Stable Minimal Hypersurfaces in Euclidean Spaces

HILÁRIO ALENCAR[1], **MANFREDO DO CARMO**[2] **and MARIA FERNANDA ELBERT**[3]

[1]UFAL, Departamento de Matemática – 57072–970 Maceió, AL

[2]IMPA, Estrada Dona Castorina 110–22460–320 Rio de Janeiro, RJ

[3]UFRJ, Departamento de Matemática, Cx. Postal 68530–21844 Rio de Janeiro, RJ

ABSTRACT

Given a domain $D \subset M^n$ in a minimal hypersurface $M^n \subset \mathbb{R}^{n+1}$, we obtain conditions on M and D which ensure that D is stable and generalize a well known result for $n = 2$.

Key words: stability, hypersurface, Gauss-Kronecker curvature, eigenvalues, self adjoint elliptic operators.

INTRODUCTION

In this paper, we want to present some results on stability of bounded domains $D \subset M^n$ of minimal hypersurfaces $x \colon M^n \to \mathbb{R}^{n+1}$ in the euclidean space $\mathbb{R}^{n+1}$. The particular case of a minimal surface in $\mathbb{R}^3$ has been understood for some time and one of its forms reads as follows:

Theorem A. (Barbosa-do Carmo (1976)). *Let $x \colon M^2 \to \mathbb{R}^3$ be a minimal surface in $\mathbb{R}^3$ and let $D \subset M^n$ be a bounded domain with smooth boundary. Assume that*

$$\int_D |K| dM < 2\pi,$$

where K is the gaussian curvature of M. Then, D is stable.

To generalize this theorem for higher dimensions, one has several possible choices of objects that generalize K. Here we choose the Gauss-Kronecker curvature $H_n = k_1, \cdots, k_n$, where $k_i, i = 1, \cdots, n$, are the principal curvatures of x. In case x is minimal, we can prove the theorem below for the statement of which we need some definitions.

We say that an immersion $x \colon M^n \to \mathbb{R}^{n+1}$ is special if

$$\frac{|H_n|}{\sum_i k_i^2} = \text{const.}$$

It can be shown that x is special with $H_n \neq 0$ if and only if a certain differential operator associated to the immersion is self-adjoint (see Lemma 2 in Section 2 below). Notice that all minimal surfaces in $\mathbb{R}^3$ are special, since $k_1 = -k_2$, hence

$$k_1 k_2 / (k_1^2 + k_2^2) = -\frac{1}{2}.$$

Correspondence to: Manfredo do Carmo

Hilário Alencar, Manfredo do Carmo and Maria Fernanda Elbert were partially supported by the National Council for Scientific and Technological Development – CNPq of Brazil.

In the last Section of this paper, we will come back to the question of examples of special surfaces.

Let $C \subset S_1^n$ be a spherical cap in the unit n-sphere S_1^n and denote by $\tilde{A}(C)$ its area in the canonical metric of S_1^n. Let C_γ be the spherical cap whose first eigenvalue of the Laplacian in the metric of S_1^n is γ.

THEOREM 1. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a minimal immersion and $D \subset M$ be a bounded domain with smooth boundary. Assume that the Gauss-Kronecker curvature H_n of $\overline{D}$ is nowhere zero and that M is special. Assume further that*

$$\int_D |H_n| dM < \tilde{A}(C_\gamma),$$

where $\gamma = \max_{i,D}\left(\sum_j k_j^2/k_i^2\right)$. Then, D is stable.

Notice that for $n = 2$, $\gamma = 2$ and $\tilde{A}(C_\gamma) = 2\pi$. Thus Theorem 1 reduces to Theorem A in the case that $K \neq 0$.

We can also prove the corresponding instability criterion which generalizes an instability result of Schwartz for minimal surfaces in $\mathbb{R}^3$ (see Barbosa-do Carmo (1976)).

It will be convenient to let $g\colon M^n \to S_1^n$ be the Gauss map of x and to denote by $\widetilde{\Delta}$ the Laplacian of M in the pullback metric by g of the metric in S_1^n.

THEOREM 2. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a minimal special immersion with $H_n \neq 0$ everywhere. If the first eigenvalue $\lambda_1^{\widetilde{\Delta}}(D)$ of the Laplacian $\widetilde{\Delta}$ in a domain $D \subset M^n$ satisfies*

$$\lambda_1^{\widetilde{\Delta}}(D) < \min_{i,D} \frac{\sum_j k_j^2}{k_i^2},$$

then D is unstable.

In both Theorems 1 and 2 we do not need that x be minimal; the proof works equally well for hypersurfaces with constant mean curvature. Actually, with appropriate definitions, we can extend Theorems 1 and 2 to hypersurfaces with constant r-mean curvature. We will not go into that here but refer to Alencar, do Carmo and Elbert (1998).

REMARK 1.1 Not much is known about stability of domains in hypersurfaces of R^{n+1}, $n > 2$. The only results that we know of are the following.

First, a result by J. Spruck (Spruck (1975)) that states the existence of an $\epsilon(n) > 0$, such that if $x\colon M^n \to \mathbb{R}^{n+1}$ is minimal and $D \subset M$ satisfies $\int_D \|B\|^n dM < \epsilon$, then D is stable; here $\|B\|$ is the norm of the second fundamental form of x.

Second, a result that generalizes an old Schwarz theorem and states that if the image of the Gauss map of a domain D of a minimal surface is contained in a hemisphere, then D is stable, and if it contains a hemisphere, then D is unstable. This follows from the fact the support function of a minimal surface satisfies the Jacobi equation and holds in a greater generality (for a simple proof of the general case see Elbert (1998) Proposition (7.2)).

SKETCHES OF THE PROOFS OF THEOREMS 1 AND 2

We start with Theorem 1. The Jacobi equation for minimal hypersurfaces in R^{n+1} is given by

(1) $$\Delta f + \|B\|^2 f = 0,$$

where Δ is the Laplacian in the induced metric and $\|B\|$ is the norm of the second fundamental form. To say that D is stable means that there is no $D' \subset D$ together with a solution of (1) that vanishes in the boundary $\partial D'$ of D'. To study (1), we use the Gauss map $g = M^n \to S_1^n$ to induce a new metric $\langle\langle\, , \rangle\rangle$ in M^n defined by

$$\langle\langle X, Y\rangle\rangle_p = \langle -dg_p X, -dg_p Y\rangle, \quad X, Y \in T_p M.$$

This is legitimate, since $\det(-dg_p) = H_n \neq 0$ everywhere. Notice that the metric has constant sectional curvature equal to one. We denote objects in this metric by a tilde; thus $\widetilde{\Delta}$ denotes the Laplacian in the metric $\langle\langle\, , \rangle\rangle$.

For simplicity, choose an orthonormal principal frame $e_1, \cdots, e_n$ with principal curvatures $k_1, \cdots, k_n$ (this is not necessary for the proof but will make the ideas more transparent). Set $(\widetilde{f}_{ij}) = \widetilde{\text{Hess}}\, f$. Then a direct computation shows that, in the new metric,

$$\Delta f = \sum_i k_i^2 \widetilde{f}_{ii} + \text{terms of first order.} \tag{2}$$

Notice that $\widetilde{\Delta} f = \sum_i \widetilde{f}_{ii}$.

LEMMA 1. *Assume that the mean curvature H of the immersion $x\colon M^n \to \mathbb{R}^{n+1}$ is constant. Then the terms of first order in* (2) *vanish.*

Thus, in the new metric, the Jacobi equation writes, in the above frame,

$$\sum_i k_i^2 \widetilde{f}_{ii} + \left(\sum_j k_j^2\right) f = 0$$

which is equivalent to

$$\widetilde{G} f + f = 0, \tag{1'}$$

where

$$\widetilde{G} f = \sum_i \frac{k_i^2}{\sum k_j^2} \widetilde{f}_{ii}.$$

We would like the operator $\widetilde{G}$ to be self-adjoint so that we could talk about eigenvalues of $\widetilde{G}$.

LEMMA 2. *A hypersurface $x\colon M^n \to \mathbb{R}^{n+1}$ with constant mean curvature is special and has $H_n \neq 0$ everywhere in $\overline{D} \subset M$ if and only if $\widetilde{G}$ is self-adjoint.*

The proof makes use of Codazzi equations and a criterion of self-adjointness of an operator of the type $\widetilde{G}$ given in (Cheng & Yau, 1977).

Detailed proofs of Lemmas 1 and 2 can be found in Alencar, do Carmo & Elbert (1998).

Now we want to estimate the first eigenvalue $\lambda_1^{\widetilde{G}}(D)$ of $\widetilde{G}$ in D. By the definition of γ, we have

$$\begin{aligned}
\lambda_1^{\widetilde{\Delta}}(D) &= \inf_f \frac{\int_D \left|\widetilde{\nabla} f\right|^2 d\widetilde{M}}{\int_D f^2 d\widetilde{M}} \\
&= \inf_f \frac{\int_D \left(\sum_i \widetilde{f}_i^2\right) d\widetilde{M}}{\int_D f^2 d\widetilde{M}} \leq \gamma \inf_f \frac{\int_D \sum_i \frac{k_i^2 \widetilde{f}_i^2}{\sum_j k_j^2} d\widetilde{M}}{\int_D f^2 d\widetilde{M}}.
\end{aligned}$$

Since $\widetilde{G}f = \sum_i \frac{k_i^2 \widetilde{f}_{ii}}{\sum_j k_j^2}$ is self-adjoint in D we have that (Cheng & Yau, 1977, Eq. 1.10)

$$-\int_D f\widetilde{G}f\, d\widetilde{M} = \int_D \frac{|B\widetilde{\nabla} f|^2}{\|B\|^2}\, d\widetilde{M} = \int_D \frac{\sum_i k_i^2 \widetilde{f}_1^2}{\sum_i k_j^2}\, d\widetilde{M}.$$

Then

$$\lambda_1^{\widetilde{\Delta}}(D) \le \gamma \inf_f \left(-\frac{\int_D f\widetilde{G}f\, d\widetilde{M}}{\int_D f^2 d\widetilde{M}} \right) = \gamma\, \lambda_1^{\widetilde{G}}(D). \tag{2}$$

Now, by hypothesis,

$$\widetilde{A}(D) \le \int_D |H_n| dM < \widetilde{A}(C_\gamma).$$

Choose a cap $C_1 \subsetneq C_\gamma$ with $\widetilde{A}(D) = \widetilde{A}(C_1)$. It is known that among all domains with the same area in a manifold with sectional curvature one, the geodesic ball minimizes the first eigenvalue of the Laplacian (Chavel (1993) p. 163). Thus

$$\lambda_1^{\widetilde{\Delta}}(D) \ge \lambda_1^{\widetilde{\Delta}}(C_1) > \lambda_1^{\widetilde{\Delta}}(C_\gamma) = \gamma,$$

hence, by (2), $\lambda_1^{\widetilde{\Delta}}(D) > 1$.

The rest of the proof is rather standard. If M is not stable, there exists a domain $D' \subset D$ and a solution v in $\overline{D'}$ of the Jacobi equation $\widetilde{G}v + v = 0$ that vanishes on $\partial D'$. It follows that

$$\int_{D'} v^2 d\widetilde{M} = -\int_{D'} v\widetilde{G}v\, dM \ge \lambda_1^{\widetilde{G}}(D') \int_{D'} v^2 d\widetilde{M},$$

hence

$$\lambda_1^{\widetilde{G}}(D) \le \lambda_1^{\widetilde{G}}(D) \le 1.$$

This is a contradiction and proves that D is stable.

We now sketch a proof of Theorem 2. With the same notation as above, set

$$\nu = \min_{i,D} \frac{\sum_j k_j^2}{k_i^2}.$$

Then

$$\begin{aligned} \lambda_1^{\widetilde{\Delta}}(D) &= \inf_f \frac{\int_D (\sum_i \widetilde{f}_i^2) dM}{\int_D f^2 dM} \\ &\ge \nu \inf_f \frac{\int_D \sum_i \frac{k_i^2 \widetilde{f}_i^2}{\sum_j l_j^2}\, d\widetilde{M}}{\int_D f^2 d\widetilde{M}} \\ &= \nu \inf_f \left(-\frac{\int_D f\widetilde{G}f\, d\widetilde{M}}{\int_D f^2 d\widetilde{M}} \right) = \nu \lambda_1^{\widetilde{G}}(D), \end{aligned}$$

hence

$$\lambda_1(\widetilde{\Delta}(D)) \ge \nu \lambda_1^{\widetilde{G}}(D). \tag{3}$$

Together with the hypotheses, (3) implies that $\lambda_1^{\widetilde{G}}(D) < 1$. Let u be the first eigenfunction of $\widetilde{G}(D)$.

Then the index form $I_D(u,u)$ is given by

$$
\begin{aligned}
I_D(u,u) &= -\int_D (u\Delta u + \|B\|^2 u^2) dM \\
&= -\int_D u \sum_i k_i^2 u_{ii}^2 + \left(\sum k_i^2\right) u^2 \frac{d\widetilde{M}}{|\det(B)|} \\
&= \int_D (\lambda_1^{\widetilde{G}}(D) u^2 - 1) \frac{\sum_i k_i^2 d\widetilde{M}}{|\det B|} < 0,
\end{aligned}
$$

because $\lambda_1^{\widetilde{G}}(D) < 1$. It follows that the variation determined by u increases area, hence D is unstable.

EXAMPLES

To find examples of special minimal (or with constant mean curvature) hypersurfaces that have nonzero principal curvatures is not an easy matter. Of course, there is a large supply of sets of nonzero values (candidates to principal curvatures) $\{k_1, \cdots, k_n\}$ that satisfy both the conditions $\sum k_1 = 0$ and $|k_1 \cdots k_n| / \sum_i k_i^2 = \text{const.} = D$. Actually, a simple algebraic argument shows that we can prescribe $k_3, \cdots, k_n$ and D so that k_1 is determined by a second degree equation and k_2 by the minimality condition. Whether such a set of values $\{k_i\}$ can be realized as principal curvatures of a hypersurface is quite another question.

Usually to find a minimal hypersurface in $\mathbb{R}^{n+1}$, $n > 2$, one introduces an additional condition (action of a group, ruled, separation of variables, etc.) that reduces the minimal equation (or the constant mean curvature equation) to an ordinary differential equation which can be solved (or analyzed qualitatively). After the reduced equation has been solved, it only remains some constants to be adjusted and this may not be enough to satisfy the condition of special.

Of course there are many examples of minimal special hypersurfaces with one vanishing principal curvature: If $M^{n+1} \subset S^n \subset \mathbb{R}^{n+1}$ is minimal hypersurface in the sphere S^n, the cone $(CM)^n \subset \mathbb{R}^{n+1}$ over M with vertex at the origin is such a minimal hypersurface. Unfortunately, $H_n \equiv 0$ and our Theorems do not apply.

As it stands, although there are reasons to believe that there exist many examples of hypersurfaces with constant mean curvature which are special and have nonzero principal curvatures, we are able to display only a few of them. Thus the question of examples should be considered essentially open and should be taken up in another paper.

References

Alencar, H.; do Carmo, M. & Elbert, M. F., (1998), Stability of hypersurfaces with constant r-mean curvature in Euclidean spaces. Preprint.

Barbosa, J. L. & do Carmo, M., (1976), On the size of a stable minimal surface in $\mathbb{R}^3$. *Amer. J. Math.*, **98:** 515–528.

Chavel, I., (1993), Riemannian Geometry: A Modern Introduction. *Cambridge University Press*, New York, USA.

Cheng, S. Y. & Yau, S. T., (1997), Hypersurfaces with constant scalar curvature. *Math. Ann.*, **225:** 195–204.

Elbert, M. F., (1998), On complete constant r-mean curvature hypersurfaces in space forms. *Doctoral Thesis*, **IMPA**.

Spruck, J., (1975), Remarks on the stability of minimal submanifolds of $\mathbb{R}^n$. *Math. Z.*, **144:** 169–174.