Stability of hypersurfaces with vanishing r-mean curvatures in euclidean spaces

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Abstract. Hypersurfaces of euclidean spaces with vanishing r-mean curvature generalize minimal hypersurfaces (case $r = 1$) and include the important case of scalar curvature ($r = 2$). They are critical points of variational problems and a notion of stability can be assigned to them. When their defining equations are elliptic, we obtain a criterion for stability of bounded domains of such hypersurfaces that generalizes a known theorem of Barbosa and do Carmo for stability of minimal surfaces.

1. Introduction

The goal of this paper is to generalize the following result. Let $x: M^2 \to \mathbb{R}^3$ be an orientable minimal surface and $D \subset M$ be a domain with compact closure and piecewise smooth boundary. Let $g: M \to S_1^2 \subset \mathbb{R}^3$ be the Gauss map of x.

Theorem A ([BdC], Theorem 1.3)**.** *Assume that the area of $g(D) \subset S_1^2$ is smaller than the area of a hemisphere of S_1^2. Then D is stable and the estimate is sharp.*

A first generalization is as follows. Let $x: M^3 \to \mathbb{R}^4$ be a hypersurface with scalar curvature $H_2 = 0$ (this generalizes the above notion of minimal surface). We will say that $D \subset M^3$ is a regular domain if it has compact closure and piecewise smooth boundary. Let $g: M^3 \to S_1^3$ be the Gauss map of x and let $D \subset M^3$ be a regular domain in M.

We recall that hypersurfaces $x: M^3 \to \mathbb{R}^4$ with $H_2 = 0$ are critical points of the functional $\int_D H_1 \, dM$ for all variations compactly supported in D (see [Re], [Ro] or [BC]). Thus the notion of stability for such hypersurfaces makes sense (see details below) and we can ask for a condition to ensure that a regular domain $D \subset M$ be stable. Surprisingly enough, the condition is essentially the same as for a minimal surface.

Theorem 1.1. *Assume that M is orientable, that the Gauss-Kronecker curvature H_3 is nowhere zero and that the area of $g(D) \subset S_1^3$ is smaller than the area of a hemisphere of S_1^3. Then D is stable and the estimate is sharp.*

Compared with Theorem A, Theorem 1.1 has one additional condition, namely that H_3 is nowhere zero. If H_3 is allowed to be identically zero, then we can find an example (discussed later in 3.9) for which the above theorem is false. Hence, some condition on H_3 is required. See, however, the question 4.2 in the last section of this paper.

Theorem 1.1 is but a particular case of a series of similar results for hypersurfaces $x\colon M^n \to \mathbb{R}^{n+1}$ with $H_{n-1} = 0$. Such hypersurfaces are critical points of the functional $\int_D H_{n-2}\, dM$ for variations compactly supported in D. Thus the notion of stability makes sense, and the following theorem holds for regular domains D in M. Let

$$g\colon M \to S_1^n \subset \mathbb{R}^{n+1}$$

be the Gauss map of x.

Theorem 1.2. *Assume that M is orientable, that the Gauss-Kronecker curvature H_n is nowhere zero and that the area of $g(D)$ is smaller than the area of a hemisphere of S_1^n. Then D is stable and the estimate is sharp.*

The proof of Theorem 1.2 works equally well for a more general situation. To state this more general result, we need a few definitions. Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface and consider the elementary symmetric functions S_r of the principal curvatures of x:

$$S_0 = 1, \quad S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \ldots k_{i_r} \quad (1 \leqq r \leqq n), \quad S_r = 0 \quad (r > n).$$

The r-th mean curvature H_r of x is defined by

$$S_r = \binom{n}{r} H_r.$$

It is known that (see e.g. [Re], [Ro] or [BC]) $H_{r+1} = 0$ iff M is a critical point of the integral $A_r = \int_M S_r\, dM$ for compactly supported variations of M.

From now on, unless explicitly stated, we assume that M is orientable.

To describe the Jacobi equation of such a critical point, it is convenient to introduce the Newton transformations defined inductively by

$$P_0 = I, \quad P_r = S_r I - BP_{r-1}.$$

Here I is the identity matrix and B_p is the linear map of T_pM, $p \in M$, associated to the second fundamental form of x. We then introduce a second order differential operator L_r that will play, in the present case, a rôle similar to that of the Laplacian in the minimal case:

$$L_r(f) = \operatorname{div}(P_r \nabla f), \tag{1}$$

where ∇f is the gradient of f in the induced metric. Notice that L_0 agrees with the Laplacian. Finally, the Jacobi equation can be written as

$$T_r f \stackrel{\text{def}}{=} L_r f - (r+2) S_{r+2} f = 0, \tag{2}$$

where f is the normal component of the variation vector field. Although $L_0 = \Delta$ is always elliptic, some conditions are necessary to ensure that L_r is elliptic. This is contained in a recent work of Hounie-Leite ([HL1], Corollary 2.3) and can be stated as follows:

Assume $H_{r+1} = 0$. *Then* L_r *is elliptic iff* $\operatorname{rank}(B) > r$.

Thus, the hypothesis that H_n is nowhere zero ensures that L_r is elliptic. By (1), the fact that L_r is elliptic is equivalent to the fact that P_r has all its eigenvalues positive or all its eigenvalues negative.

We will denote by $\theta_i(r)$ the eigenvalues of $\sqrt{P_r}B$ when P_r is positive definite and the eigenvalues of $\sqrt{-P_r}B$ when P_r is negative definite.

Assume that $H_{r+1} = 0$, and let $D \subset M$ be a regular domain. We say that D is *r-stable* if the critical point is such that $\left(\frac{d^2 A_r}{dt^2}\right)_{t=0} > 0$, for all variations with compact support in D, or $\left(\frac{d^2 A_r}{dt^2}\right)_{t=0} < 0$ for all such variations. This unusual definition requires some discussion that will be presented in a moment. If the critical point is such that $\left(\frac{d^2 A_r}{dt^2}\right)_{t=0} > 0$ for some variation with compact support in D and $\left(\frac{d^2 A_r}{dt^2}\right)_{t=0} < 0$ for some other variation of the same nature, we say that D is *r-unstable*.

We now justify our definition of stability. In the minimal case, L_r is the Laplacian that is elliptic and can be defined so that the matrix of the coefficients of its principal part is either positive definite or negative definite; once one of these choices is made, we can stick to it to the rest of our investigation. In the present case, however, $L_r = \operatorname{div}(P_r \nabla f)$ depends on the hypersurface and, even when it is elliptic, the symbol can be either positive definite or negative definite and no definite choice can be made once and for all. This is related to the fact that $A_r = \int_D S_r \, dM$ is not necessarily positive like A_0. To circumvent this difficulty, we could proceed as follows. If r is odd and L_r is elliptic for a critical point, we could, by choosing orientation, assume A_r to be positive for that critical point. This follows from Lemma (2.3)(ii) in Section 2 of this paper, i.e., if L_r is elliptic, S_r is always positive or always negative. When r is even, L_r is elliptic and P_r is negative definite, we could change the variational problem into $-A_r$. These choices, besides leaving aside the non-elliptic case, are somewhat artificial.

Our definition of stability is more adequate to the present problem. When L_r is elliptic and P_r is positive definite it gives the usual notion of minimum. When L_r is elliptic and P_r is negative definite, no minimum can exist and the maximum is the natural substitute for it; this is equivalent to looking for the minimum of the new variational problem $-A_r$. A moment's reflection shows that this agrees with the content of the Morse Index Theorem which applies to the case $H_{r+1} = 0$ when $\operatorname{rank}(B) > r$ yielding, for each compact set, a finite number of variations that give maxima, when P_r is positive definite, and a finite number of variations that give minima, when P_r is negative definite.

We need a final definition before stating our main result. We say that a hypersurface $x\colon M^n \to \mathbb{R}^{n+1}$ is *r-special* if

$$\frac{|S_n|}{\sum_{j=1}^{n} \theta_i^2(r)} = \text{constant}.$$

Theorem 1.3. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be an r-special immersion with $H_{r+1} = 0$ and $H_n \neq 0$ everywhere. Let $D \subset M$ be a regular domain and $g\colon M \to S_1^n$ be the Gauss map of x. If the area of $g(D)$ is smaller than the area of a spherical cap $C_\tau \subset S_1^n$ whose first eigenvalue for the spherical Laplacian is τ, where*

$$\tau = \max_{i,D} \left(\sum_{j=1}^{n} \theta_j^2(r)/\theta_i^2(r) \right),$$

then D is r-stable.

The estimate of Theorem 1.2 is a simple corollary of Theorem 1.3, because, as we will show in Lemma 2.4, under the condition $H_{n-1} = 0$, the eigenvalues θ_i of $\sqrt{P_{n-2}B}$ (or of $\sqrt{-P_{n-2}B}$) satisfy $\theta_i^2 = -S_n$. Thus

$$\frac{|S_n|}{\sum_{j=1}^{n} \theta_j^2} = \frac{|S_n|}{-nS_n} = \frac{1}{n},$$

hence the immersion is $(n-2)$-special. Furthermore

$$\tau = \max_{i,D} \left(\sum_j \theta_j^2/\theta_i^2 \right) = \max_{i,D}(-nS_n/-S_n) = n,$$

hence the spherical cap $C_\tau = C_n$ is a hemisphere of S_1^n, and this shows what we claimed. That the estimate of Theorem 1.2 is sharp will be shown in Section 3.

Remark 1.4. Although it is conceivable that there exist examples of r-special hypersurfaces with $H_{r+1} = 0$ and $H_n \neq 0$ everywhere, other than those of Theorem 1.2, we have not yet found them. The point of Theorem 1.3 is that it is not necessary that $|S_n|/\sum \theta_i^2(r)$ be $1/n$. It can be any constant and the proof works equally well.

Remark 1.5. As we will show in the proof of Theorem 1.3, we can prove a little more, namely we can replace the condition that area $g(D) <$ area C_τ by the condition area $g(D) \leqq$ area C_τ.

We can also prove an instability result:

Theorem 1.6. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be an r-special immersion with $H_{r+1} = 0$ and $H_n \neq 0$ everywhere. Let $D \subset M$ be a regular domain and $g\colon M^n \to S_1^n$ be the Gauss map of x. Assume, in addition, that g restricted to $\overline{D}$ is a covering onto $g(\overline{D})$. If the first eigenvalue of $g(D)$ for the spherical Laplacian is smaller than γ, where*

$$\gamma = \min_{i,D}\left(\sum_{j=1}^{n}\theta_j^2(r)/\theta_i^2(r)\right),$$

then D is r-unstable.

The following corollary is immediate from the considerations after Theorem 1.3. It generalizes a result of Schwarz for minimal surfaces (see [BdC], Theorem 2.7).

Corollary 1.7. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface with $H_{n-1} = 0$ and $H_n \neq 0$ everywhere. Let $D \subset M$ be a regular domain and let $g\colon M^n \to S_1^n$ be the Gauss map of x. Assume, in addition, that g restricted to $\overline{D}$ is a covering map onto $g(\overline{D})$. If the first eigenvalue of $g(D)$ for the spherical Laplacian is smaller than n, then D is r-unstable.*

Remark 1.8. There are many examples of hypersurfaces $x\colon M^n \to \mathbb{R}^{n+1}$ with $H_r = 0$ and $H_n \neq 0$ everywhere. For instance, all rotation hypersurfaces with $H_r = 0$ (see [HL] or [P1]) have this property. Also, in the hypersurfaces $x\colon M^{2p+1} \to \mathbb{R}^{2p+2}$ that are invariant by $0(p+1) \times 0(p+1)$ and have $H_r = 0$, in most cases, $H_n = 0$ only at one (compact) orbit, and any domain $D \subset M$ that does not meet such orbit satisfies $H_n \neq 0$ everywhere; the case $r = 2$ is fully treated in [P2] $(p = 1)$ and [Sa] $(p > 1)$.

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2. Preliminaries

Let W be an n-dimensional Riemannian manifold and let $D \subset W$ be a regular domain in W. Let us denote by $C_0^\infty(D)$ (respectively $C_c^\infty(D)$) the set of smooth functions which are zero on ∂D (respectively with compact support in D).

We now recall some properties and results concerning the first eigenvalue of an elliptic, self-adjoint, linear differential operator $T\colon C_0^\infty(D) \to C^\infty(D)$ of second order. Elliptic means that the matrix of the coefficients of the principal part of T is either positive definite or negative definite; for convenience, we assume here that this matrix is positive definite. We recall that the first eigenvalue $\lambda_1^T(D)$ of T is defined as the smallest λ that satisfies

$$T(g) + \lambda g = 0, \tag{3}$$

for some nonzero function $g \in C_0^\infty(D)$. A nonzero function g in $C_0^\infty(D)$ that satisfies (3) for $\lambda = \lambda_1^T$ is called a first eigenfunction of T in D.

Lemma 2.1. *If D and D′ are domains in M with $D \subset D'$ then $\lambda_1^T(D) \geqq \lambda_1^T(D')$ and equality holds iff $D = D'$.*

For a proof see [Sm], Lemma 2 and notice that T satisfies the unique continuation principle (see [A]).

Lemma 2.2.

$$\lambda_1^T(D) = \inf\left\{ \frac{-\int_D f T(f)\, dM}{\int_D f^2\, dM}; f \in H^1(D); f \not\equiv 0 \right\},$$

where $H^1(D)$ denotes the Sobolev space over D.

For a proof, see [Sm], Lemma 4(a). For the definition of $H^1(D)$, see [Sm], proof of Lemma 2.

Let $e_1, e_2, \dots, e_n$ be orthonormal eigenvectors of B corresponding, respectively, to the eigenvalues $k_1, k_2, \dots, k_n$. We represent by B_i the restriction of the transformation B to the subspace normal to e_i and by $S_r(B_i)$ the r-symmetric function associated to B_i. The proof of the following lemma can be found in [BC], Lemma 2.1.

Lemma 2.3. *For each $1 \leqq r \leqq n-1$, we have*:

(i) $P_r(e_i) = S_r(B_i)e_i$ *for each* $1 \leqq i \leqq n$.

(ii) $\operatorname{trace}(P_r) = \sum_{i=1}^{n} S_r(B_i) = (n-r)S_r$.

(iii) $\operatorname{trace}(BP_r) = \sum_{i=1}^{n} k_i S_r(B_i) = (r+1)S_{r+1}$.

(iv) $\operatorname{trace}(B^2 P_r) = \sum_{i=1}^{n} k_i^2 S_r(B_i) = S_1 S_{r+1} - (r+2)S_{r+2}$.

The following lemma has been used in the Introduction.

Lemma 2.4. *Let $S_{n-1} = 0$. Then $\theta_i^2(r) = -S_n$, for $r = n-2$.*

Proof. In fact, we have by definition

$$0 = S_{n-1} = k_i S_{n-2}(B_i) + S_{n-1}(B_i)$$

and thus, by Lemma 2.3 (i), we obtain, for $r = n-2$,

$$\theta_i^2(r) = k_i^2 S_{n-2}(B_i) = k_i\big(-S_{n-1}(B_i)\big) = -S_n. \quad \square$$

When working with minimal immersions one uses variations vanishing on the boundary ∂D of a regular domain D. We now explain why we use variations with support in D, rather than variations vanishing on ∂D, when describing the variational problem associated to the immersions with vanishing $r+1$-mean curvature. Denote by X a variation of D and let $E = \dfrac{\partial X}{\partial t}$ be its variational vector. Set $f = \langle E, N \rangle$, where N is the unit normal field along $x(D)$ and denote by ν the unit normal field along $x(\partial D)$ tangent to $x(D)$. The formula for the first variation of the functional

$$A_r = \int_D S_r \, dM,$$

is

$$A_0'(0)(f) = \int_D [-S_1] f \, dM + \int_{\partial D} \langle E, v \rangle \, ds,$$

$$A_r'(0)(f) = \int_D [-(r+1)S_{r+1}] f \, dM + \int_{\partial D} [\langle P_{r-1} \nabla f, v \rangle + S_r \langle E, v \rangle] \, ds, \quad r \geqq 1.$$

Here ∇f is the gradient of f, ds is the element of volume of ∂D. Notice that to eliminate the boundary term when $r = 0$ it is enough to assume that the variation fixes the boundary. However, to do the same when $r \geqq 1$, we need an additional condition, namely $\nabla f|_{\partial D} = 0$ or $\langle \nabla f, P_{r-1} v \rangle|_{\partial D} = 0$. Thus, it is convenient to consider variations with support contained in D, and, for unification purposes, we will do that even when $r = 0$.

We can use Stokes Theorem and the self-adjointness of P_r to see that L_r is self-adjoint. Thus we can define a bilinear symmetric form by

$$I_r(f, g) = -\int_M f \, T_r(g) \, dM,$$

and it can be proved that

$$A_r''(0)(f) = I_r(f, f).$$

Definition 2.5. We say that D is r-stable if $I_r(f, f) > 0$ for all $f \in C_c^\infty(D)$ or if $I_r(f, f) < 0$ for all $f \in C_c^\infty(D)$. We say that D is r-unstable if there exists a function $f \in C_c^\infty(D)$ such that $I_r(f, f) < 0$ and there exists a function $g \in C_c^\infty(D)$ such that $I_r(g, g) > 0$.

Remark 2.6. Along the paper we will work with the case that P_r is positive definite and will make some comments for the case that P_r is negative definite at appropriate places.

Because of Lemma (2.3)(iv) and the fact that $S_{r+1} = 0$, we can rewrite T_r as

$$T_r = L_r + \operatorname{trace}(B^2 P_r) = L_r + \|\sqrt{P_r} B\|^2,$$

where

$$\|\sqrt{P_r} B\|^2 = \sum_{i=1}^n \theta_i^2(r).$$

The following lemma has been proved in [T] for $r = 1$. Except for the fact that we must consider two cases, according to the positivity (or negativity) of P_r, the proof below is similar to that of [T]. We included it here for completeness.

Lemma 2.7. *The following statements are equivalent*:

(i) *$\exists f \in C_c^\infty(D)$ such that $I_r(f, f) \leqq 0$.*

(ii) $\exists f \in C_c^\infty(D)$ *such that* $I_r(f,f) < 0$.

(iii) $\exists f \in C_0^\infty(D)$ *such that* $I_r(f,f) < 0$.

Proof. (i) $\Rightarrow$ (ii): Suppose (i) is true and (ii) is not true. Then, $I_r(g,g) \geqq 0$ for all $g \in C_c^\infty(D)$ and there exists a nonzero function $f \in C_c^\infty(D)$ such that $I_r(f,f) = 0$. Let t be a real number. Since I_r is bilinear and $I_r(f,f) = 0$, we have for $g \in C_c^\infty(D)$ and for all t that

$$I_r(f+tg, f+tg) = 2tI_r(f,g) + t^2\big(I_r(g,g)\big)^2.$$

But $I_r(f+tg, f+tg) \geqq 0$ for all real t and then we must have $I_r(f,g) = 0$. Thus, $I_r(f,g) = 0$ for all $g \in C_c^\infty(D)$, which implies that $T_r(f) = 0$. Since f has compact support, it vanishes on an open set and then, by the unique continuation principle (cf. [A]), $f = 0$ on D, a contradiction.

(iii) $\Rightarrow$ (ii): Let f be the function in the statement of (iii). Set

$$D_R = \{p \in D; \operatorname{dist}(p, \partial D) > R\}.$$

For R small enough, D_R is a domain with smooth boundary. Let $\phi_R \in C_c^\infty(D)$ be such that $\phi_R = 1$ in D_R, $|\nabla\phi_R| < 2/R$ in D and $0 \leqq \phi_R \leqq 1$ in D. Define $g = \phi_R f$. Using Stokes Theorem we have

$$I_r(f,f) = \int_D \langle \nabla f, P_r \nabla f\rangle\, dM + \int_D \big((r+2)S_{r+2}\big) f^2\, dM$$

and

$$I_r(g,g) - I_r(f,f) = \int_D \big\{(\phi_R^2 - 1)\big[\langle \nabla f, P_r\nabla f\rangle + \big((r+2)S_{r+2}\big)f^2\big] + 2f\phi_R\langle \nabla f, P_r \nabla\phi_R\rangle$$
$$+ f^2\langle \nabla\phi_R, P_r\nabla\phi_R\rangle\big\}\, dM.$$

The function under the integral is zero in D_R and we claim that it is bounded independently of R on $D \backslash D_R$. Then taking R small enough, we see that $I_r(g,g) < 0$ and this proves the proposition. We have then to prove the claim. By using the positivity of P_r, we can see that

$$(\phi_R^2 - 1)\big[\langle \nabla f, P_r\nabla f\rangle + \big((r+2)S_{r+2}\big)f^2\big] \leqq |(r+2)S_{r+2}| f^2.$$

The Cauchy-Schwarz inequality yields

$$2f\phi_R\langle \nabla f, P_r\nabla\phi_R\rangle + f^2\langle\nabla\phi_R, P_r\nabla\phi_R\rangle \leqq 2|f|\,|\nabla f|\,|P_r|\,|\nabla\phi_R|$$
$$+ f^2|P_r|\,|\nabla\phi_R|^2.$$

Finally, we use that $|f| \leqq R \sup_{\bar{D}} |\nabla f|$ in $D \backslash D_R$, which is a consequence of the Mean Value Theorem, and that $|\nabla\phi_R| < \dfrac{2}{R}$ to obtain

$$2f\phi_R\langle\nabla f, P_r\nabla\phi_R\rangle + f^2\langle\nabla\phi_R, P_r\nabla\phi_R\rangle \leqq 4\Big(\sup_{\bar{D}}|\nabla f|\Big)^2|P_r| + 4|P_r|\Big(\sup_{\bar{D}}|\nabla f|\Big)^2.$$

Thus the claim is proved.

All the other implications are trivial. □

Remark 2.8. According to our convention, we have assumed P_r to be positive definite. If P_r is negative definite, the first estimate above becomes

$$(1-\phi_R^2)[\langle\nabla f, -P_r\nabla f\rangle - (r+2)S_{r+2}f^2] \leqq |-P_r|\,|\nabla f|^2 + |(r+2)S_{r+2}|f^2,$$

all the other estimates remaining the same. Thus, the same conclusion holds.

Since we are supposing that the Gauss Kronecker curvature $H_n = \det(B)$ is nowhere zero, the Gauss map is an immersion; hence it can be used to define a new Riemannian metric $\tilde{s}$ on M by setting

$$\tilde{s}(X, Y) = \langle BX, BY\rangle, \quad X, Y \in T(M),$$

where $\langle\,,\rangle$ denotes the Riemannian metric induced in M by $\mathbb{R}^{n+1}$ and $T(M)$ is the tangent bundle of M. Notice that the metric $\tilde{s}$ is the pull-back by the Gauss map of the metric of the sphere. Thus, the sectional curvature of the metric $\tilde{s}$ is one.

Let ∇f denote the gradient of f in the metric $\langle\,,\rangle$ and $\tilde{\nabla} f$ denote the gradient of f in the metric $\tilde{s}$. We have the following lemma.

Lemma 2.9. *For any smooth function f on M, we have $\nabla f = B^2\tilde{\nabla} f$.*

Proof. Let $v \in T(M)$. If df denotes the differential of f, we have, by using the self-adjointness of B, that

$$\langle\nabla f, v\rangle = df(v) = \tilde{s}(\tilde{\nabla} f, v) = \langle B\tilde{\nabla} f, Bv\rangle = \langle B^2\tilde{\nabla} f, v\rangle$$

and then $\nabla f = B^2\tilde{\nabla} f$. □

3. Proofs

3.1. Proof of Theorem 1.3. We will need a few facts from Linear Algebra. Let V be a vector space with a positive definite inner product $\langle\,,\rangle$ and let $A = V \to V$ be a self-adjoint linear map. Then it is easily checked that the norm $|A|$ of A, namely

$$|A| = \sup_X\{|AX|; X \in V, |X| = 1\},$$

is given by $|A| = \max_i|\lambda_i|$, where λ_i, $i = 1, \ldots, n$, is an eigenvalue of A. It follows that

$$|A^{-1}| = \max_i(1/|\lambda_i|).$$

Now, set $V = T_pM$, $A = \dfrac{\sqrt{P_r}B}{\|\sqrt{P_r}B\|}$, and let p vary in D to obtain

$$\max_D |A^{-1}|^2 = \max_D \left(\max_i \frac{1}{|\lambda_i|^2} \right) = \max_{i,D} \frac{1}{\lambda_i^2}\cdot$$

Since $\lambda_i = \theta_i \big/ \sqrt{\sum_j \theta_j^2}$, we conclude that

$$(4) \qquad \tau = \max_{i,D} \left(\sum_j \theta_j^2/\theta_i^2 \right) = \max_{i,D} \frac{1}{\lambda_i^2} = \max_D \left| \left(\frac{\sqrt{P_r}B}{\|\sqrt{P_r}B\|} \right)^{-1} \right|^2.$$

Now we will go into the proof proper and will use an idea of Fischer-Colbrie and Schoen [FC-S]. Actually, we will prove a slightly stronger theorem, namely if the area of $g(D)$ is smaller *or equal* than the area of a spherical cap C_τ whose first eigenvalue for the spherical Laplacian $\tilde{\Delta}$ is τ, then D is r-stable. Assume that area $g(D) \leqq$ area C_τ. Then, since symmetrization of domains in the sphere does not increase eigenvalues, we obtain that the first eigenvalue $\lambda_1(g(D))$ of $g(D)$ satisfies $\lambda_1(g(D)) \geqq \tau$. Let f be the first eigenfunction of $g(D)$, that is, $f > 0$ in $g(D)$, $f = 0$ in $\partial(g(D))$ and f satisfies

$$\tilde{\Delta} f + \lambda_1 f = 0.$$

Let $u = f \circ g$ be defined in $D \subset M$ and consider in M the pull-back metric $\tilde{s}$. With this metric, $g\colon M^n \to S_1^n$ is a local isometry and u satisfies again $\tilde{\Delta} u + \lambda_1 u = 0$. Since $\det(dg) \neq 0$, we have that $g(\operatorname{int} \bar{D}) \subset \operatorname{int}(g(\bar{D}))$, where int( ) denotes the interior of the enclosed set; it follows that $u > 0$ in (the open set) D. (Notice that u may be positive in parts of ∂D so λ_1 is not necessarily the first eigenvalue of D.)

Since $u > 0$ in D and $\tilde{\Delta} u + \lambda_1 u = 0$ we can use [FC-S], Corollary 1, to conclude that the first eigenvalue of the operator $\tilde{\Delta} + \lambda_1$ is nonnegative, that is,

$$\inf_h \left(\int_D ([\tilde{\nabla} h]^2 - \lambda_1 h^2)\, dS \right) \geqq 0,$$

where the infimum is taken over all $C_0^\infty(D)$ functions that satisfy $\int_D h^2\, dS = 1$; here $\tilde{\nabla}$ is the gradient, [ ] is the norm of a vector, and dS is the volume element in the metric $\tilde{s}$. Since $\lambda_1 \geqq \tau$, we obtain

$$\inf_h \left(\int_D ([\tilde{\nabla} h]^2 - \tau h^2)\, dS \right) \geqq 0.$$

Observe now that $dS = |S_n|\, dM$. But the immersion is r-special, hence

$$|S_n| = c\|\sqrt{P_r}B\|^2,$$

where c is a positive constant. It follows that

$$\inf_h \left(\int_D ([\tilde{\nabla} h]^2 - \tau h^2) \right) c \|\sqrt{P_r} B\|^2 \, dM \geqq 0.$$

Now, by using (4), we obtain that for any vector $X = T_p M$,

$$[X]^2 \leqq \left[\left(\frac{\sqrt{P_r} B}{\|\sqrt{P_r} B\|} \right)^{-1} \right]^2 \left[\frac{\sqrt{P_r} BX}{\|\sqrt{P_r} B\|} \right]^2 \leqq \tau \left[\frac{\sqrt{P_r} BX}{\|\sqrt{P_r} B\|} \right]^2.$$

Thus

$$\begin{aligned} 0 &\leqq \inf_h \left(\int_D ([\tilde{\nabla} h]^2 - \tau h^2) c \|\sqrt{P_r} B\|^2 \, dM \right) \\ &\leqq \inf_h \tau \left(\int_D \left(\frac{[\sqrt{P_r} B \tilde{\nabla} h]^2}{\|\sqrt{P_r} B\|^2} c \|\sqrt{P_r} B\|^2 - h^2 c \|\sqrt{P_r} B\|^2 \right) dM \right) \\ &= \inf_h \tau c \left(\int_D ([\sqrt{P_r} B \tilde{\nabla} h]^2 - \|\sqrt{P_r} B\|^2 h^2) \, dM \right). \end{aligned}$$

By Lemma 2.9, $\tilde{\nabla} h = B^{-2} \nabla h$, where ∇ is the gradient in the original metric. Furthermore P_r commutes with B and the norms of vectors in these two metrics are related by $[X] = |BX|$, with obvious notation. We finally obtain that

$$\begin{aligned} 0 &\leqq \tau c \inf_h \left(\int_D ([\sqrt{P_r} B \tilde{\nabla} h]^2 - \|\sqrt{P_r} B\|^2 h^2) \, dM \right) \\ &= \tau c \inf_h \left(\int_D (|\sqrt{P_r} \nabla h|^2 - \|\sqrt{P_r} B\|^2 h^2) \, dM \right). \end{aligned} \tag{5}$$

Since the Jacobi operator is

$$T_r u = \operatorname{div}(P_r \nabla u) + \|\sqrt{P_r} B\|^2 u,$$

the above inequality means that the first eigenvalue of the Jacobi operator T_r in D is nonnegative. This implies that $I_r(h, h) \geqq 0$, for all $h \in C_0^\infty(D)$. By Lemma 2.7, this is equivalent to the fact that $I_r(h, h) > 0$, for all $h \in C_c^\infty(D)$, that is, D is r-stable as we wished to prove. □

Remark 3.3. If P_r is negative definite $-P_r = Q_r$ is positive definite. Since, in this case,

$$\|\sqrt{Q_r} B\|^2 = \sum_j \theta_j^2(r),$$

it is easily seen that we can replace $\|\sqrt{P_r} B\|^2$ by $\|\sqrt{Q_r} B\|^2$ throughout the above proof. In this case, the operator T_r becomes

$$T_r u = \operatorname{div}(P_r \nabla u) + \operatorname{trace}(P_r B^2) u$$
$$= -\{\operatorname{div}(Q_r \nabla u) + \operatorname{trace}(Q_r B^2) u\}.$$

Then inequality (5) with $\|\sqrt{Q_r}B\|^2$ means that the index form I_r is negative definite in $C_c^\infty(D)$. Thus, according to our definition, D is r-stable.

3.4. Proof of Theorem 1.6. Let again λ_1 be the first eigenvalue and f be the first eigenfunction of $g(D)$ for the spherical Laplacian; thus $f > 0$ on $g(D)$, $f = 0$ in $\partial(g(D))$ and $\tilde{\Delta} f + \lambda_1 f = 0$ in $g(D)$. Since g restricted to $\overline{D}$ is a covering map onto $g(\overline{D})$, we have that $g(\partial D) = \partial(g(D))$. Thus $u = f \circ g$ is positive in D, $u = 0$ in ∂D and $\tilde{\Delta} u + \lambda_1 u = 0$ in D. By Stokes Theorem,

$$0 = \int_D ([\tilde{\nabla} u]^2 - \lambda_1 u^2)\, dS > \int_D ([\tilde{\nabla} u]^2 - \gamma u^2) c \|\sqrt{P_r} B\|^2\, dM,$$

where in the last inequality we have used that $\lambda_1 < \gamma$ and that the immersion is r-special.

By using the definition of γ and proceeding as in the proof of Theorem 1.3, we will obtain that

$$\frac{1}{\gamma} = \max_D \left[\frac{\sqrt{P_r} B}{\|\sqrt{P_r} B\|}\right]^2$$

and that

$$\left[\frac{\sqrt{P_r} B X}{\|\sqrt{P_r} B\|}\right]^2 \leqq \frac{1}{\gamma} [X]^2.$$

It follows that

$$\begin{aligned} 0 &> \int_D ([\tilde{\nabla} u]^2 - \gamma u^2) c \|\sqrt{P_r} B\|^2\, dM \\ &\geqq \int_D \left(\gamma \frac{[\sqrt{P_r} B \tilde{\nabla} u]^2}{\|\sqrt{P_r} B\|^2} - \gamma u^2\right) c \|\sqrt{P_r} B\|^2\, dM \\ &= \gamma c \int_D ([\sqrt{P_r} B \tilde{\nabla} u]^2 - \|\sqrt{P_r} B\|^2 u^2)\, dM \\ &= \gamma c \int_D (|\sqrt{P_r} \nabla u|^2 - \|\sqrt{P_r} B\|^2 u^2)\, dM. \end{aligned}$$

Therefore, there exists a $u \in C_0^\infty(D)$ such that $I_r(u,u) < 0$. By Lemma 2.7, this is equivalent to the fact that there exists $u \in C_c^\infty(D)$ such that $I_r(u,u) < 0$, that is, D is r-unstable as we wished to prove. □

Remark 3.5. An argument similar to that of Remark 3.3 applies here.

Remark 3.6. It follows from the proof that we can drop the condition on the restriction $g|D$ being a covering map if we replace the condition on the first eigenvalue of $g(D)$ by a similar condition on the first eigenvalue of D. Namely, the following statement holds: *If the first eigenvalue of D for the Laplacian* $\tilde{\Delta}$ *in the pull-back metric is smaller than* γ, *then D is r-unstable*. Of course, the same remark applies to Corollary 1.7.

3.7. Proof that the estimate of Theorem 1.2 is sharp. We first recall that the support function $\varphi = \langle x, N \rangle$ of a hypersurface $x\colon M^n \to \mathbb{R}^{n+1}$ with $S_{r+1} = \text{const.}$ satisfies (see [Ro], last equation on p. 227)

$$L_r\varphi + \big(S_1 S_{r+1} - (r+2)S_{r+2}\big)\varphi = -(r+1)S_{r+1}.$$

Here N is a unit normal vector and x is any position vector. In the situation of Theorem 1.3, $r+1 = n-1$, $S_{r+1} = 0$, and the above equation becomes

$$L_r\varphi - (r+2)S_n\varphi = 0.$$

But this means that φ satisfies the Jacobi equation (2).

Next, we look into the classification of rotation hypersurfaces $x\colon M^n \to \mathbb{R}^{n+1}$ with $S_{r+1} = 0$ (see [HL2] or [P]). Following [HL2], we choose x_1 as the rotation axis and let the profile curve be given as a positive function $x_{n+1} = f(x_1)$; here $(x_1, \ldots, x_{n+1})$ are coordinates in $\mathbb{R}^{n+1}$. It is shown that f is a convex function, symmetric relative to the axis x_{n+1}. For case $x\colon M^3 \to \mathbb{R}^4$ with $S_2 = 0$, the profile curve is a parabola: $f(x_1) = 1 + (x_1)^2/4$. In all other cases $x\colon M^n \to \mathbb{R}^{n+1}$ with $S_{n-1} = 0$ the profile curve behaves like a parabola and goes to infinity with x_1; more precisely,

$$f(x_1) = C|x_1|^{\frac{n-1}{n-2}}\big(1 + O(|x|^{-1})\big), \quad |x_1| \to \infty,$$

where C is a positive constant.

From the above facts, we conclude that Lindelöf's method to obtain conjugate boundaries of a rotation minimal surface ([L]) works equally well in our case. For completeness, we will present here a brief description of this method. Given a point p on the profile curve C, we draw the tangent line t to C at p and let 0 be the intersection of t with the rotation axis. From 0 we draw another tangent t_1 to C that touches C at the point q. The boundaries B_1 and B_2 generated by p and q as C goes about the rotation axis are easily seen to be conjugate boundaries: just take the support function $\varphi = \langle x, N \rangle$, the position vector x having 0 as origin and notice that the Jacobi field φN vanishes on B_1 and B_2 and nowhere else in the region of M bounded by them.

By the same argument used in [BdC] for minimal surfaces, we can see that, by choosing appropriately B_1 and B_2, an unstable domain of a rotation hypersurface with $S_{n-1} = 0$ can be obtained whose spherical image has area that is larger than the area of a hemisphere H of S_1^n and is as close as we wish to this area. Thus the estimate is sharp.

Remark 3.8. We owe the referee the following interesting observation. The rotational 3-hypersurface with vanishing scalar curvature, whose profile curve is $f(x) = 1 + (x^2/4)$, is a time-symmetric $t = 0$ slice of the $3+1$ dimensional Schwarzschild metric for a static non-rotating black hole of mass $1/2$.

3.9. Example. We now describe an example of an unstable domain D in a hypersurface $x\colon M^3 \to \mathbb{R}^4$ with $S_2 = 0$, $S_3 = 0$ and area $g(D) = 0$. This was mentioned in the Introduction and shows that some condition on the zeroes of S_3 are necessary for the validity of Theorem 1.1.

Let x_1, x_2, x_3, x_4 be coordinates in $\mathbb{R}^4$ and let M^3 be a cylinder over a curve $c(t)$ in the plane x_3x_4, that is, M is generated by a plane P_t, parallel to the plane x_1x_2, that displaces itself along the curve $c(t)$.

Choose an orthonormal frame e_1, e_2, e_3, e_4 along M with e_1, e_2 in the plane P_t, e_3 tangent to $c(t)$ and e_4 normal to M. Then $k_1 = k_2 = 0$, k_3 is the curvature of $c(t)$ and

$$T_1 f = L_1 f - 3S_3 f = L_1 f = \operatorname{div}(P_1 \nabla f).$$

Since $P_1 = S_1 I - B$ we obtain that the eigenvalues of P_1 are $k_3, k_3, 0$. Thus for $D \subset M$ and f with compact support in D we have

$$(6) \qquad -\int_D f T_1 f \, dM = -\int_D f \operatorname{div}(P_1 \nabla f) \, dM = \int_D \langle P_1 \nabla f, \nabla f \rangle \, dM$$

$$= \int_D k_3 |(\nabla f)_{P_t}|^2 \, dM$$

where we have used that $\nabla f = \sum f_i e_i$ and denoted by $(\nabla f)_{P_t}$ the projection of ∇f over P_t.

Choose now a domain $D \subset M$ bounded by two parallel planes P_{t_1}, P_{t_2} and by the intersections of P_t, $t \in [t_1, t_2]$ with the planes $x_1 = x_2 = 0$ and $x_1 = x_2 = h$ (a cylinder with height h). Assume that k_3 has one single zero in $[t_1, t_2]$ where it changes sign. Assume further that an orientation has been fixed so that

$$\int_D H \, dM = \int_D k_3 \, dM > 0$$

and choose f so that its support is contained in that part of D where $k_3 < 0$. Then, by (6)

$$I(f, f) = -\int_D f T_1 f \, dM < 0.$$

Of course, by choosing f so that its support is contained in the part of D where $k_3 > 0$, we obtain that $I(f, f) > 0$.

Thus D is unstable and it is easily seen that this fact does not depend on the chosen orientation. Since M^3 is a cylinder, the spherical image of D has area zero, and yet Theorem 1.1 does not hold in this example. Notice that we can choose the curve $c(t)$ so that the example is graph over the 3-space x_1, x_2, x_3.

4. Comments and questions

4.1. The proof of Theorem 1.3 works equally well if we consider $H_{r+1} = \text{const} > 0$ rather than $H_{r+1} = 0$. However, in order to make sure that L_r is elliptic, we must introduce the additional assumption that there is a point $p \in M$ such that all principal curvatures at p are positive. (See [BC], Proposition 3.2. The Proposition is stated for compact hypersurfaces; the compactness, however, is only used to ensure the existence of a point where all principal curvatures are positive.) In this case, P_r is always positive definite and the Jacobi equation is again

$$T_r f \stackrel{\text{def}}{=} L_r f + [\sqrt{P_r} B]^2 f = 0.$$

Further details on the variational problem for $H_{r+1} = \text{const}$ can be found in [BC], [Re] or [Ro].

The main issue here is that we have not been able to obtain explicit examples (except the round sphere) of r-special hypersurfaces with $H_{r+1} = \text{const} > 0$. To find out such examples might be an interesting question.

4.2. It is not clear what is the weakest assumption that we must set on the zeroes of H_n for the validity of Theorem 1.3. As we have seen in Example 3.9, if H_n is identically zero the theorem does not hold (the example was worked out for $n = 3$ but a similar construction can be made for an arbitrary n). We believe that the following is true: *If the set of zeroes of H_n has codimension $\geqq 2$ in M^n, and is contained in D, then Theorem* 1.3 *holds.*

4.3. Probably the most interesting question about Theorem 1.2 is to determine which complete hypersurfaces $x\colon M^n \to \mathbb{R}^{n+1}$ with $H_{n-1} = 0$ are stable (that is, each of its domain is stable). Let us look into the case $n = 3$, where we have some examples. If in Example 3.9 we take the base curve $c(t)$ to have curvature $k_3(t) > 0$, then we have an example of a complete stable $M^3 \subset R^4$; however, if k_3 changes sign, we have shown that M^3 is unstable. In analogy with the case of a complete minimal surface in $\mathbb{R}^3$, where minimality plus stability imply that the Gaussian curvature vanishes, we can ask whether the stability of a complete $x\colon M^3 \to \mathbb{R}^4$ with $H_2 = 0$ implies that H_3 is identically zero, or, more generally: *are the stable complete $x\colon M^n \to \mathbb{R}^{n+1}$ with $H_{n-1} = 0$ contained in the class of hypersurfaces that have both $H_{n-1} = 0$* and *$H_n = 0$*? If this is the case, we should be able to extract all stable hypersurfaces from the above class.

We have reasons to believe that the following conjecture is true: *there exists no complete stable $x\colon M^n \to \mathbb{R}^{n+1}$ with $H_{n-1} = 0$ and $H_n \neq 0$ everywhere.*

4.4. Let $\overline{M}^{n+1}(\delta)$ be a space of constant sectional curvature δ, $\delta \leqq 0$, and let $x\colon M^n \to \overline{M}^{n+1}(\delta)$ be an r-special hypersurface with $H_{r+1} = 0$ and $H_n \neq 0$ everywhere. Then our proof of Theorem 1.3 works to show that if the first eigenvalue of the Laplacian $\tilde{\lambda}_1$ of $D \subset M^n$, in the metric $\langle\langle\, , \rangle\rangle$ defined at the end of Section 2, satisfies $\tilde{\lambda}_1 > \tau$, where τ is defined in Theorem 1.3, then D is r-stable. We have only to observe that the Jacobi operator is now $L_r + \|\sqrt{P_r} B\|^2 + \delta \operatorname{trace} P_r$. Of course, a theorem like Theorem 1.2 (where we replace the condition on the area of $g(D)$ by the condition that $\tilde{\lambda}_1(D) > n$) also holds, although we do not know if the estimate is sharp.

By the same token, our proof of Theorem 1.6 works to show that, in the above situation, but with $\delta > 0$, if $\tilde{\lambda}_1(D) < \gamma$, and γ is defined as in Theorem 1.6, then D is r-unstable.

The question here is whether one can estimate $\tilde{\lambda}_1(D)$ in terms of say, $\int_D H_n \, dM$, or some other geometric invariant associated to D. Probably it will be necessary to estimate the sectional curvature of M in the metric $\langle\langle\, , \rangle\rangle$. For that, one might use the expression obtained for this sectional curvature in ([dCD], Theorem 1.2 (iii) and Remark (1.5)); in low dimensions, this should be manageable. A further question is to extend Theorem 1.3 to the case $\delta > 0$.

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