=N(R)+1}^{N(Q)-1} \widehat{\chi}(t_n)[\xi(t_{n+1})^2 - \xi(t_n)^2]$$

$$- 2\pi \widehat{\chi}(t_{N(Q)})[\xi(Q)^2 - \xi(t_{N(Q)})^2]$$

$$- e^{F(Q)} \int_{C(R,Q)} (\xi^2)''(r) e^{-f} d\Sigma.$$

Since $\widehat{\chi}(t_n) = \omega_n + \widehat{\chi}(t_{n-1})$, we have

$$\widehat{\chi}(t_{N(R)})[\xi(t_{N(R)+1})^2 - \xi(R)^2] + \sum_{n=N(R)+1}^{N(Q)-1} \widehat{\chi}(t_n)[\xi(t_{n+1})^2 - \xi(t_n)^2]$$

$$+ \widehat{\chi}(t_{N(Q)})[\xi(Q)^2 - \xi(t_{N(Q)})^2]$$

$$= \widehat{\chi}(t_{N(R)})\xi(t_{N(R)+1})^2 - \widehat{\chi}(t_{N(R)})\xi(R)^2 + \sum_{n=N(R)+1}^{N(Q)-1} \widehat{\chi}(t_n)\xi(t_{n+1})^2$$

$$- \sum_{n=N(R)}^{N(Q)-2} \widehat{\chi}(t_{n+1})\xi(t_{n+1})^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2 - \widehat{\chi}(t_{N(Q)})\xi(t_{N(Q)})^2$$

$$= -\widehat{\chi}(t_{N(R)})\xi(R)^2 + \sum_{n=N(R)}^{N(Q)-1} \widehat{\chi}(t_n)\xi(t_{n+1})^2 - \sum_{n=N(R)}^{N(Q)-1} \widehat{\chi}(t_{n+1})\xi(t_{n+1})^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2$$

$$= -\widehat{\chi}(t_{N(R)})\xi(R)^2 - \sum_{N(R)+1}^{N(Q)} [\widehat{\chi}(t_n) - \widehat{\chi}(t_{n-1})]\xi(t_n)^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2$$

$$= -\widehat{\chi}(t_{N(R)})\xi(R)^2 - \sum_{N(R)+1}^{N(Q)} \omega_n\xi(t_n)^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2.$$

The estimate (2.2) then follows.

$\square$

**Definition 2.3.** Let $(\Sigma, \langle\cdot,\cdot\rangle, e^{-f})$ be a Riemannian surface with weighted measure $e^{-f}d\Sigma$ and $\Delta_f u = e^f \operatorname{div}(e^{-f}u) = \Delta u - \langle\nabla f, \nabla u\rangle$ be its weighted (drifted) Laplacian, where $\Delta$ denotes the Laplacian and $\nabla$ denotes the gradient on $\Sigma$. If $W$ is a locally integrable function and $a \in \mathbb{R}$, we say that the operator $\Delta_f - aK - W$ is nonnegative if the quadratic form

$$(2.4) \quad Q(\xi,\xi) = -\int_\Sigma \xi\left[\Delta_f\xi - aK\xi - W\xi\right]e^{-f}d\Sigma = \int_\Sigma (|\nabla\xi|^2 + aK\xi^2 + W\xi^2)e^{-f}d\Sigma$$

is nonnegative for every $\xi \in C_0^\infty(\Sigma)$.

**Proposition 2.1.** *Let $\Sigma$ be complete, noncompact Riemannian surface, let $f : \Sigma \to \mathbb{R}$ and $W : \Sigma \to \mathbb{R}$ be locally integrable functions, and let $F : [0,\infty) \to \mathbb{R}$ be a function such that $F(Q) \le \inf_{B(Q)} f(x)$, where $B(Q)$ is the geodesic ball of $\Sigma$ with center at a fixed reference point $x_0 \in \Sigma$ and radius $Q > 0$. If the operator $\Delta_f - aK - W$ is nonnegative, then*

$$e^{F(Q)}\int_{B(Q)} W_-\xi(r)^2 e^{-f}d\Sigma + e^{F(Q)}\int_{B(Q)} [(2a-1)(\xi'(r))^2 + 2a\xi(r)\xi''(r)]e^{-f}d\Sigma$$

$$(2.5)$$

$$+ 2\pi a\sum_{n=1}^{N(Q)} \omega_n\xi(t_n)^2 \le 2\pi a\xi(0)^2 + e^{F(Q)}\int_{B(Q)} W_+\xi(r)^2 e^{-f}d\Sigma,$$

*for every admissible function with support in $B(Q)$. In particular, if $a \in (1/4, \infty)$ and taking $\xi(t) = (1 - t/Q)^{2\alpha}$ for $\alpha > 4a/(4a - 1)$, we have, for every $\varepsilon \in (0, 1)$,*

$$
\begin{aligned}
(2.6) \quad & (1-\varepsilon)^{2\alpha} e^{F(Q)} \int_{B(\varepsilon Q)} W_- e^{-f} d\Sigma + \alpha[(4a-1)\alpha - 2a](1-\varepsilon)^{2\alpha-2} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma \\
& + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \leq 2\pi a + e^{F(Q)} \int_{B(Q)} W_+ e^{-f} d\Sigma.
\end{aligned}
$$

*Here, $W_+ = \max\{W, 0\}$, $W_- = \max\{-W, 0\}$, $\{t_n\}_{n=1}^{\overline{N}}$ is the set of discontinuities of the function $\widehat{\chi}$, $\omega_n = \widehat{\chi}(t_n^-) - \widehat{\chi}(t_n^+)$, and $N(Q)$ is the largest integer $n$ such that $t_n \leq Q$.*

*Proof.* First notice that $W = W_+ - W_-$. Applying the inequality (2.4) to the admissible function $\xi(r(x))$ gives

$$
\int_{B(Q)} W_- \xi(r)^2 e^{-f} d\Sigma \leq \int_{B(Q)} [(\xi'(r))^2 + aK\xi(r)^2] e^{-f} d\Sigma + \int_{B(Q)} W_+ \xi(r)^2 e^{-f} d\Sigma.
$$

Considering $\xi(Q) = 0$ and using (2.3), we have

$$
\begin{aligned}
\int_{B(Q)} W_- \xi(r)^2 e^{-f} d\Sigma \leq & \int_{B(Q)} (\xi'(r))^2 e^{-f} d\Sigma + 2\pi a e^{-F(Q)} \left[ \xi(0)^2 - \sum_{n=1}^{N(Q)} \omega_n \xi(t_n)^2 \right] \\
& - a \int_{B(Q)} (\xi^2)''(r) e^{-f} d\Sigma + \int_{B(Q)} W_+ \xi(r)^2 e^{-f} d\Sigma. \\
= & \, 2\pi a e^{-F(Q)} \xi(0)^2 + \int_{B(Q)} [(1 - 2a)(\xi'(r))^2 - 2a\xi(r)\xi''(r)] e^{-f} d\Sigma \\
& - 2\pi a \sum_{n=1}^{N(Q)} \omega_n \xi(t_n)^2 + \int_{B(Q)} W_+ \xi(r)^2 e^{-f} d\Sigma.
\end{aligned}
$$

This proves (2.5). By taking $\xi(r) = (1 - r/Q)^\alpha$, where $\alpha > 1$, we have

$$
\xi'(r) = -\frac{\alpha}{Q} \left(1 - \frac{r}{Q}\right)^{\alpha-1} \leq 0, \text{ and } \xi''(r) = \frac{\alpha(\alpha - 1)}{Q^2} \left(1 - \frac{r}{Q}\right)^{\alpha-2} \geq 0,
$$

which implies that $\xi$ is admissible. Moreover,

$$
(1 - 2a)(\xi'(r))^2 - 2a\xi(r)\xi''(r) = -\frac{\alpha[(4a - 1)\alpha - 2a]}{Q^2} \left(1 - \frac{r}{Q}\right)^{2\alpha-2}.
$$

This gives

$$
(2.7) \quad
\begin{aligned}
e^{F(Q)} \int_{B(Q)} W_- & \left(1 - \frac{r}{Q}\right)^{2\alpha} e^{-f} d\Sigma + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \\
& + \alpha[(4a-1)\alpha - 2a] \frac{e^{F(Q)}}{Q^2} \int_{B(Q)} \left(1 - \frac{r}{Q}\right)^{2\alpha - 2} e^{-f} d\Sigma \\
& \leq 2\pi a + e^{F(Q)} \int_{B(Q)} W_+ e^{-f} d\Sigma.
\end{aligned}
$$

Taking $a > 1/4$ and $\alpha > \frac{4a}{4a-1}$, all the terms in the left hand side of (2.7) are nonnegative. In order to conclude the proof of the proposition, notice that, for every $\varepsilon \in (0,1)$,

$$
\begin{aligned}
\int_{B(Q)} W_- \left(1 - \frac{r}{Q}\right)^{2\alpha} e^{-f} d\Sigma &\geq \int_{B(\varepsilon Q)} W_- \left(1 - \frac{r}{Q}\right)^{2\alpha} e^{-f} d\Sigma \\
&\geq (1-\varepsilon)^{2\alpha} \int_{B(\varepsilon Q)} W_- e^{-f} d\Sigma.
\end{aligned}
$$

Analogously, since, for $r \in [0, \varepsilon Q]$, $(1-\varepsilon)^\beta < (1-r/Q)^\beta < 1$ if $\beta > 0$ and $1 < (1-r/Q)^\beta < \frac{1}{(1-\varepsilon)^\beta}$ if $\beta < 0$, we have

$$
\frac{e^{F(Q)}}{Q^2} \int_{B(Q)} \left(1 - \frac{r}{Q}\right)^{2\alpha - 2} e^{-f} d\Sigma \geq (1-\varepsilon)^{2\alpha - 2} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma.
$$

Replacing these two estimates in (2.7) gives

$$
\begin{aligned}
(1-\varepsilon)^{2\alpha} e^{F(Q)} \int_{B(\varepsilon Q)} & W_- e^{-f} d\Sigma + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \\
& + \alpha[(4a-1)\alpha - 2a](1-\varepsilon)^{2\alpha - 2} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma \\
& \leq 2\pi a + e^{F(Q)} \int_{B(Q)} W_+ e^{-f} d\Sigma.
\end{aligned}
$$

$\square$

## 3. $f$-INDEX

Let $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$ be a weighted three-dimensional Riemannian manifold. We say that a surface $\Sigma$ immersed in $M^3$ is $f$-minimal, if its mean curvature satisfies

$$
H = \langle \overline{\nabla} f, \nu \rangle,
$$

where $\overline{\nabla}$ denotes the gradient of $M^3$ and $\nu$ is the outward unitary normal vector field of the immersion. Complete $f$-minimal surfaces are the critical points of the weighted area functional

$$\int_\Sigma e^{-f} d\Sigma$$

under all the compactly supported normal variations. Taking the second derivative, we obtain

$$\frac{d^2}{dt^2} \left( \int_\Sigma e^{-f} d\Sigma \right) \Big|_{t=0} = - \int_\Sigma \xi [\Delta_f \xi + (|A|^2 + \mathrm{Ric}_f(\nu, \nu))\xi] e^{-f} d\Sigma$$

$$:= - \int_\Sigma \xi L_f \xi e^{-f} d\Sigma,$$

for every variation of the form $\xi\nu$, where $\xi : \Sigma \to \mathbb{R}$ is a smooth compactly supported function. Here,

$$L_f \xi = \Delta_f \xi + (|A|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu))\xi$$

is the $L_f$-stability operator,

$$\Delta_f \xi = e^f \operatorname{div}(e^{-f} \nabla \xi) = \Delta \xi - \langle \nabla f, \nabla \xi \rangle$$

is the weighted (drifted) Laplacian, $|A|^2$ is the squared norm of the second fundamental form of $\Sigma$, $\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}} f$, $\overline{\mathrm{Ric}}$ is the Ricci tensor of $M^3$, and $\overline{\mathrm{Hess}} f$ is the Hessian tensor of $f$ in $M^3$. We refer the reader to Cheng, Mejia and Zhou, see [8], to more detailed discussions and calculations.

Since, $L_f$ is an elliptic operator, we can consider the spectrum of $L_f$. In a more general setting, let $L = \Delta_f - W$ be an elliptic differential operator, where $W$ is a locally integrable function. Given a bounded domain $\Omega \subset \Sigma$, define

$$\mathrm{Ind}^L(\Omega) = \#\{\text{negative eigenvalues of } L \text{ on } C_0^\infty(\Omega)\}$$

and the $f$-index of $\Sigma$ as

$$\mathrm{Ind}_f(\Sigma) := \mathrm{Ind}^L(\Sigma) = \sup_{\Omega \subset \Sigma} \mathrm{Ind}^L(\Omega).$$

The $f$-index is the dimension of the maximal subspace of $C_0^\infty(\Sigma)$ such that the quadratic form

$$Q_L(\xi, \xi) = - \int_\Sigma \xi \left[ \Delta_f \xi - W \xi \right] e^{-f} d\Sigma = \int_\Sigma \left[ |\nabla \xi|^2 + W \xi^2 \right] e^{-f} d\Sigma$$

is negative. We will need the following result, whose proof is in [1]:

**Proposition 3.1.** *Let $(\Sigma, \langle\cdot,\cdot\rangle, e^{-f})$ be a weighted complete Riemannian manifold and let $W$ be a locally integrable function on $\Sigma$. Then the operator $L = \Delta_f - W$ has finite $f$-index if and only if there exists a locally integrable function $P$ with compact support such that the operator $\Delta_f - W - P$ is nonnegative.*

Now we are ready to present the main result of this section. In the following we will consider the $f$-index of the stability operator

$$L_f = \Delta_f + (\overline{\mathrm{Ric}}_f(\nu, \nu) + |A|^2).$$

This theorem will the core of the proof of Theorem 1.1 and Theorem 1.3.

**Theorem 3.1.** *If a complete $f$-minimal surface $\Sigma$ of a weighted three-dimensional Riemannian manifold $(M^3, \langle\cdot,\cdot\rangle, e^{-f})$, for $\inf_\Sigma f = -\infty$, has finite $f$-index and satisfies*

(i) *$\overline{\mathrm{Scal}} + \overline{\mathrm{Hess}}f(\nu, \nu) \geq 0$, then $\Sigma$ is homeomorphic to $\mathbb{C}$ or $\mathbb{C}\backslash\{0\}$;*

(ii) *$\overline{\mathrm{Scal}} + \overline{\mathrm{Hess}}f(\nu, \nu) \geq -\delta$, for some $\delta > 0$, and*

$$\lim_{Q\to\infty} e^{F(Q)} \int_{B(Q)} e^{-f} d\Sigma < \infty,$$

*for some function $F : [0, \infty) \to \mathbb{R}$ such that $F(Q) \leq \inf_{B(Q)} f$, then $\Sigma$ has finite topology.*

*Moreover, in both situations, for every $\varepsilon \in (0, 1)$, we have*

$$(3.1) \qquad \lim_{Q\to\infty} e^{F(Q)} \int_{B(\varepsilon Q)} |A|^2 e^{-f} d\Sigma < \infty, \quad and \quad \lim_{Q\to\infty} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma < \infty.$$

*Here, $B(Q)$ is the geodesic ball of $\Sigma$ with center in a reference point $x_0 \in \Sigma$ and radius $Q > 0$, $\overline{\mathrm{Scal}}$ is the scalar curvature of $M^3$, $\overline{\mathrm{Hess}}f$ is the Hessian tensor of $f$ in $M^3$, and $\nu$ is the unitary normal vector field of the immersion.*

*Proof.* Since $\Sigma$ has finite index, by Proposition 3.1, there exists a locally integrable function $P$, with compact support, such that $L_f - P$ is nonnegative. Let us apply Proposition 2.1 to $L_f - P$. Let $\{t_n\}_{n=1}^{\overline{N}}$ be the discontinuities of $\widehat{\chi}(s)$. Choose $N = \overline{N}$ if $\overline{N} < \infty$ and consider $N$ as any fixed integer if $\overline{N} = \infty$. By taking $Q$ large enough, inequality (2.6)

gives

$$(1-\varepsilon)^{2\alpha}e^{F(Q)}\int_{B(\varepsilon Q)}\left[\frac{1}{2}|A|^2+\frac{1}{2}\langle\overline{\nabla}f,\nu\rangle^2+\overline{\mathrm{Scal}}+\overline{\mathrm{Hess}}f(\nu,\nu)+\delta\right]e^{-f}d\Sigma$$

$$+\alpha(3\alpha-2)(1-\varepsilon)^{2\alpha-2}\frac{e^{F(Q)}}{Q^2}\int_{B(\varepsilon Q)}e^{-f}d\Sigma$$

(3.2)

$$+2\pi\sum_{n=1}^{N}\omega_n\left(1-\frac{t_n}{Q}\right)^{2\alpha}$$

$$\leq 2\pi+\delta e^{F(Q)}\int_{B(Q)}e^{-f}d\Sigma+e^{F(Q)}\int_{\Sigma}Pe^{-f}d\Sigma.$$

Notice that, since $P$ has compact support and it is locally integrable, then last in integral in the right hand side of (3.2) is finite. On the other hand, since $\inf_{\Sigma}f=-\infty$, we have that $\lim_{Q\to\infty}F(Q)=-\infty$, which implies that $\lim_{Q\to\infty}e^{F(Q)}=0$. Therefore,

$$\lim_{Q\to\infty}e^{F(Q)}\int_{B(Q)}Pe^{-f}d\Sigma=0.$$

By taking $Q\to\infty$ and $N\to\overline{N}$, we obtain

$$\sum_{n=1}^{\overline{N}}\omega_n<\infty.$$

Since $\omega_n\geq 1$, we get $\overline{N}<\infty$. On the other hand, Lemma 2.3, p.8, implies

$$1-\sum_{n=1}^{\overline{N}}\omega_n\leq\chi(\Sigma).$$

Therefore, by using these facts and taking the limit when $Q\to\infty$ in (3.2),

$$(1-\varepsilon)^{2\alpha}\lim_{Q\to\infty}e^{F(Q)}\int_{B(\varepsilon Q)}\left[\frac{1}{2}|A|^2+\frac{1}{2}\langle\overline{\nabla}f,\nu\rangle^2+\overline{\mathrm{Scal}}+\overline{\mathrm{Hess}}f(\nu,\nu)+\delta\right]e^{-f}d\Sigma$$

(3.3)

$$+\alpha(3\alpha-2)(1-\varepsilon)^{2\alpha-2}\lim_{Q\to\infty}\frac{e^{F(Q)}}{Q^2}\int_{B(\varepsilon Q)}e^{-f}d\Sigma$$

$$\leq 2\pi\chi(\Sigma)+\delta\lim_{Q\to\infty}e^{F(Q)}\int_{B(Q)}e^{-f}d\Sigma.$$

Since, by hypothesis, the left-hand side is nonnegative, we have

$$2\pi\chi(\Sigma)+\delta\lim_{Q\to\infty}e^{F(Q)}\int_{B(Q)}e^{-f}d\Sigma\geq 0.$$

Thus, if $\delta = 0$, $\chi(\Sigma) \geq 0$, which implies that $\Sigma$ is homeomorphic to $\mathbb{C}$ or $\mathbb{C}\backslash\{0\}$. On the other hand, if $\delta > 0$, then

$$\chi(\Sigma) \geq -\frac{1}{2\pi} \lim_{Q \to \infty} e^{F(Q)} \int_{B(Q)} e^{-f} d\Sigma > -\infty$$

by hypothesis, i.e., $\Sigma$ has finite topology. The inequalities in (3.1) comes from (3.3) and noticing that each integral in the right hand side of this inequality is nonnegative. $\quad\square$

Now, we prove Theorem 1.1 and Theorem 1.3 of the Introduction.

*Proof of Theorem 1.1.* Since

$$\langle x, V \rangle \leq \|x\| \quad \text{and} \quad r(x) \geq \|x - x_0\| \geq \|x\| - \|x_0\|,$$

we have

$$\langle x, V \rangle \leq r(x) + \|x_0\|.$$

This gives

$$\inf_{B(Q)} -\langle V, x \rangle \geq -Q - \|x_0\| =: F(Q).$$

By using $F(Q) = -Q - \|x_0\|$, the proof is a direct consequence of Theorem 3.1, item (i). $\quad\square$

*Proof of Theorem 1.3.* Notice that

$$r(x) \geq \|x - x_0\|.$$

This gives

$$\begin{aligned}
r(x)^2 &\geq \|x - x_0\|^2 \\
&\geq (\|x\| - \|x_0\|)^2 \\
&= \|x\|^2 - 2\|x\|\|x_0\| + \|x_0\|^2 \\
&\geq (1 - \eta)\|x\|^2 + (1 - 1/\eta)\|x_0\|^2,
\end{aligned}$$

for every $\eta \in (0, 1)$, where in the last inequality we used the Peter-Paul inequality $2ab \leq \eta a^2 + (1/\eta)b^2$. This gives

$$\frac{1}{4}\|x\|^2 \leq \frac{1}{4(1-\eta)} r(x)^2 + \frac{1}{4\eta}\|x_0\|^2,$$

i.e., for $f(x) = -\frac{1}{4}\|x\|^2$, we can consider

$$F(Q) = -\frac{\beta}{4}Q^2 + \frac{1}{4\eta}\|x_0\|^2,$$

where $\beta = \frac{1}{1-\eta} > 1$. The result then follows by applying Theorem 3.1, item (ii) to this choice of $F(Q)$. $\qquad\qquad\square$

## 4. THE BOTTOM OF THE SPECTRUM OF THE STABILITY OPERATOR

Since $L_f = \Delta_f + (\overline{\mathrm{Ric}}_f(\nu, \nu) + |A|^2)$ is an elliptic operator, we can consider the spectrum of $L_f$ and, inspired in the Colding-Minicozzi article [10], section 9, we can define the bottom of the spectrum as follows.

**Definition 4.1.** Let $\Sigma \subset (M^3, \langle \cdot, \cdot \rangle, e^{-f})$ be a $f$-minimal surface. We define the bottom of the spectrum of the $L_f$-operator on $\Sigma$ by

$$\mu_1 = \inf_{\xi} \frac{-\int_{\Sigma} \xi L_f \xi e^{-f} d\Sigma}{\int_{\Sigma} \xi^2 e^{-f} d\Sigma} = \inf_{\xi} \frac{\int_{\Sigma} \left[ |\nabla \xi|^2 - (|A|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu))\xi^2 \right] e^{-f} d\Sigma}{\int_{\Sigma} \xi^2 e^{-f} d\Sigma},$$

where the infimum is taken over every smooth function with compact support in $\Sigma$.

Since the squared norm of the second fundamental form satisfies

$$|A|^2 = H^2 - 2(K - \overline{K}(T\Sigma))$$
$$= \langle \overline{\nabla} f, N \rangle^2 - 2K + 2\overline{K}(T\Sigma),$$

where $\overline{K}(T\Sigma)$ is the sectional curvature of $M^3$ at the plane $T\Sigma$, and $\Delta_f \xi = e^f \, \mathrm{div}(e^{-f} \nabla \xi)$, then the Definition 4.1 is equivalent to

$$0 \leq \int_{\Sigma} \left[ |\nabla \xi|^2 + K\xi^2 - \left( \frac{1}{2}|A|^2 + \frac{1}{2}\langle \overline{\nabla} f, N \rangle^2 + \overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) \right) \xi^2 - \mu_1 \xi^2 \right] e^{-f} d\Sigma,$$

for every smooth function $\xi$ with compact support in $\Sigma$.

The following result, for general $f$-minimal surfaces, is the core of the proof of Theorem 1.2 and Theorem 1.4.

**Theorem 4.1.** *Let $\Sigma$ be a complete, $f$-minimal surface of a weighted three-dimensional Riemannian manifold $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$ such that $\inf_{\Sigma} f = -\infty$ and $\overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) \geq -\delta$, for some $\delta \in \mathbb{R}$. If the bottom $\mu_1$ of the spectrum of $L_f$ satisfies*

(i) *$\mu_1 \geq \delta$, then $\Sigma$ is homeomorphic to $\mathbb{C}$ or $\mathbb{C}\backslash\{0\}$ and, moreover, if $\mu_1 > \delta$, then, for every $\varepsilon \in (0, 1)$, and for every function $F : [0, \infty) \to \mathbb{R}$ such that $F(Q) \leq \inf_{B(Q)} f$, it holds*

$$\lim_{Q \to \infty} e^{F(Q)} \int_{B(\varepsilon Q)} e^{-f} d\Sigma < \infty;$$

(ii) $\mu_1 \in (-\infty, \delta)$ *and*

$$\lim_{Q \to \infty} e^{F(Q)} \int_{B(Q)} e^{-f} d\Sigma < \infty$$

*for some function $F : [0, \infty) \to \mathbb{R}$ such that $F(Q) \leq \inf_{B(Q)} f$, then $\Sigma$ has finite topology.*

*Moreover, in both situations, for every $\varepsilon \in (0, 1)$ we have*

$$(4.1) \qquad \lim_{Q \to \infty} e^{F(Q)} \int_{B(\varepsilon Q)} \left[ \frac{1}{2} |A|^2 + (\overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) + \delta) \right] e^{-f} d\Sigma < \infty.$$

*Here, $B(Q)$ is the geodesic ball of $\Sigma$ with center in a reference point $x_0 \in \Sigma$ and radius $Q > 0$, $\overline{\mathrm{Scal}}$ is the scalar curvature of $M^3$, $\overline{\mathrm{Hess}} f$ is the Hessian tensor of $f$ in $M^3$, and $\nu$ is the unitary normal vector field of the immersion.*

*Proof.* To prove item (i), let us apply Proposition 2.1 to

$$W = -\left( \frac{1}{2} |A|^2 + \frac{1}{2} \langle \overline{\nabla} f, \nu \rangle^2 + \overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) + \mu_1 \right)$$

$$= -\left( \frac{1}{2} |A|^2 + \frac{1}{2} \langle \overline{\nabla} f, \nu \rangle^2 + \overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) + \delta + (\mu_1 - \delta) \right).$$

By using the hypothesis, we obtain that $W_+ \equiv 0$. Using inequality (2.6), we have

$$(1 - \varepsilon)^{2\alpha} e^{F(Q)} \int_{B(\varepsilon Q)} \left( \frac{1}{2} |A|^2 + \frac{1}{2} \langle \overline{\nabla} f, \nu \rangle^2 + \overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) + \mu_1 \right) e^{-f} d\Sigma$$

$$+ \alpha(3\alpha - 2)(1 - \varepsilon)^{2\alpha - 2} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\mu + 2\pi \sum_{n=1}^{N(Q)} \omega_n \left( 1 - \frac{t_n}{Q} \right)^{2\alpha} \leq 2\pi.$$

Choose $N = \overline{N}$ if $\overline{N} < \infty$ and consider $N$ as any fixed integer if $\overline{N} = \infty$. By taking $Q$ large enough and taking $Q \to \infty$, we obtain

$$(1 - \varepsilon)^{2\alpha} \lim_{Q \to \infty} e^{F(Q)} \int_{B(\varepsilon Q)} \left( \frac{1}{2} |A|^2 + \frac{1}{2} \langle \overline{\nabla} f, \nu \rangle^2 \right) e^{-f} d\Sigma$$

$$(4.2) \qquad + (1 - \varepsilon)^{2\alpha} \lim_{Q \to \infty} e^{F(Q)} \int_{B(\varepsilon Q)} \left( \overline{\mathrm{Scal}} + \overline{\mathrm{Hess}} f(\nu, \nu) + \delta + (\mu_1 - \delta) \right) e^{-f} d\Sigma$$

$$+ \alpha(3\alpha - 2)(1 - \varepsilon)^{2\alpha - 2} \lim_{Q \to \infty} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\mu + 2\pi \sum_{n=1}^{N} \omega_n \leq 2\pi.$$

Since all the terms in the left hand side are nonnegative, taking $N \to \overline{N}$, we obtain that

$$\sum_{n=1}^{\overline{N}} \omega_n \leq 1.$$

Using that $\omega_n \geq 1$, thus $\overline{N} = 0$ and $\Sigma$ is homeomorphic to $\mathbb{C}$ or $\overline{N} = 1$, $\omega_1 = 1$ and $\Sigma$ is homeomorphic to $\mathbb{C}\setminus\{0\}$. On the other hand, if $\mu_1 > \delta$, then, by (4.2),

$$(1 - \varepsilon)^{2\alpha}(\mu_1 - \delta) \lim_{Q \to \infty} e^{F(Q)} \int_{B(\varepsilon Q)} e^{-f} d\Sigma \leq 2\pi.$$

This concludes the proof of item (i). In order to prove item (ii), we apply Proposition 2.1 to

$$W = -\left(\frac{1}{2}|A|^2 + \frac{1}{2}\langle\overline{\nabla}f, \nu\rangle^2 + \overline{\mathrm{Scal}} + \overline{\mathrm{Hess}}f(\nu, \nu) + \mu_1\right)$$

$$= -\left(\frac{1}{2}|A|^2 + \frac{1}{2}\langle\overline{\nabla}f, \nu\rangle^2 + \left(\overline{\mathrm{Scal}} + \overline{\mathrm{Hess}}f(\nu, \nu) + \delta\right)\right) + (\delta - \mu_1).$$

Here, unlike item (i), $W_+ = \delta - \mu_1 > 0$. By using the hypothesis and inequality (2.6), we have

(4.3)
$$(1 - \varepsilon)^{2\alpha} e^{F(Q)} \int_{B(\varepsilon Q)} \left(\frac{1}{2}|A|^2 + \frac{1}{2}\langle\overline{\nabla}f, \nu\rangle^2 + \overline{\mathrm{Scal}} + \overline{\mathrm{Hess}}f(\nu, \nu) + \delta\right) e^{-f} d\Sigma$$

$$+ \alpha(3\alpha - 2)(1 - \varepsilon)^{2\alpha - 2}\frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\mu + 2\pi \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha}$$

$$\leq 2\pi + (\delta - \mu_1)e^{F(Q)} \int_{B(Q)} e^{-f} d\Sigma < \infty.$$

Choose $N = \overline{N}$ if $\overline{N} < \infty$ and consider $N$ as any fixed integer if $\overline{N} = \infty$. By taking $Q$ large enough and taking $Q \to \infty$, we obtain

$$\sum_{n=1}^{\overline{N}} \omega_n < \infty.$$

Since, by the Lemma 2.3,

$$1 - \sum_{n=1}^{\overline{N}} \omega_n \leq \chi(\Sigma),$$

we have that $\chi(\Sigma) > -\infty$, i.e., $\Sigma$ has finite topology. The finiteness of the integral (4.1) comes directly from the estimates (4.2) and (4.3).                    □

*Remark* 4.1. In the limit case that $\mu_1 = \delta$ we still can prove that

(4.4)
$$\lim_{Q \to \infty} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma < \infty.$$

This is an immediate consequence of the estimate (4.2).

*Remark* 4.2. If $\Sigma$ is homeomorphic to $\mathbb{C}\backslash\{0\}$, then both limits in (4.2) and (4.3) are equal to zero. In fact, if $\mathbb{C}\backslash\{0\}$, then

$$(1-\varepsilon)^{2\alpha} \lim_{Q\to\infty} e^{F(Q)} \int_{B(\varepsilon Q)} \left(\frac{1}{2}|A|^2 + \frac{1}{2}\langle\overline{\nabla}f, \nu\rangle^2\right) e^{-f} d\Sigma$$

$$+ \lim_{Q\to\infty} e^{F(Q)} \int_{B(\varepsilon Q)} \left[\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu,\nu) + \delta + (\mu_1 - \delta)\right] e^{-f} d\Sigma$$

$$+ \alpha(3\alpha-2)(1-\varepsilon)^{2\alpha-2} \lim_{Q\to\infty} \frac{e^{F(Q)}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\mu \leq 0,$$

which implies that all the limits are equal to zero.

A translating soliton is a $f$-minimal surface for the weight $f(x) = -\langle x, V\rangle$. Since $f(x) = -\langle x, V\rangle$ is not necessarily bounded from below, in the next we will use Theorem 4.1 to prove Theorem 1.2 of the Introduction:

*Proof of Theorem 1.2.* It is an immediate consequence of Theorem 4.1, item (ii), and (4.1), by taking $F(Q) = -Q - \|x_0\|$ and $\delta = 0$. $\qquad\square$

A self-expander is a $f$-minimal surface for $f(x) = -\frac{1}{4}\|x\|^2$. As a particular case of Theorem 4.1, we obtain the proof of Theorem 1.4:

*Proof of Theorem 1.4.* The proof of item (i) comes from item (i) of Theorem 4.1 by taking $\delta = 1/2$. The proof of item (ii) follows by applying Theorem 4.1, item (ii) choosing $F(Q) = -\frac{\beta}{4}Q^2 + \frac{\beta-1}{4\beta}\|x_0\|^2$, $\beta > 1$, and for $\delta = 1/2$. In its turn, item (iii) is a direct consequence of (4.1) using our choice of $F(Q)$. $\qquad\square$

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