Remarks on the Growth of Functions and the Weak Stability of Hypersurfaces with Constant Mean Curvature

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ABSTRACT
We show that a weakly stable complete noncompact hypersurface $M^n$ of $\mathbb{R}^{n+1}$, $n \leq 5$, with constant mean curvature is a hyperplane provided certain conditions hold.

Key words: Constant mean curvature, stability, growth of functions.

INTRODUCTION
We want to consider the following question. In (Alencar & do Carmo, (1994), Theorem 4) we proved a result on strongly stable hypersurfaces of $\mathbb{R}^{n+1}$ with constant mean curvature $H$. The question is whether the theorem can be extended for the weakly stable case. We recall that $M$ is weakly stable if for all piecewise smooth functions $f: M \to \mathbb{R}$ with compact support and mean value zero, i.e., $\int_M f dM = 0$, we have

$$\int_M |\nabla f|^2 \geq \int_M |A|^2 f^2 dM.$$ 

Here $|\nabla f|^2$ is the gradient of $f$ in the induced metric and $|A|^2$ is the square of the norm of the linear map $A$ associated to the second fundamental form.

We start with a proposition that will give a test function for weak stability.

PROPOSITION 1. Let $M$ be a complete noncompact Riemannian manifold, and let $g: M \to \mathbb{R}$, $g \geq 0$
be a $C^\infty$ function. Let $x_0 \in M$ and denote by $\rho(x) = d(x, x_0)$, where $d$ is the geodesic distance in $M$. Then there exists a function

$$\xi \in C^0(M),$$

$\xi$ piecewise linear, with $\xi(x) \leq 1$ if $\rho(x) \leq R$ (R a fixed number),

$$\xi(x) = 0 \text{ if } \rho(x) \geq 4R,$$

$|\nabla \xi|$ or bounded, and $\int_M g \xi = 0$.

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Choose \( 0 < \delta \leq R/4 \), and \( a > 0 \). Define a family \( \xi_a \) of functions parametrized by \( a \) as follows:

\[
\xi_a(x) = 1, \quad 0 \leq x \leq R - \delta,
\]

\[
\xi_a(x) = \frac{R - x}{\delta}, \quad R - \delta \leq x \leq R,
\]

\[
\xi_a(x) = 0, \quad R \leq x \leq 2R - \delta,
\]

\[
\xi_a(x) = \frac{(2R - \delta)a - ax}{\delta}, \quad 2R - \delta \leq x \leq 2R,
\]

\[
\xi_a(x) = -a, \quad 2R \leq x \leq 3R,
\]

\[
\xi_a(x) = \frac{ax - (3R + \delta)}{\delta}, \quad 3R \leq x \leq 3R + \delta,
\]

\[
\xi_a(x) = 0, \quad 3R + \delta \leq x \leq \infty.
\]

Then

\[
\int_{M} g\xi_a = \int_{B(R)} g\xi_a + \int_{B(2R) - B(2R - \delta)} g\xi_a - a \int_{B(3R) - B(2R)} g + \int_{B(3R + \delta) - B(3R)} g\xi_a.
\]

The first term is positive and the last three terms are negative. Clearly, if \( a \) is small enough, the integral is positive and if \( a \) is large the integral is negative. Thus there exists an \( a \) such that

\[
\int_{M} g\xi_a = 0.
\]

Furthermore, since

\[
0 = \int_{M} g\xi_a \leq \int_{B(R)} g - a \int_{B(3R) - B(2R)} g,
\]

such an \( a \) is bounded by

\[
a \leq \frac{\int_{B(R)} g}{\int_{B(3R) - B(2R)} g}.
\]

We will need that the \( a \) found in the above proof be bounded as \( R \to \infty \). We will say that the positive real functions \( f \) and \( h \) have the same order if

\[
\lim_{R \to \infty} \sup \frac{f(R)}{h(R)} = c > 0
\]

for fixed \( \delta \) and \( \Delta \) (see Hardy, G. H., (1954), p. 2). This definition implies (but it is stronger than) our definition.

**Proposition 2.** Notations being as in Proposition 1, let \( f(R) = \int_{B(R)} g \). Then the number \( a \) found in the proof of Proposition 1 is bounded as \( R \to \infty \) if there exists a positive function \( h(R) \) which has the same order as \( f \) and, in addition, has the property that for every sequence \( R_i, i = 1, \ldots \), going to infinity,

\[
(1) \quad \lim_{R_i \to \infty} \frac{h(nR_i)}{h(mR_i)} = \gamma_{n,m} < 1, \quad \text{if} \ n < m.
\]

**Proof** The fact that

\[
\lim_{R \to \infty} \sup f(R)/h(R) = c > 0
\]

implies that for every \( \delta > 0 \)

\[
(2) \quad f(R) \leq \varphi h(R), \quad \varphi = c(1+\delta).
\]
Furthermore, there exists a sequence $Q_i$ of real numbers going to infinity such that

$$\lim_{Q_i \to \infty} f(Q_i)/h(Q_i) = c.$$  

Now, take a sequence $\{R_i\} = \{Q_i/3\}$ and compute

$$\lim_{R_i \to \infty} a(R_i) \leq \lim_{R_i \to \infty} \frac{f(R_i)}{f(3R_i) - f(2R_i)} \leq \lim_{R_i \to \infty} \frac{\tau h(R_i)}{\tau h(3R_i) - \tau h(2R_i)} \leq \lim_{R_i \to \infty} \frac{f(3R_i)}{\tau h(3R_i)} \frac{h(2R_i)}{h(3R_i)} = \frac{\gamma_1}{c - \gamma_2} = \frac{\gamma_1}{1 + \delta - \gamma_2},$$

where we have used (2), (3) and (1). Since we can choose $\delta$ small enough so that the denominator is strictly positive, $a(R_i)$ is bounded.

**Remark 2** There are many functions $h(R)$ that satisfy $\lim_{R \to \infty} h(mR)/h(nR) < 1$, if $m < n$. For instance, for every positive $\alpha$, $R^\alpha$ is such a function and so is $e^{R\alpha}$. Indeed,

$$\lim_{R \to \infty} \left(\frac{n}{m}\right)^\alpha = \begin{cases} 0, & \text{if } n < m, \\ \infty, & \text{if } n > m. \end{cases}$$

$$\lim_{R \to \infty} \frac{e^{nR\alpha}}{e^{mR\alpha}} = \lim_{R \to \infty} e^{(n-m)R} = 0, \quad \text{if } n < m.$$

In fact, one easily checks that $e^R$ also satisfies the above condition.

On the other hand, $\log(R)$ does not satisfy the condition, since

$$\lim_{R \to \infty} \frac{\log nR}{\log mR} = \lim_{R \to \infty} \frac{\log n}{\log m} + 1 = 1.$$

As an application of the above ideas, we will show that the question posed in the beginning of this note has an affirmative answer provided that $f(R) = \int_{B(R)} |\phi|^{1+q}dM$ grows with the same order as a positive function $h(R)$ that satisfies

$$\lim_{R \to \infty} h(mR)/h(nR) < 1, \quad \text{if } m < n$$

(recall that $\phi = -A + HI$). More precisely,

**Theorem.** Let $M^n$, $n \leq 5$, be a complete noncompact hypersurface of $\mathbb{R}^{n+1}$ with constant mean curvature $H$. Assume that $M$ is (weakly) stable and that

$$\lim_{R \to \infty} \frac{\int_{B(R)} |\phi|^2dM}{R^{2+2q}} = 0, \quad q \leq \frac{2}{6n+1}.$$  

In addition, assume that for some $q$, the function

$$f(R) = \int_{B(R)} |\phi|^{1+q}dM$$
satisfies the following: There exists a positive function \( h(R) \) such that

\[
\lim_{R \to \infty} \sup R / h(R) = \epsilon > 0
\]

and

\[
\lim_{R \to \infty} \frac{h(mR)}{h(nR)} < 1, \quad \text{if m < n}.
\]

Then \( M^n \) is a hyperplane.

**Proof.** Set in Proposition 1, \( g = |\phi|^{1+q} \) with the \( q \) given in the statement of the theorem. Since \( \int_M \xi_a |\phi|^{1+q} = 0 \), the integrand can be used as a test function in the (weak) stability. Proceeding as in loc. cit. we arrive at

\[
\int_M \xi_a^{2+2q} \phi |\phi|^{2+2q} \leq \beta_3 \int_M |\phi|^2 |\nabla \xi_a|^{2+2q},
\]

(which is Eq. (17) of loc. cit. where we changed \( f \) to \( \xi_a \) to conform to our present notation).

By using the definition of \( \xi_a \) and setting \( \delta = R/4 \), we obtain

\[
\int_{B(R-\delta)} |\phi|^{2+2q} \leq \int_{B(R-\delta)} \xi_a^{2+2q} \phi |\phi|^{2+2q} \\
\leq \int_M \xi_a^{2+2q} \phi |\phi|^{2+2q} \\
\leq \beta_3 \int_{B(4R)} |\phi|^2 |\nabla \xi_a|^{2+2q} \\
\leq \beta_3 \left( \left( \frac{4}{R} \right)^{2+2q} + \left( \frac{8a}{R} \right)^{2+2q} \right) \int_{B(4R)} |\phi|^2 \\
\leq \beta_3 \left( \frac{4^{2+2q} + (8a)^{2+2q}}{R^{2+2q}} \right) \frac{1}{R^{2+2q}} \int_{B(4R)} |\phi|^2.
\]

Now, let \( R \) go to infinity. Since \( a \) is bounded and \( \lim_{R \to \infty} 1 / R^{2+2q} \int_M |\phi|^2 = 0 \), we see that \( |\phi| \equiv 0 \), and since \( M \) is complete noncompact, \( M \) is a hyperplane.

**References**
