Remarks on the Growth of Functions and the Weak Stability of Hypersurfaces with Constant Mean Curvature

H. ALENCAR and M. DO CARMO

IMPA – Instituto de Matemática Pura e Aplicada
22460–320 Rio de Janeiro, RJ, Brasil

ABSTRACT

We show that a weakly stable complete noncompact hypersurface M^n of $\mathbb{R}^{n+1}$, $n \leq 5$, with constant mean curvature is a hyperplane provided certain conditions hold.

Key words: Constant mean curvature, stability, growth of functions.

INTRODUCTION

We want to consider the following question. In (Alencar & do Carmo, (1994), Theorem 4) we proved a result on strongly stable hypersurfaces of $\mathbb{R}^{n+1}$ with constant mean curvature H. The question is wether the theorem can be extended for the weakly stable case. We recall that M is weakly stable if for all piecewise smooth functions $f\colon M \to R$ with compact support and mean value zero, i.e., $\int_M f dM = 0$, we have

$$\int_M |\nabla f|^2 \geq \int_M |A|^2 f^2 dM.$$

Here $|\nabla f|^2$ is the gradient of f in the induced metric and $|A|^2$ is the square of the norm of the linear map A associated to the second fundamental form.

We start with a proposition that will give a test function for weak stability.

PROPOSITION 1. *Let M be a complete noncompact Riemannian manifold, and let*

$$g\colon M \to \mathbb{R}, \quad g \geq 0,$$

be a C^∞ function. Let $x_0 \in M$ and denote by $\rho(x) = d(x, x_0)$, where d is the geodesic distance in M. Then there exists a function

$$\xi \in C_0^0(M),$$

ξ piecewise linear, with $\xi(x) \leq 1$ if $\rho(x) \leq R$ (R a fixed number),

$$\xi(x) = 0 \quad if \quad \rho(x) \geq 4R,$$

$|\nabla \xi|$ or bounded, and $\int_M g\xi = 0$.

H. Alencar and M. do Carmo were partially supported by the National Council for Scientific and Technological Development – CNPq of Brazil.

PROOF. Choose $0 < \delta \leq R/4$, and $a > 0$. Define a family ξ_a of functions parametrized by a as follows:

$$\begin{aligned}
\xi_a(x) &= 1, && 0 \leq x \leq R-\delta,\\
\xi_a(x) &= \frac{R-x}{\delta}, && R-\delta \leq x \leq R,\\
\xi_a(x) &= 0, && R \leq x \leq 2R-\delta,\\
\xi_a(x) &= \frac{(2R-\delta)a - ax}{\delta}, && 2R-\delta \leq x \leq 2R,\\
\xi_a(x) &= -a, && 2R \leq x \leq 3R,\\
\xi_a(x) &= \frac{ax-(3R+\delta)}{\delta}, && 3R \leq x \leq 3R+\delta,\\
\xi_a(x) &= 0, && 3R+\delta \leq x \leq \infty
\end{aligned}$$

Then

$$\int_M g\xi_a = \int_{B(R)} g\xi_a + \int_{B(2R)-B(2R-\delta)} g\xi_a - a\int_{B(3R)-B(2R)} g + \int_{B(3R+\delta)-B(3R)} g\xi_a.$$

The first term is positive and the last three terms are negative. Clearly, if a is small enough, the integral is positive and if a is large the integral is negative. Thus there exists an a such that $\int_M g\xi_a = 0$. Furthermore, since

$$0 = \int_M g\xi_a \leq \int_{B(R)} g - a\int_{B(3R)-B(2R)} g,$$

such an a is bounded by

$$a \leq \int_{B(R)} g / \int_{B(3R)-B(2R)} g.$$

We will need that the a found in the above proof be bounded as $R \to \infty$. We will say that the positive real functions f and h *have the samr order* if

$$\lim_{R\to\infty} \sup f(R)/h(R) = c > 0$$

REMARK 1. The usual definition that f and h have the same order is that, for large R,

$$0 \leq \delta \leq f(R)/h(R) \leq \Delta,$$

for fixed δ and Δ (see Hardy, G. H., (1954), p. 2). This definition implies (but it is stronger than) our definition.

PROPOSITION 2. *Notations being as in Proposition 1, let $f(R) = \int_{B(R)} g$. Then the number a found in the proof of Proposition 1 is bounded as $R \to \infty$ if there exists a positive function $h(R)$ which has the same order as f and, in addition, has the property that for every sequence $R_i, i = 1, \cdots$, going to infinity,*

$$\lim_{R_i\to\infty} \frac{h(nR_i)}{h(mR_i)} = \gamma_{n,m} < 1, \quad if\ n < m. \tag{1}$$

PROOF The fact that

$$\lim_{R\to\infty} \sup f(R)/h(R) = c > 0$$

implies that for every $\delta > 0$

$$f(R) \leq \bar{c}h(R), \quad \bar{c} = c(1+\delta). \tag{2}$$

Furthermore, there exists a sequence Q_i of real numbers going to infinity such that

(3) $$\lim_{Q_i\to\infty} f(Q_i)/h(Q_i) = c.$$

Now, take a sequence $\{R_i\} = \{Q_i/3\}$ and compute

$$\begin{aligned}
\lim_{R_i\to\infty} a(R_i) &\leq \lim_{R_i\to\infty} \frac{f(R_i)}{f(3R_i) - f(2R_i)} \\
&\leq \lim \frac{\bar{c}h(R_i)}{f(3R_i) - \bar{c}h(2R_i)} \\
&\leq \lim \frac{\dfrac{\bar{c}h(R_i)}{\bar{c}h(3R_i)}}{\dfrac{f(3R_i)}{\bar{c}h(3R_i)} - \dfrac{h(2R_i)}{h(3R_i)}} \\
&= \frac{\gamma_{13}}{\dfrac{c}{c} - \gamma_{23}} = \frac{\gamma_{13}}{\dfrac{1}{1+\delta} - \gamma_{23}},
\end{aligned}$$

where we have used (2), (3) and (1). Since we can choose δ small enough so that the denominator is strictly positive, $a(R_i)$ is bounded. REMARK 2 There are many functions $h(R)$ that satisfy

$$\lim_{R_i} \frac{h(nR_i)}{h(mR_i)} < 1, \quad \text{if} \quad m < n.$$

For instance, for every positive α, R^α is such a function and so is e^{R^α}. Indeed,

$$\lim \frac{(nR)^\alpha}{(mR)^\alpha} = \left(\frac{n}{m}\right)^\alpha < 1, \qquad \text{if} \quad n < m,$$

$$\lim \frac{e^{nR^\alpha}}{e^{mR^\alpha}} = \lim \frac{1}{e^{(m-n)R^\alpha}} = 0, \quad \text{if} \quad n < m.$$

In fact, one easily checks that e^{e^R} also satisfies the above condition.

On the other hand, $\log(R)$ does not satisfy the condition, since

$$\lim_{R\to\infty} \frac{\log nR}{\log mR} = \lim \frac{\dfrac{\log n}{\log R} + 1}{\dfrac{\log m}{\log R} + 1} = 1.$$

As an application of the above ideas, we will show that the question posed in the beginning of this note has an affirmative answer provided that $f(R) = \int_{B(R)} |\phi|^{1+q} dM$ grows with the same order as a positive function $h(R)$ that satisfies

$$\lim \frac{h(mR)}{h(nR)} < 1, \quad \text{if } m < n$$

(recall that $\phi = -A + HI$). More precisely,

THEOREM. *Let M^n, $n \leq 5$, be a complete noncompact hypersurface of $\mathbb{R}^{n+1}$ with constant mean curvature H. Assume that M is (weakly) stable and that*

$$\lim_{R\to\infty} \frac{\int_{B_R} |\phi|^2 dM}{R^{2+2q}} = 0, \quad q \leq \frac{2}{6n+1}.$$

In addition, assume that for some q, the function

$$f(R) = \int_{B(R)} |\phi|^{1+q} dM$$

satisfies the following: There exists a positive function $h(R)$ such that

$$\lim_{R\to\infty} \sup f(R)/h(R) = c > 0$$

and

$$\lim_{R\to\infty} \frac{h(mR)}{h(nR)} < 1, \quad \text{if } m < n.$$

Then M^n is a hyperplane.

PROOF. Set in Proposition 1, $g = |\phi|^{1+q}$ with the q given in the statement of the theorem. Since $\int_M \xi_a |\phi|^{1+q} = 0$, the integrand can be used as a test function in the (weak) stability. Proceeding as in loc. cit. we arrive at

$$\int_M \xi_a^{2+2q} |\phi|^{2+2q} \leq \beta_3 \int_M |\phi|^2 |\nabla \xi_a|^{2+2q}$$

(which is Eq. (17) of loc. cit. where we changed f to ξ_a to conform to our present notation).

By using the definition of ξ_a and setting $\delta = R/4$, we obtain

$$\begin{aligned}
\int_{B(R-\delta)} |\phi|^{2+2q} &\leq \int_{B(R-\delta)} \xi_a^{2+2q} |\phi|^{2+2q} \\
&\leq \int_M \xi_a^{2+2q} |\phi|^{2+2q} \\
&\leq \beta_3 \int_{B(4R)} |\phi|^2 |\nabla \xi_a|^{2+2q} \\
&\leq \beta_3 \left(\left(\frac{4}{R}\right)^{2+2q} + \left(\frac{8a}{R}\right)^{2+2q} \right) \int_{B(4R)} |\phi|^2 \\
&\leq \beta_3 (4^{2+2q} + (8a)^{2+2q}) \frac{1}{R^{2+2q}} \int_{B(4R)} |\phi|^2.
\end{aligned}$$

Now, let R go to infinity. Since a is bounded and $\lim_{R\to\infty} \frac{1}{R^{2+2q}} \int_M |\phi|^2 = 0$, we see that $|\phi| \equiv 0$, and since M is complete noncompact, M is a hyperplane.

References

ALENCAR, H. & DO CARMO, M., (1994) Hypersurfaces with constant mean curvature in space forms. *An. Acad. Bras. Ci.*, **66** (3): 265-274.

HARDY, G. H., (1954), Orders of Infinity, Cambridge University Press.