SOME REMARKS ON THE EXISTENCE OF HYPERSURFACES OF CONSTANT MEAN CURVATURE WITH A GIVEN BOUNDARY, OR ASYMPTOTIC BOUNDARY, IN HYPERBOLIC SPACE

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Consider a smooth Jordan curve Γ in $\mathbf{H}^3$. What, and where, are the H-surfaces with boundary Γ? If H is sufficiently small, $H \neq 0$, then it is not difficult to find two small H-surfaces with boundary Γ (*cf.* Section II). By H-surface we mean a surface of constant mean curvature H.

In this paper we will describe situations where we can find two solutions. For example, if Γ is on a sphere of radius R, then there are two H-surfaces in the ball bounded by the sphere (with boundary Γ) for any H, $0 < H < \tanh(R)$. In particular, for $0 < H \leq 1$. Using this fact, we show that for any smooth Jordan curve Γ_∞ at the sphere at infinity of $\mathbf{H}^3$ (denoted $S(\infty)$), there are two complete H-surfaces with asymptotic boundary Γ_∞, for any H, $0 < H < 1$.

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Many of our results remain true in arbitrary dimension (Γ codimension two) with the solutions integral currents.

1. An existence Theorem

Let B be a domain in $\mathbf{H}^{n+1}$ or $\mathbf{R}^{n+1}$ with smooth boundary S and assume S is mean convex;

$$H(S) = \inf_{x \in S} H(x) > 0.$$

THEOREM 1. – *Let* Γ *be a smooth codimension one submanifold of* S *that is homologous to zero in* S*. Then for* $0 < H < H(s)$*, there are two integral currents in* B*, with boundary* Γ *and mean curvature* H*, their intersection is* Γ*.*

Proof. – We will work with integral currents mod two. We refer to integral n-currents mod two as n-chains and integral $n+1$-currents mod two as domains. Their mass will be called area and volume respectively, denoted by $|M|$ and $|Q|$, M and n-chain, Q a domain.

Let $S_1 \subset S$ be a smooth submanifold with $\partial S_1 = \Gamma$. Consider domains Q in B with $\partial Q = S_1 \cup M$, M an n-chain, $\partial M = \Gamma$ and $|M| \leq |S_1|$. Let J denote the functional on such domains Q:

$$Q \mapsto nH|Q| + |M|.$$

It is well known if Q is a minimum of this functional then at all smooth points of M in the interior of B, M has mean curvature H [3]. We seek a minimum of J with $M - \Gamma$ contained in the interior of B.

Q and ∂Q have bounded mass, so the fundamental theorem of geometric measure theory yields a limit Q of a sequence Q_i, $Q_i \subset B$, $\partial Q_i = S_1 \cup M_i$, $|M_i| \leq |S_1|$, such that $J(Q_i)$ converges to its minimum value [2]. Since area and volume are lower semi continuous, Q is a minimum of the functional J.

In general, such a Q may touch the obstacle; here the obstacle is S, and at such points M is not of constant mean curvature H. But since $H < H_0(S)$, M can not touch S, except along Γ. This is proved in Lemma 5 of [3] in dimension three but the same proof works in all dimensions; if M touched S one could strictly reduce $J(Q)$ by cutting off a piece of Q is a neighborhood of the point of contact.

This gives us an n-chain M, $\partial M = \Gamma$, and at all smooth points of interior M, M has mean curvature H.

Now Γ is homologous to zero in S, hence in B, so there is a least area n-chain $\sum$, $\partial \sum = \Gamma$, interior $\sum \subset B$. $\sum$ is a solution to the classical Plateau problem in the setting of geometric measure theory, with S a barrier (since S is mean convex).

Let E_1, E_2 be the connected components of $B - \sum$, $\partial E_1 = S_1 \cup \sum$. Notice that the M we obtained above by minimizing J, is contained in E_1. For it Q_1 satisfies $\partial Q_1 = S_1 \cup M$ with $Q_1 \cap \sum \neq \emptyset$, one can cut off the part of Q_1 outside E_1 and strictly reduce J; this strictly reduces volume of Q_1 and the part of M in E_2 that one replaces by a part of $\sum$, with the same boundary, has at least as much area; cf. *Fig.* 1.

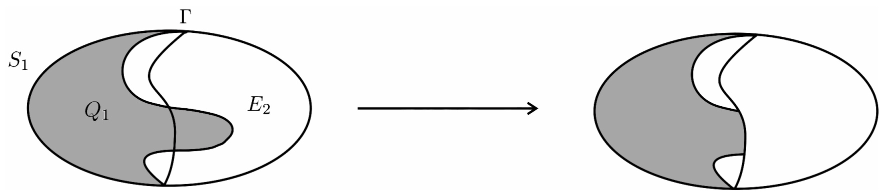

Fig. 1

Now to obtain a distinct n-chain M_2 in B, $\partial M_2 = \Gamma$, M_2 an H-surface, one begins with the other component S_2 of $S - \Gamma$ and one minimizes J among domains Q with $\partial Q = M + S_2$, $|M| \leq |S_2|$. The minimum is then contained in E_2, which proves Theorem 1.

Remark. – If one assumes Γ is a oriented codimension two closed submanifold of B (not necessarily on ∂B), homologous to zero in B, then one always has two H-chains in B with boundary Γ, for $0 < H < H(\partial B)$ (but one no longer controls where they are). One does the same proof as above, replacing the mass of Q by the algebraic volume of Q, *i.e.*, one chooses a fixed (oriented) n-chain M_0 in B, $\partial M_0 = \Gamma$, and for any M_1 with boundary Γ, $|Q|$ becomes the algebraic volume of the domain bounded by $M_0 + M_1$ [3]. Then a minimum of the functional $|M_1| + nH|Q|$ yields an H-surface M_1 in B with boundary Γ. Now change the orientation of M_0 and minimize the new functional (only the algebraic volume term changes). This yields a second H-surface M_2 with $\partial M_2 = \Gamma$ and M_1 is geometrically distinct from M_2 (if $|Q(M_1)| < 0$ then $|Q(M_2)| > 0$.

Since compact balls in $\mathbf{H}^{n+1}$ always have $H > 1$, one can always find two distinct H-surfaces with boundary Γ, for $0 < H \leq 1$.

Corollary 1.1 – *Suppose $B = B(R)$ is the ball of radius R and $r < R$. Let (Q, M) be a minimum of J given by Theorem 1. Then the area of $M \cap B(r)$ is at most $|S(r)|$, (here $S(r) = \partial B(r)$).*

Proof. – Let $\tilde{Q}$ be the domain $Q \cap (B(R) - \operatorname{int} B(r))$. Then $\partial\tilde{Q} = S_1 \cup \tilde{M} \cup \tilde{S}$ where $\tilde{S} \subset S(r)$,

$$\partial\tilde{S} = \partial(M \cap B(r)) \qquad \text{and} \qquad \tilde{M} = M \cap (B(R) - B(r)).$$

If $|M \cap B(r)| > |\tilde{S}|$ then $|\tilde{Q}| < |Q|$ and $|\tilde{M} \cup \tilde{S}| < |M|$ hence $J(\tilde{Q}) < J(Q)$, which contradicts Q being a minimum of J. Thus $|M \cap B(r)| \leq |\tilde{S}|$ and $|\tilde{S}| \leq |S(r)|$.

2. Two small H-surfaces for H small

An H-surface M with boundary Γ is called a small H-surface if M is contained in some ball $B(r)$ with $r < \frac{1}{\tanh H}$ (or $r < \frac{1}{H}$ in euclidean space). From the maximum principle it follows that $M \subset \bigcap_{B \in \mathcal{A}} B$, where $\mathcal{A}$ is the family of balls $B(\rho)$, $\rho \leq \frac{1}{\tanh H}$, such that $\Gamma \subset B(\rho)$.

Theorem 2. – *Let $\Gamma \subset \mathbf{H}^{n+1}$ or (or $\mathbf{R}^{n+1}$, $n \leq 6$, Γ a smooth closed condimension two submanifold. Then for H sufficiently small, there exist at least two small H-surfaces with boundary Γ.*

Proof. Let $\sum$ be an oriented n-manifold in $\mathbf{H}^{n+1}$, $\partial \sum = \Gamma$ and $\sum$ of least area; the geometric measure theory solution to Plateau's problem has no singularities when $n \leq 6$ so $\sum$ is smooth.

Let $0 < \alpha < 1$ and $C_0^{2,\alpha}(\sum)$ denote the functions u on $\sum$ of class $C^{2,\alpha}$ which vanish on $\partial \sum = \Gamma$. Let $C^{\alpha}(\sum)$ denote the functions on $\sum$ of class C^{α}.

One has the mean curvature map of normal variations of $\sum$:

$$H : C_0^{2,\alpha}\left(\sum\right) \to C^{\alpha}\left(\sum\right),$$
$$H(u)(x) = H(\exp_x(u(x)n(x))),$$

$n(x)$ a unit normal vector field along $\sum$. H is the mean curvature of the surface in the normal bundle of $\sum$ obtained by going a distance $u(x)$ form x, along the geodesic at x having $n(x)$ as tangent. H is well defined in a neighborhood of $u = 0$; *i.e.* for small variations of $\sum$.

Now H is a map between Banach spaces and $L = dH$ at $u = 0$ is the linearized operator of the mean curvature of $\sum$. This is an elliptic operator and its kernel is trivial since $\sum$ is area minimizing. Thus L is an isomorphism and H is a diffeomorphism of a neighborhood $\mathcal{U}$ of $u = 0$ to a neighborhood V of $H(0) = 0$. For t sufficiently small, one has the constant functions t and $-t$ in V hence $H^{-1}(t)$ and $H^{-1}(-t)$ are two t-surfaces for t small.

Remark. Notice that the above argument works for any elliptic differential operator of order two on $\sum$ with no Jacobi fields; *i.e.* non trivial functions in the kernel of the linearized operator. For example, any compact surface $\sum \subset \mathbf{H}^3$, $\partial \sum = \Gamma \neq \phi$, with constant Gaussian curvature between -1 and 0 [5]. One obtains a foliation of t-surfaces with boundary Γ, by letting $t \to 0$.

3. H-surfaces with asymptotic boundary at infinity

We now suppose Γ is a smooth closed codimension one submanifold of S_∞, the sphere at infinity of $\mathbf{H}^{n+1}$. M. Anderson has proved that there is a minimal integral n-current in $\mathbf{H}^{n+1}$ whose asymptotic boundary is Γ; *i.e.* one can solve the Plateau problem at ∞ [1]. We will prove that for $0 < H < 1$, there are at least two H integral n-currents with asymptotic boundary Γ. Notice that for $H \geq 1$, there is no embedded H-surface M with $\partial_\infty M = \Gamma$: M would separate $\mathbf{H}^{n+1}$ into two components, let A be the component to which the mean curvature of M points. Let z be a point of $S_\infty - \Gamma$ in the asymptotic boundary of A. Let $S(t)$ be the horospheres tangent to S_∞ at z, parametrized so that $S(t) \to z$ as $t \to 0$. Then for t near 0, $S(t) \cap M = \emptyset$. Let t increase until $S(t)$ touches M for the first time. Then the maximum principle at this contact point yields $M = S(t)$; a contradiction.

Theorem 3. – *Let $\Gamma \subset S(\infty)$ be as above, and $0 < H < 1$. There are at least two H-chains in $\mathbf{H}^{n+1}$, with asymptotic boundary Γ.*

Before proving Theorem 3 we need some preliminaries. Let Ω_1 and Ω_2 be the connected components of $S_\infty - \Gamma$. Let B_1 be an equidistant ball in $\mathbf{H}^{n+1}$ with $\partial_\infty(B_1) \subset \text{int}\,\Omega_1$ and $\partial B_1 = S_1$ an equidistant sphere of mean curvature H, whose mean curvature vector points to the exterior of B_1 (cf. *Fig.* 2). Similarly let B_2 be an equidistant ball with $\partial_\infty B_2 \subset \text{int}\,\Omega_2$, $\partial B_2 = S_2$ an equidistant sphere of mean curvature H, whose mean curvature vector points to the interior of B_2.

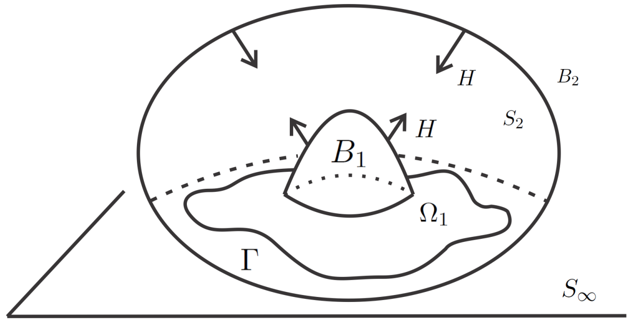

Fig. 2

LEMMA 3.1. *Let M be a compact connected n-current of constant mean curvature H and ∂M contained in the domain E of $\mathbf{H}^{n+1}$ bounded by $S_1 \cup S_2$. Then $M \subset E$ as well.*

Proof of 3.1. – Assume on the contrary that $M \cap \text{ext}\,(B_2) \neq \phi$. Notice that B_2 is foliated by hypersurfaces $L(t)$, $2 \leq t < \infty$, with $L(2) = S_2$ and $L(t)$ isometric to S_2. To see this one fixes a point $p \in \partial_\infty(\text{int}\,B_1)$ and the homotheties of $\mathbf{H}^{n+1}$ from p (in the upper-half space model of $\mathbf{H}^{n+1}$) induce isometries of $\mathbf{H}^{n+1}$ and the images of S_2 foliate B_2.

Now M is compact so there is some $L(t)$, for t large, such that $L(t) \cap M = \phi$. Let t decrease to 2 and then by compacity of M there is a first point of contact of an $L(s)$ with M. At this first point of contact the mean curvature vectors of $L(s)$ and M are equal, since each $L(t)$ for $t > s$, has a mean curvature vector that points into the component of $\mathbf{H}^{n+1} - L(t)$ containing M. But then the maximum principle for constant mean curvature hypersurfaces implies $M = L(s)$ and this is a contradiction since $\partial M \subset \text{int } E$.

Similarly, M can not enter int B_1 so $\partial E = S_1 \cup S_2$ is a barrier for solutions to the Plateau problem we are considering, for boundary data inside E.

Proof of Theorem 3 – Let Γ be an $n-1$ submanifold of infinity. Choose B_1, B_2 (the equidistant balls of our previous discussion) so that Γ is contained in int $\partial_\infty(E)$. Let $p \in E$ and let $C(\Gamma)$ denote the cone in $\mathbf{H}^{n+1}$ composed of all geodesics starting at p and asymptotic to a point of Γ. Since E is mean convex, we have $C(\Gamma) \subset E$ (cf. *Fig.* 3).

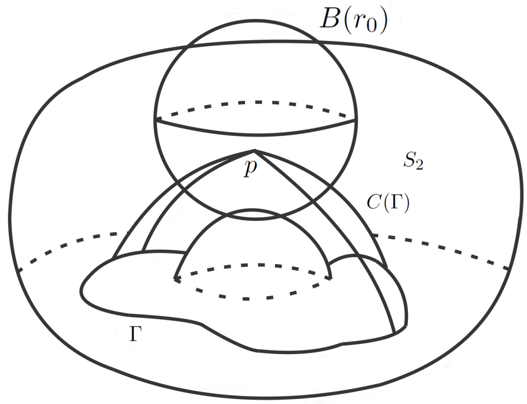

Fig. 3

We consider geodesic balls $B(r)$ of $\mathbf{H}^{n+1}$, centered at p, and denote by $\Gamma(r)$ the cycle $C(\Gamma) \cap S(r)$, $S(r) = \partial B(r)$.

Choose r_0 sufficiently large so that $B(r_0)$ has a non trivial intersection with the equidistant spheres S_1 and S_2, *i.e.*, we require that $B(r_0)$ intersects each S_1, S_2 is an open set. Clearly for $r > r_0$, $\partial(B(r) \cap S_1)$ is a topological $n-1$ sphere that generates the $n-1$ dimensional homology of $E - B(r_0)$. Each such cycle $\partial(B(r) \cap S_1)$, for $r > r_0$, is not homologous to zero in $E - B(r_0)$.

Next apply Theorem 1 to $\Gamma(R)$ for $R > r_0$, to obtain an n-chain mod 2, $M(R)$ contained in $B(R)$, such that $M(R)$ has mean curvature H and $\partial M(R) = \Gamma(R)$. By Lemma 3.1, $M(R) \subset E$, since $\Gamma(R) \subset E$.

Since $\Gamma(R)$ is not null homologous in $E - B(r_0)$, it follows that $M(R) \cap B(r_0) \neq \emptyset$. By Corollary 1.1, one has uniform area bounds for $M(R) \cap B(r_0)$ that depend only on r_0; *i.e.* there exists $C(r_0) > 0$ such that

$$|M(R) \cap B(r_0)| \leq C(r_0), \qquad \text{for} \quad R > r_0.$$

Observe that we also have uniform lower area bounds for $|M(R) \cap B(r_0)|$ depending only upon r_0.

Indeed, $M(R)$ must intersect each geodesic of $\mathbf{H}^{n+1}$ issue from a point of $B(r_0) \cap S_1$ and orthogonal to S_1 at this point. Otherwise $\Gamma(R)$ would be null homologous in $E - B(r_0)$, which is not the case. Now the (geodesic) projection of $M(R) \cap B(r_0)$ onto S_1 is of bounded area distortion; the amount one can increase area of $M(R) \cap B(r_0)$ is bounded above by a constant depending only upon S_1, S_2 and dist (S_1, S_2). Hence there exists a $K(r_0) > 0$, and

$$|M(R) \cap B(r_0)| \geq K(r_0)|B(r_0) \cap S_1| > 0,$$

for all $R > r_0$.

Since one has uniform area bounds above and below for $M(R) \cap B(r_0)$, the compactness theorem for integral currents mod two yields a subsequence $M(R_i)$, $R \to \infty$, such that $M(R_i) \cap B(r_0)$ converges in $B(r_0)$ to a constant mean curvature n-chain. Now repeat the above argument with the sequence R_i, working in the ball $B(r_0 + 1)$ to obtain a subsequence converging to a constant mean curvature n-chain $B(r_0 + 1)$. Continue inductively in each $B(r_0 + n)$ and then take a diagonal subsequence; this subsequence converges to a solution to the Plateau problem.

Now it is simple two construct two complete solutions M_1, M_2 when $H \neq 0$. First construct a complete area minimizing current $\sum$ with asymptotic boundary Γ. Then for $H \neq 0$, construct M_1 in one of the components of $\mathbf{H}^{n+1} - \sum$ so that M_1 has mean curvature H and asymptotic boundary Γ. M_2 is then constructed in the other component of $\mathbf{H}^{n+1} - \sum$. This completes the proof of Theorem 3.

Remark 1. – B. Nelli and J. Spruck have studied the non parametric problem in $\mathbf{H}^{n+1}$ [4]. They establish the existence of a graph (in a suitable coordinate system) of prescribed mean curvature $H < 1$ and $\Gamma_\infty \subset S_\infty$ "mean convex" in a particular model of S_∞ (this is not invariant by isometry).

2. After a first preprint of this paper appeared we learned of a preprint of [6]. He proves the existence of one integral n-current M, $\partial_\infty M = \Gamma_\infty$, M an H-surface, for any H, $0 < H < 1$, and he studies its regularity.

References

[1] Anderson (M. T.) – Complete minimal varieties in hyperbolic space, *Invent. Math.*, Vol. 69, 1982, p. 477–494.

[2] Federer (H.) – *Geometric Measure Theory.* – Berlin, Springer, 1979.

[3] Gulliver (R. D. H.) – The Plateau problem for surfaces of prescribed mean curvature in a Riemannian manifold, *J. Diff. Geom.*, Vol. 8, 1973, p. 317–330.

[4] Nelli (B.) and Spruck (J.) – *On existence and uniqueness of mean curvature hypersurfaces in hyperbolic space* (Preprint).

[5] Rosenberg (H.) and Spruck (J.) – On the existence of convex hypersurfaces of constant Gaussian curvature in hyperbolic space, *Journ. Diff. Geom.*, Vol. 40, 1994, p. 379–409.

[6] Tonegawa (Y.) – *Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space* (Preprint).