

STABILITY PROPERTIES OF COMPLETE SELF-SHRINKING SURFACES IN \mathbb{R}^3

HILÁRIO ALENCAR, GREGÓRIO SILVA NETO & DETANG ZHOU

ABSTRACT. This paper studies rigidity for immersed self-shrinkers of the mean curvature flow of surfaces in the three-dimensional Euclidean space \mathbb{R}^3 . We prove that an immersed self-shrinker with finite L -index must be proper and of finite topology. As one of consequences, there is no stable two-dimensional self-shrinker in \mathbb{R}^3 without assuming properness. We conclude the paper by giving an affirmative answer to a question of Mantegazza.

1. INTRODUCTION

A n -dimensional self-shrinker of the mean curvature flow in \mathbb{R}^{n+1} is a hypersurface Σ which satisfies

$$H(x) = \frac{1}{2} \langle x, \nu(x) \rangle,$$

where $H(x)$ is the mean curvature of Σ at $x \in \Sigma$ and ν is its outward unitary normal vector field. Here we are using the convention of [8] such that the mean curvature of a n -dimensional round sphere of radius R is n/R , and $H = \text{trace } A$, where $A(Y) = \bar{\nabla}_Y \nu$, for $Y \in T\Sigma$, and $\bar{\nabla}$ is the connection of \mathbb{R}^{n+1} .

Self-shrinkers are known as type I singularities of the mean curvature flow. They can also be seen as the critical points of the weighted area functional

$$\int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma$$

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under all the compactly supported normal variations. Taking the second variation of the weighted area functional, we obtain

$$\begin{aligned} \frac{d^2}{dt^2} \left(\int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma \right) \Big|_{t=0} &= - \int_{\Sigma} \xi \left[\mathcal{L}\xi + \left(|A|^2 + \frac{1}{2} \right) \xi \right] e^{-\frac{1}{4}\|x\|^2} d\Sigma \\ &=: - \int_{\Sigma} \xi L\xi e^{-\frac{1}{4}\|x\|^2} d\Sigma \end{aligned}$$

for every variation of the form $\xi\nu$, where ξ is any smooth function with compact support. Here $\mathcal{L}\xi = \Delta\xi - \frac{1}{2}\langle x, \nabla\xi \rangle$ is the so called drifted Laplacian and $L\xi = \mathcal{L}\xi + (|A|^2 + \frac{1}{2})\xi$. We say that a complete self-shrinker is L -stable, if

$$\frac{d^2}{dt^2} \left(\int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma \right) \Big|_{t=0} \geq 0$$

for all the compactly supported normal variations. We refer to Cheng, Mejia and the third author, see [6], for the calculations and a more detailed discussion of this subject.

It was shown by Colding and Minicozzi (see Theorem 0.5 of [9]), that there is no L -stable smooth complete n -dimensional self-shrinkers without boundary and with polynomial volume growth in \mathbb{R}^{n+1} . This non-existence result was extended to n -dimensional self-shrinkers with sub exponential volume growth by Impera and Rimoldi, see [16]. Our first result is to prove the volume growth hypothesis is not necessary for the two-dimensional case:

Theorem 1.1. *There is no complete L -stable two-dimensional self-shrinkers in \mathbb{R}^3 .*

Remark 1.1. It is natural to ask whether there exists self-shrinkers with volume growth faster than polynomial. Halldorsson [14] proved the existence of complete self-shrinking curves Γ contained in an annulus of \mathbb{R}^2 centered at the origin and which is dense in the annulus. Since this self-shrinker is not proper, by the result of Cheng and the third author [7], it has volume growth faster than polynomial. This implies that the cylinders $\Gamma \times \mathbb{R}^{n-1}$ are self-shrinkers with volume growth faster than polynomial in \mathbb{R}^{n+1} (see also Proposition 1.1 of the paper [5] of Cheng and Ogata).

In order to study the L -instability of a self-shrinker, we use the concept of L -index. Given a bounded domain $\Omega \subset \Sigma$, define

$$\text{Ind}^L(\Omega) = \#\{\text{negative eigenvalues of } L \text{ on } C_0^\infty(\Omega)\}$$

and the L -index of Σ as

$$\text{Ind}_L(\Sigma) := \sup_{\Omega \subset \Sigma} \text{Ind}^L(\Omega).$$

The L -index is the maximal dimension of the subspace in $C_0^\infty(\Sigma)$ such that the quadratic form

$$Q_L(\xi, \xi) = - \int_{\Sigma} \xi L \xi e^{-\frac{1}{4}\|x\|^2} d\Sigma$$

is negative. Intuitively, this is the maximal dimension of the subspaces in $C_0^\infty(\Sigma)$ such that the compact variations decreases the weighted area. In this subject, our main result is

Theorem 1.2. *Let Σ be a two-dimensional self-shrinker of \mathbb{R}^3 . If Σ has finite L -index then*

- i) Σ is proper;
- ii) Σ has finite topology;
- ii) the squared norm $|A|^2$ of the second fundamental form satisfies

$$\int_{\Sigma} |A|^2 e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty.$$

As a consequence of Theorem 1.2 we have

Corollary 1.1. *Let $\Sigma \subset \mathbb{R}^3$ be a complete self-shrinker with L -index at most 4. Then Σ is a plane or a cylinder $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}$.*

Remark 1.2. In [8], Colding and Minicozzi introduced the notion of F -stability considering the variations of the functional

$$F_{x_0, t_0}(\Sigma) = \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} d\Sigma.$$

They proved that a self-shrinker with polynomial volume growth is a critical point of the functional F_{x_0, t_0} under all the normal variations of the volume and under all the variations x_s and t_s of the translations x_0 and t_0 . A self-shrinker with polynomial volume growth is said F -stable if, for every normal variation Σ_s of Σ , there exist variations x_s and t_s of x_0 and t_0 that make

$$\left. \frac{d^2}{ds^2} (F_{x_s, t_s}(\Sigma_s)) \right|_{s=0} \geq 0.$$

They also proved in Theorem 4.31, p.480, of [8] that the hyperplanes which passes through the origin are the only F -stable self-shrinkers with polynomial volume growth.

On the other hand, since there is no L -stable self-shrinker of polynomial volume growth (i.e., when x_0 and t_0 are fixed and varies Σ alone), the instability then comes from the directions of variation given by the translations x_0 and t_0 . This gives there are at most $n + 2$ directions of instability and thus, initially, F -stable hypersurfaces of \mathbb{R}^{n+1} has L -index at most $n + 2$. Therefore, if we want to remove the hypothesis of polynomial volume growth in the definition of F -stability, the class of hypersurfaces which must be considered is the one of hypersurfaces with index at most $n + 2$.

Cao and Li proved in [3] that properly immersed n -dimensional self-shrinkers in \mathbb{R}^{n+k} with the squared norm of the second fundamental form satisfying $|A|^2 \leq 1/2$ are of the form $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, for $k \in \{0, 1, \dots, n\}$. As a immediate consequence of Theorem 1.2 and the result of Cao and Li, we can observe that

Corollary 1.2. *The only self-shrinking surfaces in \mathbb{R}^3 with finite index and such that its squared norm of the second fundamental form satisfies $|A|^2 \leq 1/2$ are the sphere $\mathbb{S}^2(2)$, the plane passing through the origin, and the cylinder $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}$.*

In [8], section 9, Colding and Minicozzi defined the bottom of the spectrum of the operator L by

$$\mu_1 = \inf_{\xi} \frac{-\int_{\Sigma} \xi L\xi e^{-\frac{1}{4}\|x\|^2} d\Sigma}{\int_{\Sigma} \xi^2 e^{-\frac{1}{4}\|x\|^2} d\Sigma} = \inf_{\xi} \frac{\int_{\Sigma} [|\nabla\xi|^2 - (|A|^2 + \frac{1}{2})\xi^2] e^{-\frac{1}{4}\|x\|^2} d\Sigma}{\int_{\Sigma} \xi^2 e^{-\frac{1}{4}\|x\|^2} d\Sigma},$$

where the infimum is taken over all smooth functions ξ with compact support in Σ . Notice that it is possible to have $\mu_1 = -\infty$. Our next result is

Theorem 1.3. *Let $\Sigma \subset \mathbb{R}^3$ be a two-dimensional complete self-shrinker. If $\mu_1 \geq -\frac{1}{2}$, then Σ is homeomorphic to \mathbb{C} . Moreover, it holds*

$$\int_{\Sigma} (|A|^2 + H^2) e^{-\frac{1}{4}\|x\|^2} d\Sigma \leq 4\pi.$$

In [8], Colding and Minicozzi proved that complete, embedded, self-shrinkers with polynomial volume growth satisfies $\mu_1 \leq -\frac{1}{2}$ (Theorem 9.2, p.797). Moreover, they proved that, if H change sign, then $\mu_1 < -1$ (Theorem 9.36, p.802 of [8]). In this paper, without any embeddedness hypothesis, but assuming that Σ has finite weighted volume, we have

Theorem 1.4. *Let $\Sigma \subset \mathbb{R}^3$ be a two-dimensional complete self-shrinker with finite weighted volume. If $\mu_1 \in (-\infty, -1/2)$, then Σ has finite topology. Moreover, it holds*

$$\int_{\Sigma} (|A|^2 + H^2) e^{-\frac{1}{4}\|x\|^2} d\Sigma \leq 4\pi\chi(\Sigma) - (2\mu_1 + 1) \int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty,$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ .

As a consequence of Theorem 1.3 and Theorem 1.4 we have

Corollary 1.3. *Let $\Sigma \subset \mathbb{R}^3$ be a two-dimensional self-shrinker with polynomial volume growth. If Σ has infinite topology, then $\mu_1 = -\infty$.*

Remark 1.3. We are using, in Corollary 1.3, the equivalence between properness, polynomial volume growth, and

$$\int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty$$

proved by Cheng and the third author (see Theorem 1.3, p.688-689 of [7]).

We will include here a result answering an open question proposed by Mantegazza in [19] for n -dimensional self-shrinkers in \mathbb{R}^{n+1} removing the volume growth condition or properness. In Proposition 2.10 in [22], White proved that hyperplanes of multiplicity one are the only ones realizing the minimum of Gaussian densities of all proper nonempty mean curvature flows.

Later, Mantegazza reinterpreted and detailed a little bit more in the proof in his book (see Lemma 3.2.17, p.66 of [19]) showing that, for self-shrinkers, the properness can be replaced by the integrability condition

$$(1.1) \quad \int_{\Sigma} e^{-\|x\|} dV < \infty.$$

He also asked whether the hypothesis (1.1) can be removed (see Remark 3.2.18, p.67 of [19]). Here we prove that the integrability condition in Lemma 3.2.17, p.66, is superfluous which gives a positive answer to his question. Namely,

Theorem 1.5. *Among all the smooth, complete, n -dimensional self-shrinkers Σ in \mathbb{R}^{n+1} , the hyperplanes of multiplicity one through the origin are the only minimizers of the functional*

$$F(\Sigma) := \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} dV.$$

Hence, for all such hypersurfaces the value of this integral is at least 1.

Remark 1.4. The results we present here, except for Theorem 1.5, will be proven in a more general context, namely for f -minimal surfaces in three-dimensional weighted Riemannian manifolds $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$ satisfying

$$\overline{\text{Scal}} + \overline{\text{Hess}f}(\nu, \nu) \geq k$$

for some $k \in \mathbb{R}$. Here, $\overline{\text{Scal}}$ is the scalar curvature of M^3 and $\overline{\text{Hess}f}(\nu, \nu)$ is the hessian tensor of f in M^3 applied to the outward unitary normal vector field ν of Σ in M^3 .

Remark 1.5. The case of two-dimensional translating solitons and self-expanders is treated by the first two authors in [1].

2. PRELIMINARIES

Let Σ be a Riemannian surface with Gaussian curvature K . Let $r(x)$ be the Riemannian (intrinsic) distance in Σ between $x \in \Sigma$ to a fixed point $x_0 \in \Sigma$ and let $B(s)$ be the open geodesic ball in Σ of center x_0 and radius s . Denote by $L(s)$ the length of the boundary of $B(s)$. This length function is a priori only defined for $s \in \mathbb{R}_+ \setminus E$, where the set E of exceptional values is closed, and has Lebesgue measure zero. For $t < s$, denote for $C(t, s) = B(s) \setminus \overline{B}(t)$, where $\overline{B}(t)$ is the closure of $B(t)$. Denoting by $\chi(B(t))$ the Euler characteristic of $B(t)$, define

$$\widehat{\chi}(s) = \sup\{\chi(B(t)) \mid t \in [s, \infty)\}.$$

This function is continuous on the left, nonincreasing from $[0, \infty)$ to \mathbb{Z} , and with at most countably many discontinuities. Let

$$\{t_j\}_{j=1}^{\overline{N}} = \{0 < t_1 < t_2 < \cdots < t_n < \cdots\}$$

be the set of discontinuities, with $\overline{N} \in \mathbb{N} \cup \{\infty\}$, $\overline{N} = 0$ when the sequence is empty, and $\overline{N} = \infty$ when the sequence is infinite. Notice that this sequence depends on the reference point x_0 . At each discontinuity t_n , $n \geq 1$, the function $\widehat{\chi}$ has a jump

$$\omega_n = \widehat{\chi}(t_n^-) - \widehat{\chi}(t_n^+), \quad \omega_n \in \mathbb{N}, \omega_n \geq 1.$$

This implies $\widehat{\chi}(s) = 1$, for $s \in [0, t_1]$, and

$$\widehat{\chi}(s) = 1 - (\omega_1 + \cdots + \omega_n) \leq -(n - 1),$$

for $s \in (t_n, t_{n+1}]$.

One of the key facts we use in the proof of our results is the following inequalities, which were proved first by Fiala [11] for the set $\mathbb{R}_+ \setminus E$ and were extended to \mathbb{R}_+ by the work Hartman [15], Shiohama and Tanaka [20] and [21]. For a more details, we refer to the paper [2] of Bérard and Castillon.

Lemma 2.1 (Fiala's inequality). *On the set $\mathbb{R}_+ \setminus E$, the function L is of class C^1 and its extension to \mathbb{R}_+ satisfies*

i)

$$L'(t) \leq 2\pi\chi(B(t)) - \int_{B(t)} K d\Sigma,$$

where $\chi(B(t))$ is the Euler characteristic of $B(t)$;

ii)

$$L(b) - L(a) \leq L(b^-) - L(a) \leq \int_a^b L'(t) dt,$$

whenever $0 \leq a < b$.

The proof of the following Lemma can found in [2].

Lemma 2.2. *Let Σ be a complete Riemannian surface. Let $\{t_n\}_{n=1}^{\bar{N}}$ be the set of discontinuities of the function $\bar{\chi}$, with jumps ω_n , relative to some reference point $x_0 \in \Sigma$. Let $\chi(\Sigma)$ be the Euler characteristic of Σ , with $\chi(\Sigma) = -\infty$ if Σ does not have finite topology. Then,*

$$1 - \sum_{n=1}^{\bar{N}} \omega_n \leq \chi(\Sigma).$$

We will also need the well known coarea formula, which we state here for the sake of completeness.

Lemma 2.3 (Coarea formula). *Let $h : \Sigma \rightarrow \mathbb{R}$ be a Lipschitz function. If $h^{-1}((-\infty, t])$ is compact for all $t \in \mathbb{R}$, then*

$$\int_{\{h \leq t\}} g |\nabla_{\Sigma} h| d\Sigma = \int_{-\infty}^t \left[\int_{\{h=u\}} g dA \right] du,$$

for every $g : \Sigma \rightarrow \mathbb{R}$ locally integrable. In particular,

$$\frac{d}{dt} \left[\int_{\{h \leq t\}} g |\nabla_{\Sigma} h| d\Sigma \right] = \int_{\{h=t\}} g dA.$$

Corollary 2.1. *For every $g : \Sigma \rightarrow \mathbb{R}$ locally integrable,*

$$\int_{B(t)} g d\Sigma = \int_{-\infty}^t \left[\int_{\partial B(u)} g ds \right] du$$

where ds is the length element of $\partial B(u)$. In particular,

$$\frac{d}{dt} \left[\int_{B(t)} g d\Sigma \right] = \int_{\partial B(t)} g ds.$$

Definition 2.1. Let $0 \leq R < S$. We say that a function $\xi : [R, S] \rightarrow \mathbb{R}$ is admissible in the interval $[R, S]$ if

- i) ξ is of class C^1 and piecewise C^2 in $[R, S]$;
- ii) $\xi \geq 0$, $\xi' \leq 0$ and $\xi'' \geq 0$.

The next lemma uses the ideas of the proof of Theorem 3.4, p.223 by Gulliver and Lawson, see [13], see also Lemma 2.2, p.1245 of [2] and Lemma 1.8, p.276 of Castillon's paper [4].

Lemma 2.4. *Fix $x_0 \in \Sigma$ and let $r(x)$ be the distance to x_0 in Σ . If $f : \Sigma \rightarrow \mathbb{R}$ is a locally integrable function, such that $\inf_{\Sigma} f > -\infty$ then, for every $0 \leq R < Q$ and for any admissible function ξ on $[R, Q]$,*

$$(2.1) \quad \int_{C(R,Q)} K \xi(r)^2 e^{-f} d\Sigma \leq e^{-\inf_{\Sigma} f} [\xi^2 G + 2\xi \xi' L - 2\pi \widehat{\chi}(R) \xi^2] \Big|_R^Q - \int_{C(R,Q)} (\xi^2)''(r) e^{-f} d\Sigma.$$

Proof. Let

$$G(t) = \int_{B(t)} K d\Sigma \text{ and } H(t) = \int_R^t G(u) du.$$

Since $f \geq \inf_{\Sigma} f$ in $C(R, Q)$, we have $e^{-f} \leq e^{-\inf_{\Sigma} f}$. This gives

$$\int_{C(R,Q)} K \xi(r)^2 e^{-f} d\Sigma \leq e^{-\inf_{\Sigma} f} \int_{C(R,Q)} K \xi(r)^2 d\Sigma.$$

On the other hand, by using the coarea formula (see Corollary 2.1), we have

$$\begin{aligned}
\int_{C(R,Q)} K\xi(r)^2 d\Sigma &= \int_R^Q \xi(t)^2 \int_{S(t)} K ds dt \\
&= \int_R^Q \xi(t)^2 G'(t) dt = \xi^2 G \Big|_R^Q - \int_R^Q (\xi^2)' G dt \\
&= \xi^2 e^{-F} G \Big|_R^Q - \int_R^Q (\xi^2)' H' dt \\
&= \xi^2 e^{-F} G \Big|_R^Q - (\xi)' H \Big|_R^Q + \int_R^Q (\xi^2)'' H dt.
\end{aligned}$$

By using the Fiala's inequality of Lemma 2.1, we obtain

$$\begin{aligned}
H(t) &= \int_R^t G(u) du \leq \int_R^t [2\pi\chi(B(u)) - L'(u)] du \\
&\leq 2\pi\widehat{\chi}(R)(t - R) - L(t) + L(R).
\end{aligned}$$

Since ξ is admissible, then $(\xi^2)' = 2\xi\xi' \leq 0$ and $(\xi^2)'' = 2(\xi')^2 + 2\xi\xi'' \geq 0$. Thus, using that $H(R) = 0$,

$$\begin{aligned}
\int_{C(R,Q)} K\xi(r)^2 d\Sigma &\leq \xi^2 G \Big|_R^Q - (\xi^2)'(Q)[2\pi\widehat{\chi}(R)(Q - R) - L(Q) + L(R)] \\
&\quad + 2\pi\widehat{\chi}(R) \int_R^Q (\xi^2)''(t)(t - R) dt + L(R) \int_R^Q (\xi^2)''(t) dt \\
&\quad - \int_R^Q (\xi^2)''(t)L(t) dt \\
&= \xi^2 G \Big|_R^Q - 2\pi\widehat{\chi}(R)(\xi^2)'(Q)(Q - R) \\
&\quad + L(Q)(\xi^2)'(Q) - L(R)(\xi^2)'(Q) \\
&\quad + 2\pi\widehat{\chi}(R) \left[(\xi^2)'(Q)(Q - R) - \int_R^Q (\xi^2)'(t) dt \right] \\
&\quad + L(R)(\xi^2)'(Q) - L(R)(\xi^2)'(R) - \int_R^Q (\xi^2)''(t)L(t) dt \\
&= \xi^2 G \Big|_R^Q + (\xi^2)'L \Big|_R^Q - 2\pi\widehat{\chi}(R)(\xi^2) \Big|_R^Q - \int_R^Q (\xi^2)''(t)L(t) dt \\
&= [\xi^2 G + (2\xi\xi')L - 2\pi\widehat{\chi}(R)\xi^2] \Big|_R^Q - \int_R^Q (\xi^2)''(t)L(t) dt.
\end{aligned}$$

Thus,

$$\int_{C(R,Q)} K\xi(r)^2 e^{-f} d\Sigma \leq e^{-\inf_{\Sigma} f} [\xi^2 G + (2\xi\xi')L - 2\pi\widehat{\chi}(R)\xi^2] \Big|_R^Q - e^{-\inf_{\Sigma} f} \int_R^Q (\xi^2)''(t)L(t) dt.$$

By using the coarea formula again and the fact that $(\xi^2)''(t) \geq 0$, we have

$$\begin{aligned}
e^{-\inf_{\Sigma} f} \int_R^Q (\xi^2)''(t) L(t) dt &= e^{-\inf_{\Sigma} f} \int_R^Q (\xi^2)''(t) \int_{S(t)} ds dt \\
&= e^{-\inf_{\Sigma} f} \int_{C(R,Q)} (\xi^2)''(r) |\nabla_{\Sigma} r| d\Sigma \\
&= e^{-\inf_{\Sigma} f} \int_{C(R,Q)} (\xi^2)''(r) d\Sigma \\
&\geq \int_{C(R,Q)} (\xi^2)''(r) e^{-f} d\Sigma.
\end{aligned}$$

This concludes the proof of the lemma. \square

Lemma 2.5. *Let $\{t_n\}_{n=1}^{\overline{N}}$ be the discontinuities of the function $\widehat{\chi}$. Let $N(R)$ be the largest integer n such that $t_n \leq R$. Let ξ be an admissible function in the interval $[R, Q]$. If $f : \Sigma \rightarrow \mathbb{R}$ is a locally integrable function such that $\inf_{\Sigma} f > -\infty$, then*

$$\begin{aligned}
(2.2) \quad e^{\inf_{\Sigma} f} \int_{C(R,Q)} K \xi(r)^2 e^{-f} d\Sigma &\leq [\xi^2 G + (\xi^2)' L] \Big|_R^Q + 2\pi \widehat{\chi}(t_{N(R)}) \xi(R)^2 \\
&\quad - \sum_{n=N(R)+1}^{N(Q)} 2\pi \omega_n \xi(t_n)^2 - 2\pi \widehat{\chi}(t_{N(Q)}) \xi(Q)^2 \\
&\quad - e^{\inf_{\Sigma} f} \int_{C(R,Q)} (\xi^2)''(r) e^{-f} d\Sigma.
\end{aligned}$$

In particular, if $R = 0$ and assuming that $\xi(Q) = 0$, then

$$(2.3) \quad \int_{B(Q)} K \xi(r)^2 e^{-f} d\Sigma \leq 2\pi e^{-\inf_{\Sigma} f} \left[\xi(0)^2 - \sum_{n=1}^{N(Q)} \omega_n \xi(t_n)^2 \right] - \int_{B(Q)} (\xi^2)''(r) e^{-f} d\Sigma.$$

Proof. Applying Lemma 2.4, we have

$$\begin{aligned}
e^{\inf_{\Sigma} f} \int_{C(R,Q)} K\xi(r)^2 e^{-f} d\Sigma &= e^{\inf_{\Sigma} f} \int_{C(R,t_{N(R)+1})} K\xi(r)^2 e^{-f} d\Sigma \\
&+ \sum_{n=N(R)+1}^{N(Q)-1} e^{\inf_{\Sigma} f} \int_{C(t_n,t_{n+1})} K\xi(r)^2 e^{-f} d\Sigma \\
&+ e^{\inf_{\Sigma} f} \int_{C(t_{N(Q)},Q)} K\xi(r)^2 e^{-f} d\Sigma \\
&\leq [\xi^2 G + 2\xi\xi' L]_R^Q - 2\pi\widehat{\chi}(t_{N(R)})[\xi(t_{N(R)+1})^2 - \xi(R)^2] \\
&\quad - 2\pi \sum_{n=N(R)+1}^{N(Q)-1} \widehat{\chi}(t_n)[\xi(t_{n+1})^2 - \xi(t_n)^2] \\
&\quad - 2\pi\widehat{\chi}(t_{N(Q)})[\xi(Q)^2 - \xi(t_{N(Q)})^2] \\
&\quad - e^{\inf_{\Sigma} f} \int_{C(R,Q)} (\xi^2)''(r) e^{-f} d\Sigma.
\end{aligned}$$

Since $\widehat{\chi}(t_n) = \omega_n + \widehat{\chi}(t_{n-1})$, we have

$$\begin{aligned}
&\widehat{\chi}(t_{N(R)})[\xi(t_{N(R)+1})^2 - \xi(R)^2] + \sum_{n=N(R)+1}^{N(Q)-1} \widehat{\chi}(t_n)[\xi(t_{n+1})^2 - \xi(t_n)^2] \\
&\quad + \widehat{\chi}(t_{N(Q)})[\xi(Q)^2 - \xi(t_{N(Q)})^2] \\
&= \widehat{\chi}(t_{N(R)})\xi(t_{N(R)+1})^2 - \widehat{\chi}(t_{N(R)})\xi(R)^2 + \sum_{n=N(R)+1}^{N(Q)-1} \widehat{\chi}(t_n)\xi(t_{n+1})^2 \\
&\quad - \sum_{n=N(R)}^{N(Q)-2} \widehat{\chi}(t_{n+1})\xi(t_{n+1})^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2 - \widehat{\chi}(t_{N(Q)})\xi(t_{N(Q)})^2 \\
&= -\widehat{\chi}(t_{N(R)})\xi(R)^2 + \sum_{n=N(R)}^{N(Q)-1} \widehat{\chi}(t_n)\xi(t_{n+1})^2 - \sum_{n=N(R)}^{N(Q)-1} \widehat{\chi}(t_{n+1})\xi(t_{n+1})^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2 \\
&= -\widehat{\chi}(t_{N(R)})\xi(R)^2 - \sum_{N(R)+1}^{N(Q)} [\widehat{\chi}(t_n) - \widehat{\chi}(t_{n-1})]\xi(t_n)^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2 \\
&= -\widehat{\chi}(t_{N(R)})\xi(R)^2 - \sum_{N(R)+1}^{N(Q)} \omega_n \xi(t_n)^2 + \widehat{\chi}(t_{N(Q)})\xi(Q)^2.
\end{aligned}$$

This concludes the proof of estimate (2.2). \square

Definition 2.2. Let $(\Sigma, \langle \cdot, \cdot \rangle, e^{-f})$ be a Riemannian surface with weighted measure $e^{-f} d\Sigma$ and $\Delta_f u = e^f \operatorname{div}(e^{-f} u) = \Delta u - \langle \nabla f, \nabla u \rangle$ be its weighted Laplacian, where Δ denotes the Laplacian and ∇ denotes the gradient on Σ . If W is a locally integrable function and $a \in \mathbb{R}$, we say that the operator $\Delta_f - aK - W$ is nonnegative if the quadratic form

$$(2.4) \quad \begin{aligned} Q(\xi, \xi) &= - \int_{\Sigma} \xi [\Delta_f \xi - aK\xi - W\xi] e^{-f} d\Sigma \\ &= \int_{\Sigma} [|\nabla \xi|^2 + aK\xi^2 + W\xi^2] e^{-f} d\Sigma \geq 0, \end{aligned}$$

for every Lipschitz function with compact support in Σ (or equivalently on C^1 functions with compact support).

Proposition 2.1. *Let Σ be a complete, noncompact Riemannian surface, $f : \Sigma \rightarrow \mathbb{R}$ and $W : \Sigma \rightarrow \mathbb{R}$ be locally integrable functions such that $\inf_{\Sigma} f > -\infty$. If $\Delta_f - aK - W$ is nonnegative, then*

$$(2.5) \quad \begin{aligned} e^{\inf_{\Sigma} f} \int_{B(Q)} W_- \xi(r)^2 e^{-f} d\Sigma + e^{\inf_{\Sigma} f} \int_{B(Q)} [(2a-1)(\xi'(r))^2 + 2a\xi(r)\xi''(r)] e^{-f} d\Sigma \\ + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \xi(t_n)^2 \leq 2\pi a \xi(0)^2 + e^{\inf_{\Sigma} f} \int_{B(Q)} W_+ \xi(r)^2 e^{-f} d\Sigma, \end{aligned}$$

for every admissible function with support in $B(Q)$, where $B(Q)$ is the geodesic ball of Σ with center at a fixed reference point $x_0 \in \Sigma$ and radius $Q > 0$. Here, $W_+ = \max\{W, 0\}$, $W_- = \max\{-W, 0\}$, $\{t_n\}_{n=1}^{\bar{N}}$ is the set of discontinuities of the function $\widehat{\chi}$, $\omega_n = \widehat{\chi}(t_n^-) - \widehat{\chi}(t_n^+)$, and $N(Q)$ is the largest integer n such that $t_n \leq Q$. In particular, if $a \in (1/4, \infty)$, then taking $\xi(t) = (1 - t/Q)^{2\alpha}$ for $\alpha > 4a/(4a-1)$, we have, for every $\varepsilon \in (0, 1)$,

$$(2.6) \quad \begin{aligned} (1-\varepsilon)^{2\alpha} e^{\inf_{\Sigma} f} \int_{B(\varepsilon Q)} W_- e^{-f} d\Sigma + \alpha[(4a-1)\alpha - 2a](1-\varepsilon)^{2\alpha-2} \frac{e^{\inf_{\Sigma} f}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma \\ + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \leq 2\pi a + e^{\inf_{\Sigma} f} \int_{B(Q)} W_+ e^{-f} d\Sigma. \end{aligned}$$

Proof. First notice that $W = W_+ - W_-$. Applying the inequality (2.4) to the admissible function $\xi(r(x))$ gives

$$\int_{B(Q)} W_- \xi(r)^2 e^{-f} d\Sigma \leq \int_{B(Q)} [(\xi'(r))^2 + aK\xi(r)^2] e^{-f} d\Sigma + \int_{B(Q)} W_+ \xi(r)^2 e^{-f} d\Sigma.$$

Considering $\xi(Q) = 0$ and using (2.3), we have

$$\begin{aligned} \int_{B(Q)} W_- \xi(r)^2 e^{-f} d\Sigma &\leq \int_{B(Q)} (\xi'(r))^2 e^{-f} d\Sigma + 2\pi a e^{-\inf_\Sigma f} \left[\xi(0)^2 - \sum_{n=1}^{N(Q)} \omega_n \xi(t_n)^2 \right] \\ &\quad - a \int_{B(Q)} (\xi^2)''(r) e^{-f} d\Sigma + \int_{B(Q)} W_+ \xi(r)^2 e^{-f} d\Sigma. \\ &= 2\pi a e^{-\inf_\Sigma f} \xi(0)^2 + \int_{B(Q)} [(1-2a)(\xi'(r))^2 - 2a\xi(r)\xi''(r)] e^{-f} d\Sigma \\ &\quad - 2\pi a \sum_{n=1}^{N(Q)} \omega_n \xi(t_n)^2 + \int_{B(Q)} W_+ \xi(r)^2 e^{-f} d\Sigma. \end{aligned}$$

This proves (2.5). By taking $\xi(r) = (1 - r/Q)^\alpha$, where $\alpha > 1$, we have

$$\xi'(r) = -\frac{\alpha}{Q} \left(1 - \frac{r}{Q}\right)^{\alpha-1} \leq 0, \quad \text{and} \quad \xi''(r) = \frac{\alpha(\alpha-1)}{Q^2} \left(1 - \frac{r}{Q}\right)^{\alpha-2} \geq 0,$$

which implies that ξ is admissible. Moreover,

$$(1-2a)(\xi'(r))^2 - 2a\xi(r)\xi''(r) = -\frac{\alpha[(4a-1)\alpha-2a]}{Q^2} \left(1 - \frac{r}{Q}\right)^{2\alpha-2}.$$

This gives

$$\begin{aligned} (2.7) \quad e^{\inf_\Sigma f} \int_{B(Q)} W_- \left(1 - \frac{r}{Q}\right)^{2\alpha} e^{-f} d\Sigma &+ 2\pi a \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \\ &+ \alpha[(4a-1)\alpha-2a] \frac{e^{\inf_\Sigma f}}{Q^2} \int_{B(Q)} \left(1 - \frac{r}{Q}\right)^{2\alpha-2} e^{-f} d\Sigma \\ &\leq 2\pi a + e^{\inf_\Sigma f} \int_{B(Q)} W_+ e^{-f} d\Sigma. \end{aligned}$$

Taking $a > 1/4$ and $\alpha > \frac{4a}{4a-1}$, all the terms in the left hand side of (2.7) are nonnegative.

In order to conclude the proof of the proposition, notice that, for every $\varepsilon \in (0, 1)$,

$$\begin{aligned} \int_{B(Q)} W_- \left(1 - \frac{r}{Q}\right)^{2\alpha} e^{-f} d\Sigma &\geq \int_{B(\varepsilon Q)} W_- \left(1 - \frac{r}{Q}\right)^{2\alpha} e^{-f} d\Sigma \\ &\geq (1-\varepsilon)^{2\alpha} \int_{B(\varepsilon Q)} W_- e^{-f} d\Sigma. \end{aligned}$$

Analogously, since, for $r \in [0, \varepsilon Q]$, $(1-\varepsilon)^\beta < (1-r/Q)^\beta < 1$ if $\beta > 0$ and $1 < (1-r/Q)^\beta < \frac{1}{(1-\varepsilon)^\beta}$ if $\beta < 0$, we have

$$\frac{e^{\inf_\Sigma f}}{Q^2} \int_{B(Q)} \left(1 - \frac{r}{Q}\right)^{2\alpha-2} e^{-f} d\Sigma \geq (1-\varepsilon)^{2\alpha-2} \frac{e^{\inf_\Sigma f}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma.$$

Replacing these two estimates in (2.7) gives

$$\begin{aligned} & (1 - \varepsilon)^{2\alpha} e^{\inf_{\Sigma} f} \int_{B(\varepsilon Q)} W_- e^{-f} d\Sigma + 2\pi a \sum_{n=1}^{N(Q)} \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \\ & \quad + \alpha[(4a - 1)\alpha - 2a](1 - \varepsilon)^{2\alpha-2} \frac{e^{\inf_{\Sigma} f}}{Q^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma \\ & \leq 2\pi a + e^{\inf_{\Sigma} f} \int_{B(Q)} W_+ e^{-f} d\Sigma. \end{aligned}$$

□

3. f -STABILITY

If Σ is a complete, orientable, f -minimal surface of a weighted manifold $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$, then its mean curvature H satisfies $H = \langle \bar{\nabla} f, \nu \rangle$, where $\bar{\nabla}$ denotes the gradient of M^3 and ν is the outward unitary normal vector field of the immersion. The f -minimal surfaces are the critical points of the weighted area functional

$$\int_{\Sigma} e^{-f} d\Sigma$$

under all the compactly supported normal variations. Taking the second derivative, we have

$$\begin{aligned} \frac{d^2}{dt^2} \left(\int_{\Sigma} e^{-f} d\Sigma \right) \Big|_{t=0} &= - \int_{\Sigma} \xi [\Delta_f \xi + (|A|^2 + \text{Ric}_f(\nu, \nu)) \xi] e^{-f} d\Sigma \\ &:= - \int_{\Sigma} \xi L_f \xi e^{-f} d\Sigma, \end{aligned}$$

for every variation of the form $\xi\nu$, where $\xi : \Sigma \rightarrow \mathbb{R}$ is a smooth compactly supported function. Here,

$$L_f \xi = \Delta_f \xi + (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu)) \xi$$

is the L_f -stability operator, $\Delta_f \xi = e^f \text{div}(e^{-f} \nabla \xi) = \Delta \xi - \langle \nabla \xi, \nabla f \rangle$ is the weighted (drifted) Laplacian, ∇ is the gradient of Σ , $|A|^2$ is the squared norm of the second fundamental form of Σ , $\overline{\text{Ric}}_f = \overline{\text{Ric}} + \overline{\text{Hess}} f$, $\overline{\text{Ric}}$ is the Ricci tensor of M^3 , and $\overline{\text{Hess}} f$ is the Hessian tensor of f in M^3 . We refer the reader to [6] to more detailed discussions and calculations.

We say that a f -minimal surface is L_f -stable if

$$\frac{d^2}{dt^2} \left(\int_{\Sigma} e^{-f} d\Sigma \right) \Big|_{t=0} \geq 0$$

for every compactly supported variation. Since the squared norm of the second fundamental form satisfies

$$\begin{aligned} |A|^2 &= H^2 - 2(K - \overline{K}(T\Sigma)) \\ &= \langle \overline{\nabla}f, \nu \rangle^2 - 2K + 2\overline{K}(T\Sigma), \end{aligned}$$

where $\overline{K}(T\Sigma)$ is the sectional curvature of M^3 at the plane $T\Sigma$, if Σ is L_f -stable, then

$$\begin{aligned} 0 &\leq - \int_{\Sigma} \xi [\Delta_f \xi + (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu))\xi] e^{-f} d\Sigma \\ &= \int_{\Sigma} \left[|\nabla \xi|^2 + K\xi^2 - \left(\frac{1}{2}|A|^2 + \frac{1}{2}\langle \overline{\nabla}f, \nu \rangle^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right) \xi^2 \right] e^{-f} d\Sigma \\ &= \int_{\Sigma} \left[|\nabla \xi|^2 + K\xi^2 - \left(\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right) \xi^2 \right] e^{-f} d\Sigma, \end{aligned}$$

for every smooth function ξ with compact support in Σ .

If Σ is closed (i.e., compact without boundary), an argument similar to Fischer-Colbrie and Schoen, see [12], gives

Theorem 3.1. *If a L_f -stable, closed, f -minimal surface Σ of a weighted three-dimensional Riemannian manifold $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$, satisfies*

$$\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \geq 0,$$

then Σ is homeomorphic to the sphere \mathbb{S}^2 or the flat torus T^2 . Moreover,

- i) if $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) > 0$ then Σ is homeomorphic to \mathbb{S}^2 ;
- ii) if Σ is homeomorphic to T^2 then Σ is totally geodesic and $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \equiv 0$.

Here $\overline{\text{Scal}}$ is the scalar curvature of M^3 , $\overline{\text{Hess}}f$ is the Hessian tensor of f in M^3 , and ν is the outward unitary normal vector field of the immersion.

Proof. Since Σ is L_f -stable, we have

$$\int_{\Sigma} \left(\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right) \xi^2 e^{-f} d\Sigma \leq \int_{\Sigma} [|\nabla \xi|^2 + K\xi^2] e^{-f} d\Sigma.$$

Taking $\xi \equiv 1$ and using the Gauss-Bonnet theorem, we have

$$\begin{aligned} \int_{\Sigma} \left(\frac{1}{2}|A|^2 + \overline{\text{Scal}} + \frac{1}{2}H^2 + \overline{\text{Hess}}f(\nu, \nu) \right) e^{-f} d\Sigma &\leq \int_{\Sigma} K e^{-f} d\Sigma. \\ &\leq (\max_{\Sigma} e^{-f}) \int_{\Sigma} K d\Sigma \\ &= 2\pi (\max_{\Sigma} e^{-f}) \chi(\Sigma). \end{aligned}$$

Since the left hand side is nonnegative by hypothesis, we have that $\chi(\Sigma) = 1$ or $\chi(\Sigma) = 0$. In the first case Σ is homeomorphic to \mathbb{S}^2 and, in the second case, Σ is homeomorphic to T^2 . Moreover, if $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) > 0$, then $\chi(\Sigma) = 1$ and Σ is homeomorphic to \mathbb{S}^2 . If Σ is homeomorphic to T^2 , then $\chi(\Sigma) = 0$, which gives $|A|^2 \equiv 0$ and $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \equiv 0$. \square

Remark 3.1. Theorem 3.1 can be compared with Theorem 1, p.1066, of [18], where Liu proved a topological classification result for closed, orientable hypersurfaces in oriented, complete, m -dimensional Riemannian manifolds (M^m, g, e^{-f}) satisfying $\text{Ric}_f \geq 0$.

In the following we will consider the case where Σ is complete and noncompact. For L_f -stable f -minimal surfaces with $\inf_{\Sigma} f > -\infty$, we have

Theorem 3.2. *If a L_f -stable, complete, f -minimal surface Σ of a weighted three-dimensional Riemannian manifold $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$, for $\inf_{\Sigma} f > -\infty$, satisfies*

$$\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \geq 0,$$

then Σ is homeomorphic to \mathbb{C} or $\mathbb{C} \setminus \{0\}$. Moreover,

- i) the f -volume of Σ has at most quadratic growth;
- ii) if Σ is homeomorphic to $\mathbb{C} \setminus \{0\}$, then Σ is totally geodesic and $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \equiv 0$.
- iii) it holds

$$\int_{\Sigma} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right] e^{-f} d\Sigma \leq 2\pi\chi(\Sigma)e^{-\inf_{\Sigma} f} < \infty.$$

In particular,

$$\int_{\Sigma} |A|^2 e^{-f} d\Sigma < \infty.$$

Moreover, if there exists $k > 0$ such that $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \geq k$, then the f -volume of Σ is finite.

Here $\overline{\text{Scal}}$ is the scalar curvature of M^3 , $\overline{\text{Hess}}f$ is the Hessian tensor of f in M^3 , and ν is the outward unitary normal vector field of the immersion.

Proof. Since $\inf_{\Sigma} f > -\infty$, we can use Proposition 2.1. Let $\{t_n\}_{n=1}^{\overline{N}}$ be the discontinuities of $\widehat{\chi}(s)$. Choose $N = \overline{N}$ if $\overline{N} < \infty$ and consider N as any fixed integer if $\overline{N} = \infty$. By

taking Q large enough, inequality (2.6) gives

$$(3.1) \quad \begin{aligned} (1 - \varepsilon)^{2\alpha} \int_{B(\varepsilon Q)} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right] e^{-f} d\Sigma \\ + \alpha(3\alpha - 2)\varepsilon^2(1 - \varepsilon)^{2\alpha-2} \frac{1}{(\varepsilon Q)^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma \\ + 2\pi e^{-\inf_{\Sigma} f} \sum_{n=1}^N \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \leq 2\pi e^{-\inf_{\Sigma} f}. \end{aligned}$$

By taking $Q \rightarrow \infty$, we obtain, from the second integral in the left hand side of (3.1), that $\int_{B(Q)} e^{-f} d\Sigma$ has at most quadratic growth and, by taking $N \rightarrow \bar{N}$,

$$\sum_{n=1}^{\bar{N}} \omega_n < \infty.$$

Since $\omega_n \geq 1$, we get $\bar{N} < \infty$. On the other hand, Lemma 2.2, p.7, implies

$$1 - \sum_{n=1}^{\bar{N}} \omega_n \leq \chi(\Sigma)$$

which gives, by taking $\varepsilon \rightarrow 0$,

$$\int_{\Sigma} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right] e^{-f} d\Sigma \leq 2\pi\chi(\Sigma)e^{-\inf_{\Sigma} f} < \infty.$$

Moreover, since the left hand side of the last inequality is nonnegative, we have

$$\sum_{n=1}^{\bar{N}} \omega_n \leq 1$$

and thus $\bar{N} = 0$ and Σ is homeomorphic to \mathbb{C} or $\bar{N} = 1$, $\omega_1 = 1$ and Σ is homeomorphic to $\mathbb{C} \setminus \{0\}$. Moreover, if Σ is homeomorphic to $\mathbb{C} \setminus \{0\}$, then $\chi(\Sigma) = 0$, which gives

$$0 \leq \int_{\Sigma} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right] e^{-f} d\Sigma \equiv 0.$$

This implies that Σ is totally geodesic, $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \equiv 0$ and $\langle \nabla f, N \rangle \equiv 0$. \square

Remark 3.2. The hypothesis $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \geq 0$ of Theorem 3.2 can be replaced by the weaker one

$$\int_{\Sigma} (\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu))_- e^{-f} d\Sigma < \infty.$$

In this case, however, we obtain the weaker result that Σ has finite topology, i.e., its Euler characteristic $\chi(\Sigma) > -\infty$. Here, $W_- = \max\{-W, 0\}$ is the negative part of the function W .

If Σ is a L -stable, complete self-shrinker of the mean curvature flow in \mathbb{R}^3 , then Σ is a L_f -stable f -minimal surface of \mathbb{R}^3 for $f = \frac{1}{4}\|x\|^2$. Applying Theorem 3.2 to this situation we can prove Theorem 1.1:

Proof of Theorem 1.1. By using Theorem 3.2, and noticing that $\inf_{\Sigma} \frac{1}{4}\|x\|^2 \geq 0$, we have

$$\frac{1}{2} \int_{\Sigma} \left(|A|^2 + \frac{1}{4} \langle x, \nu \rangle^2 + 1 \right) e^{-\frac{1}{4}\|x\|^2} d\Sigma \leq 2\pi\chi(\Sigma) < \infty.$$

In particular,

$$\int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty.$$

On the other hand, Colding and Minicozzi proved in [8] (see Theorem 0.5 of [9]) that there is no L -stable self-shrinker with polynomial volume growth. Since polynomial volume growth is equivalent to $\int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty$ by Theorem 1.3, p.688-689, of [7], we conclude the proof of the Theorem. We remark that this conclusion can be also derived more directly by using Theorem 3, p.4042, of [6]. \square

4. f -INDEX

Let $L = \Delta_f - W$, where W is a locally integrable function. Given a bounded domain $\Omega \subset \Sigma$, define

$$\text{Ind}^L(\Omega) = \#\{\text{negative eigenvalues of } L \text{ on } C_0^\infty(\Omega)\}$$

and the f -index of Σ as

$$\text{Ind}_f(\Sigma) := \text{Ind}^L(\Sigma) = \sup_{\Omega \subset \Sigma} \text{Ind}^L(\Omega).$$

The f -index is the maximal dimension of the linear subspaces of C_0^∞ such that the quadratic form

$$Q_f(\xi, \xi) = - \int_{\Sigma} \xi L \xi e^{-f} d\Sigma = - \int_{\Sigma} \xi [\Delta_f \xi - W \xi] e^{-f} d\Sigma = \int_{\Sigma} [|\nabla \xi|^2 + W \xi^2] e^{-f} d\Sigma$$

is negative.

We will need the following result, due to Devyver [10], and which was enunciated in the following form by Impera and Rimoldi in [16], Proposition 5, p.29:

Proposition 4.1. *Let $(\Sigma, \langle \cdot, \cdot \rangle, e^{-f})$ be a weighted complete manifold, and let $L = \Delta_f - W$, where $W \in L_{loc}^\infty(\Sigma)$. The following facts are equivalent:*

- i) L has finite Morse index;

- ii) *there exists a positive smooth function $\varphi \in W_{loc}^{1,2}$ which satisfies $L\varphi = 0$ outside a compact set;*
- iii) *$\lambda_1^L(M \setminus \Omega) \geq 0$ for some $\Omega \subset \Sigma$, i.e., L is nonnegative in $M \setminus \Omega$.*

We will also need the following result, which use some ideas of Proposition 4.1., p.1259, of [2]:

Proposition 4.2. *Let $(\Sigma, \langle \cdot, \cdot \rangle, e^{-f})$ be a weighted complete Riemannian manifold and let W be a locally integrable function on Σ . Then the operator $L = \Delta_f - W$ has finite f -index if and only if there exists a locally integrable function P with compact support such that the operator $\Delta_f - W - P$ is nonnegative.*

Proof. Suppose that $L = \Delta_f - W$ has finite index. By the Proposition 4.1, there exists a compact set $K \subset \Sigma$ such that L is nonnegative in $\Sigma \setminus K$. Let us find a function P , with support in a compact neighborhood of K , such that $L - P$ is nonnegative in the whole Σ . Let ϕ be a smooth function with compact support, such that $0 \leq \phi \leq 1$, and $\phi \equiv 1$ in a compact neighborhood of K . Given any smooth function ξ with compact support in Σ , write

$$\xi = \phi\xi + (1 - \phi)\xi.$$

We have

$$\begin{aligned} & \int_{\Sigma} [|\nabla\xi|^2 + W\xi^2] e^{-f} d\Sigma \\ &= \int_{\Sigma} [|\nabla((1 - \phi)\xi)|^2 + 2\langle \nabla(\phi\xi), \nabla((1 - \phi)\xi) \rangle + |\nabla(\phi\xi)|^2] e^{-f} d\Sigma \\ & \quad + \int_{\Sigma} W [(1 - \phi)^2\xi^2 + 2\phi(1 - \phi)\xi^2 + (\phi\xi)^2] e^{-f} d\Sigma \\ &= \int_{\Sigma} [|\nabla((1 - \phi)\xi)|^2 + W((1 - \phi)\xi)^2] e^{-f} d\Sigma \\ & \quad + \int_{\Sigma} W [\phi^2 + 2\phi(1 - \phi)] \xi^2 e^{-f} d\Sigma \\ & \quad + \int_{\Sigma} [2\langle \nabla(\phi\xi), \nabla((1 - \phi)\xi) \rangle + |\nabla(\phi\xi)|^2] e^{-f} d\Sigma. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \langle \nabla(\phi\xi), \nabla((1 - \phi)\xi) \rangle &= \langle \xi\nabla\phi + \phi\nabla\xi, -\xi\nabla\phi + (1 - \phi)\nabla\xi \rangle \\ &= -\xi^2|\nabla\phi|^2 + \xi(1 - 2\phi)\langle \nabla\phi, \nabla\xi \rangle + \phi(1 - \phi)|\nabla\xi|^2 \end{aligned}$$

and

$$|\nabla(\phi\xi)|^2 = \phi^2|\nabla\xi|^2 + 2\phi\xi\langle\nabla\phi, \nabla\xi\rangle + \xi^2|\nabla\phi|^2,$$

we have

$$\begin{aligned} 2\langle\nabla(\phi\xi), \nabla((1-\phi)\xi)\rangle + |\nabla(\phi\xi)|^2 &= -\xi^2|\nabla\phi|^2 + 2\xi(1-\phi)\langle\nabla\xi, \nabla\phi\rangle \\ &\quad + 2\phi\left(1 - \frac{1}{2}\phi\right)|\nabla\xi|^2. \end{aligned}$$

On the other hand, since $\operatorname{div}_f(u\nabla v) = u\Delta_f v + \langle\nabla u, \nabla v\rangle$, we obtain

$$\begin{aligned} 2\xi(1-\phi)\langle\nabla\xi, \nabla\phi\rangle &= -\frac{1}{2}\langle\nabla(\xi^2), \nabla((1-\phi)^2)\rangle \\ &= -\frac{1}{2}\operatorname{div}_f(\xi^2\nabla((1-\phi)^2)) + \frac{1}{2}\phi^2\Delta_f((1-\phi)^2) \end{aligned}$$

This gives

$$\begin{aligned} \int_{\Sigma} [|\nabla\xi|^2 + W\xi^2] e^{-f} d\Sigma &= \int_{\Sigma} [|\nabla((1-\phi)\xi)|^2 + W((1-\phi)\xi)^2] e^{-f} d\Sigma \\ &\quad + \int_{\Sigma} W[\phi^2 + 2\phi(1-\phi)] \xi^2 e^{-f} d\Sigma - \int_{\Sigma} \xi^2|\nabla\phi|^2 e^{-f} d\Sigma \\ &\quad + \frac{1}{2}\int_{\Sigma} \xi^2\Delta_f((1-\phi)^2) e^{-f} d\Sigma + 2\int_{\Sigma} \phi\left(1 - \frac{1}{2}\phi\right) |\nabla\xi|^2 e^{-f} d\Sigma \\ &= \int_{\Sigma} [|\nabla((1-\phi)\xi)|^2 + W((1-\phi)\xi)^2] e^{-f} d\Sigma \\ &\quad + 2\int_{\Sigma} \phi\left(1 - \frac{1}{2}\phi\right) |\nabla\xi|^2 e^{-f} d\Sigma \\ &\quad - \int_{\Sigma} \left[\phi(\phi-2)W + |\nabla\phi|^2 - \frac{1}{2}\Delta_f((1-\phi)^2) \right] \xi^2 e^{-f} d\Sigma. \end{aligned}$$

Defining

$$P = \phi(\phi-2)W + |\nabla\phi|^2 - \frac{1}{2}\Delta_f((1-\phi)^2)$$

we can see that P is locally integrable, has compact support in Σ , and

$$\begin{aligned} \int_{\Sigma} [|\nabla\xi|^2 + W\xi^2 + P\xi^2] e^{-f} d\Sigma &= \int_{\Sigma} [|\nabla((1-\phi)\xi)|^2 + W((1-\phi)\xi)^2] e^{-f} d\Sigma \\ &\quad + 2\int_{\Sigma} \phi\left(1 - \frac{1}{2}\phi\right) |\nabla\xi|^2 e^{-f} d\Sigma \geq 0. \end{aligned}$$

Conversely, suppose there exists a locally integrable function P with compact support and such that $L - P$ is nonnegative. Let $K \subset \Sigma$ be a compact neighborhood of the support

of P . Given any function ξ with compact support in $\Sigma \setminus K$, we have

$$\int_{\Sigma} [|\nabla \xi|^2 + W\xi^2] e^{-f} d\Sigma = \int_{\Sigma} [|\nabla \xi|^2 + W\xi^2 + P\xi^2] e^{-f} d\Sigma \geq 0.$$

This gives that L is nonnegative in $M \setminus K$. By Proposition 4.1, we conclude that L has finite index. \square

In the following we will consider the f -index of the stability operator

$$L_f = \Delta_f + (\overline{\text{Ric}}_f(\nu, \nu) + |A|^2).$$

Now we are ready to present the main result of this section:

Theorem 4.1. *If a complete f -minimal surface Σ of a weighted three-dimensional Riemannian manifold $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$, for $\inf_{\Sigma} f > -\infty$, has finite f -index and satisfies*

$$\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \geq 0,$$

then Σ has finite topology (i.e., the Euler characteristic $\chi(\Sigma) > -\infty$). Moreover,

- i) the f -volume of Σ has quadratic growth;
- ii) it holds

$$\int_{\Sigma} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right] e^{-f} d\Sigma < \infty.$$

In particular,

$$\int_{\Sigma} |A|^2 e^{-f} d\Sigma < \infty.$$

Moreover, if there exists $k > 0$ such that $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \geq k$, then the f -volume of Σ is finite.

Here $\overline{\text{Scal}}$ is the scalar curvature of M^3 , $\overline{\text{Hess}}f$ is the Hessian tensor of f in M^3 , and ν is the outward unitary normal vector field of the immersion.

Proof. Since Σ has finite index, by Proposition 4.2, there exists a locally integrable function P , with compact support, such that $L_f + P$ is nonnegative. Since $\inf_{\Sigma} f > -\infty$, we can use Proposition 2.1. Let $\{t_n\}_{n=1}^{\overline{N}}$ be the discontinuities of $\widehat{\chi}(s)$. Choose $N = \overline{N}$ if $\overline{N} < \infty$ and consider N as any fixed integer if $\overline{N} = \infty$. By taking Q large enough,

inequality (2.6) gives

$$\begin{aligned}
(1 - \varepsilon)^{2\alpha} \int_{B(\varepsilon Q)} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right] e^{-f} d\Sigma \\
+ \alpha(3\alpha - 2)\varepsilon^2(1 - \varepsilon)^{2\alpha-2} \frac{1}{(\varepsilon Q)^2} \int_{B(\varepsilon Q)} e^{-f} d\Sigma \\
+ 2\pi e^{-\inf_{\Sigma} f} \sum_{n=1}^N \omega_n \left(1 - \frac{t_n}{Q}\right)^{2\alpha} \leq 2\pi e^{-\inf_{\Sigma} f} + \int_{\Sigma} P e^{-f} d\Sigma.
\end{aligned}
\tag{4.1}$$

Notice that, since P has compact support and it is locally integrable, then the last integral in the right hand side of (4.1) is finite. By taking $Q \rightarrow \infty$, we obtain, by the second integral in the left hand side of (4.1), that $\int_{B(Q)} e^{-f} d\Sigma$ has at most quadratic growth and, by taking $N \rightarrow \bar{N}$,

$$\sum_{n=1}^{\bar{N}} \omega_n < \infty.$$

Since $\omega_n \geq 1$, we get $\bar{N} < \infty$. On the other hand, Lemma 2.2, p.7, implies

$$1 - \sum_{n=1}^{\bar{N}} \omega_n \leq \chi(\Sigma)$$

which gives, by taking $\varepsilon \rightarrow 0$,

$$\begin{aligned}
\int_{\Sigma} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \right] e^{-f} d\Sigma \\
\leq 2\pi\chi(\Sigma)e^{-\inf_{\Sigma} f} + \int_{\Sigma} P e^{-f} d\Sigma < \infty.
\end{aligned}$$

Moreover, since the left hand side of the last inequality is nonnegative,

$$\chi(\Sigma) \geq -e^{\inf_{\Sigma} f} \int_{\Sigma} P e^{-f} d\Sigma > -\infty,$$

i.e., Σ has finite topology. □

Remark 4.1. The case when $\inf_{\Sigma} f = \infty$ is treated by the first two authors in [1].

Now, we conclude this section by proving Theorem 1.2 and Corollary 1.1 in the Introduction.

Proof of Theorem 1.2. Applying Theorem 4.1, we obtain that Σ has finite topology and it holds

$$\frac{1}{2} \int_{\Sigma} \left(|A|^2 + \frac{1}{4}\langle x, \nu \rangle^2 + 1 \right) e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty.$$

This gives

$$\int_{\Sigma} |A|^2 e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty \quad \text{and} \quad \int_{\Sigma} e^{-\frac{1}{4}\|x\|^2} d\Sigma < \infty.$$

By the last statement and using Theorem 1.3, p.688-689, of [7], we have that Σ is proper. \square

Proof of Corollary 1.1. Since the L -index is at most 4, then by Theorem 1.2 we have that Σ is proper. The conclusion comes from the result of Impera, see [17], which proves that a properly immersed m -dimensional self-shrinker of \mathbb{R}^{m+1} is a hyperplane with L -index one or it has L -index at least $m + 2$, with the equality holding only on the cylinders $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{m-k}$. \square

5. THE BOTTOM OF THE SPECTRUM OF THE STABILITY OPERATOR

Definition 5.1. Let $\Sigma \subset (M^3, \langle \cdot, \cdot \rangle, e^{-f})$ be a f -minimal surface. We define the bottom of the spectrum of the L_f -operator on Σ by

$$\mu_1 = \inf_{\xi} \frac{-\int_{\Sigma} \xi L_f \xi e^{-f} d\Sigma}{\int_{\Sigma} \xi^2 e^{-f} d\Sigma} = \inf_{\xi} \frac{\int_{\Sigma} [|\nabla \xi|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu)) \xi^2] e^{-f} d\Sigma}{\int_{\Sigma} \xi^2 e^{-f} d\Sigma},$$

where the infimum is taken over every smooth function with compact support in Σ . Here $L_f \xi = \Delta_f \xi + (\overline{\text{Ric}}_f(\nu, \nu) + |A|^2) \xi$.

Remark 5.1. Following our previous discussions, Definition 5.1 is equivalent to the inequalities

$$\begin{aligned} 0 &\leq \int_{\Sigma} \left[|\nabla \xi|^2 + K \xi^2 - \left(\frac{1}{2} |A|^2 + \frac{1}{2} \langle \overline{\nabla} f, \nu \rangle^2 + \overline{\text{Scal}} + \overline{\text{Hess}} f(\nu, \nu) \right) \xi^2 - \mu_1 \xi^2 \right] e^{-f} d\Sigma \\ &= \int_{\Sigma} \left[|\nabla \xi|^2 + K \xi^2 - \left(\frac{1}{2} |A|^2 + \frac{1}{2} H^2 + \overline{\text{Scal}} + \overline{\text{Hess}} f(\nu, \nu) \right) \xi^2 - \mu_1 \xi^2 \right] e^{-f} d\Sigma \end{aligned}$$

for every smooth function ξ with compact support in Σ .

The proof of the next result follows the same steps of the proof of Theorem 3.2. Theorem 1.3 and Theorem 1.4 are direct applications of this theorem by taking $f(x) = \frac{1}{4}\|x\|^2 \geq 0$.

Theorem 5.1. *Let Σ be a complete f -minimal surface of a weighted manifold $(M^3, \langle \cdot, \cdot \rangle, e^{-f})$, such that $\inf_{\Sigma} f > -\infty$ and $\overline{\text{Scal}} + \overline{\text{Hess}} f(\nu, \nu) \geq \delta$, for some $\delta \in \mathbb{R}$.*

- i) If the bottom of the spectrum of L_f satisfies $\mu_1 \geq -\delta$, then Σ is homeomorphic to \mathbb{C} or $\mathbb{C} \setminus \{0\}$, the f -volume of Σ has at most quadratic volume growth, and it holds

$$\int_{\Sigma} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) - \delta \right] e^{-f} d\Sigma \leq 2\pi\chi(\Sigma)e^{-\inf_{\Sigma} f} < \infty.$$

Moreover,

- a) if Σ is homeomorphic to $\mathbb{C} \setminus \{0\}$, then Σ is totally geodesic and $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \equiv \delta$;
 b) if there exists $\epsilon > 0$ such that $\overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) \geq \delta + \epsilon$, then Σ is homeomorphic to \mathbb{C} and it has finite f -volume, i.e., $\int_{\Sigma} e^{-f} d\Sigma < \infty$.
 ii) If the bottom of the spectrum of L_f satisfies $\mu_1 \in (-\infty, -\delta)$ and the f -volume of Σ is finite, i.e., $\int_{\Sigma} e^{-f} d\Sigma < \infty$, then Σ has finite topology. Moreover,

$$\begin{aligned} \int_{\Sigma} \left[\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) - \delta \right] e^{-f} d\Sigma \\ \leq 2\pi\chi(\Sigma)e^{-\inf_{\Sigma} f} - (\mu_1 + \delta) \int_{\Sigma} e^{-f} d\Sigma < \infty. \end{aligned}$$

In both situations, we have, in particular, that

$$\int_{\Sigma} |A|^2 e^{-f} d\Sigma < \infty.$$

Here $\overline{\text{Scal}}$ is the scalar curvature of M^3 , $\overline{\text{Hess}}f$ is the Hessian tensor of f in M^3 , and ν is the outward unitary normal vector field of the immersion.

Proof. In fact, we can use, in Proposition 2.1, p.12,

$$(5.1) \quad W = - \left(\frac{1}{2}|A|^2 + \frac{1}{2}H^2 + \overline{\text{Scal}} + \overline{\text{Hess}}f(\nu, \nu) - \delta \right) - (\mu_1 + \delta)$$

in the proof of item i) and to observe that $W_+ \equiv 0$. In the case ii) we observe that $W_+ = -(\mu_1 + \delta)$. The conclusion comes following the same steps of the proof of Theorem 3.2. \square

Theorem 1.4 is an immediate consequence of Theorem 5.1. The proof of Theorem 1.3 we proceed as follows.

Proof of Theorem 1.3. The proof is an immediate consequence of Theorem 5.1, except by the situation that Σ homeomorphic to $\mathbb{C} \setminus \{0\}$ cannot happen. In fact, if Σ is homeomorphic to $\mathbb{C} \setminus \{0\}$, then $\chi(\Sigma) = 0$ and thus, by the item ii) of Theorem 5.1, $|A|^2 \equiv 0$, i.e., Σ is totally geodesic. Since the totally geodesic surfaces of the Euclidean space are the

planes, which are homeomorphic to \mathbb{C} , we conclude that the case of Σ homeomorphic to $\mathbb{C} \setminus \{0\}$ cannot happen. \square

We conclude the paper proving Theorem 1.5.

Proof of Theorem 1.5. Assume for the sake of contradiction the hyperplane P with multiplicity one through the origin is not the unique minimizer among the self-shrinkers. Then there exists a self-shrinker M such that $F(\Sigma)$ is at most $F(P) = 1$. From Theorem 1.3, p.688-689 of [7], the finiteness of $F(\Sigma)$ implies that Σ is proper which implies that it has polynomial volume growth which implies the integrability in the Lemma. The rest of the proof follows from Proposition 2.10 in [22]. \square

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