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Stability and geometric properties of constant weighted mean curvature hypersurfaces in gradient Ricci solitons

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Abstract In this paper, we study stability properties of hypersurfaces with constant weighted mean curvature (CWMC) in gradient Ricci solitons. The CWMC hypersurfaces generalize the f-minimal hypersurfaces and appear naturally in the isoperimetric problems in smooth metric measure spaces. We obtain a result about the relationship between the properness and extrinsic volume growth under the assumption of a limitation for the weighted mean curvature of the immersion. Moreover, we estimate Morse index for CWMC hypersurfaces in terms of the dimension of the space of parallel vector fields restricted to hypersurface.

Keywords Hypersurface · Weighted volume · Weighted mean curvature · Stability · Index

Mathematics Subject Classification 58J50 · 53C42 · 58E30

1 Introduction

In many problems in geometric analysis, it is natural to consider a Riemannian manifold $(\overline{M}^{n+1}, g)$ endowed with a measure $e^{-f}\mathrm{d}\mu$ that has a smooth positive density e^{-f} with respect to the Riemannian measure $d\mu$ induced by the metric g. A *smooth metric measure space* is a triple

$$\overline{M}_f^{n+1} = (\overline{M}^{n+1}, g, e^{-f}\mathrm{d}\mu).$$

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The smooth function $f : \overline{M} \to \mathbb{R}$ is called the *potential function*.

The smooth metric measure spaces arose in the study of diffusion processes on manifolds in the works of Bakry and Émery [1]. A natural extension of the Ricci tensor in this new context is the Bakry-Émery Ricci tensor given by

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\nabla}^2 f,$$

where $\overline{\nabla}^2 f$ is the Hessian of the potential function f on $\overline{M}^{n+1}$. It is known that a complete smooth metric measure space $\overline{M}_f^{n+1}$ satisfying $\overline{\mathrm{Ric}}_f \geq kg$, for some constant $k > 0$, is not necessarily compact. In fact, the shrinking Gaussian soliton

$$\left(\mathbb{R}^{n+1}, g_{\mathrm{can}}, e^{-\frac{|x|^2}{4}} \mathrm{d}x\right)$$

is noncompact complete and $\overline{\mathrm{Ric}}_f = \dfrac{1}{2} g_{\mathrm{can}}$, where g_{can} is canonical metric. The shrinking cylinder soliton

$$\left(\mathbb{S}^k_{\sqrt{2(k-1)}} \times \mathbb{R}^{n+1-k}, g, e^{-\frac{|x|^2}{4}} \mathrm{d}\theta \mathrm{d}x\right)$$

with product metric g, potential function $f(\theta, x) = \frac{|x|^2}{4}$, $\theta \in \mathbb{S}^k_{\sqrt{2(k-1)}}$, $x \in \mathbb{R}^{n+1-k}$, is another example of a noncompact complete smooth metric measure space with $\overline{\mathrm{Ric}}_f = \dfrac{1}{2} g$. The gradient Ricci solitons are natural generalizations of the Einstein metrics and were introduced by Hamilton in [13]. Indeed, a complete smooth metric measure space $\overline{M}_f^{n+1}$ is a *gradient Ricci soliton* if there exists a real constant k such that

$$\overline{\mathrm{Ric}}_f = kg.$$

If $k > 0$, the gradient Ricci soliton is called *shrinking soliton*. When the potential function is a constant, the gradient Ricci solitons are simply Einstein metrics. It is still important to mention that gradient Ricci solitons play an important role in Hamilton's Ricci flow and they correspond to self-similar solutions and often arise as type I singularity models, see [12].

Let $x\colon M^n \to \overline{M}_f^{n+1}$ be an isometric immersion of a Riemannian orientable manifold M^n into smooth metric measure space $\overline{M}_f^{n+1}$. The function $f\colon \overline{M} \to \mathbb{R}$, restricted to M, induces a weighted measure $e^{-f} d\sigma$ on M. Thus we have an induced smooth metric measure space $M_f^n = (M, g, e^{-f} \mathrm{d}\sigma)$.

The *second fundamental form* A of x is defined by

$$A(X, Y) = (\overline{\nabla}_X Y)^{\perp}, \qquad X, Y \in T_pM, \quad p \in M,$$

where $\perp$ symbolizes the projection above the normal bundle of M. The *weighted mean curvature vector* of M is defined by

$$\mathbf{H}_f = \mathbf{H} + (\overline{\nabla} f)^{\perp},$$

and the *weighted mean curvature* H_f is such that $\mathbf{H}_f = -H_f \eta$, where $\mathbf{H} = \mathrm{tr} A$ and η is unit outside normal vector field. The hypersurface M is called *f-minimal* when its weighted mean curvature vector $\mathbf{H}_f$ vanishes identically; when there exists real constant C such that $H_f = C$, we say that the hypersurface M has *constant weighted mean curvature* (CWMC).

The *weighted volume* of a measurable set $\Omega \subset M$ is given by

$$\mathrm{Vol}_f(\Omega) = \int_\Omega e^{-f}\,\mathrm{d}\sigma. \tag{1}$$

Let $B_r^{\overline{M}}$ be the geodesic ball of $\overline{M}$ with center in a fixed point $o \in \overline{M}$ and radius $r > 0$. It is said that the weighted volume of M has *polynomial growth* if there exists positive numbers α and C such that

$$\mathrm{Vol}_f(B_r^{\overline{M}} \cap M) \le Cr^\alpha \tag{2}$$

for any $r \ge 1$. When $\alpha = n$ in (2), M is said to have *Euclidean volume growth*.

We can consider either f-minimal or CWMC hypersurfaces in gradient Ricci solitons. We would like to point out that a self-shrinker to the mean curvature flow is a f-minimal hypersurface of the shrinking Gaussian soliton $\left(\mathbb{R}^{n+1}, g_{\mathrm{can}}, e^{-\frac{|x|^2}{4}}\mathrm{d}x\right)$. In [7], Cheng and Zhou proved that for f-minimal hypersurfaces in the shrinking Gaussian soliton $\mathbb{R}^{n+1}$, the conditions of proper immersion, Euclidean volume growth, polynomial volume growth, and finite weighted volume are equivalent to each other. Those equivalences are still being valid for f-minimal hypersurfaces immersed in a complete shrinking gradient Ricci soliton $\overline{M}_f$ satisfying $\overline{\mathrm{Ric}}_f = \frac{1}{2}g$, where g is Riemannian metric and f is a convex function (see [6], Corollary 1). In this direction, the following result was obtained:

Theorem 1 *Let M^n be a complete hypersurface isometrically immersed in a complete smooth metric measure space $\overline{M}^m_f$.*

(i) If $|\mathbf{H}_f| < \infty$ and $\mathrm{Vol}_f(M) < \infty$, then M^n is proper.
(ii) If $\overline{\mathrm{Ric}}_f = \frac{1}{2}g$, $f \in C^\infty(\overline{M})$ is convex function,

$$\sup_{x\in M}\langle \mathbf{H}_f, \overline{\nabla} f\rangle < \infty,$$

and M^n is proper, then $\mathrm{Vol}_f(M) < \infty$ and M^n has polynomial volume growth.

Let $L^2(e^{-f}\mathrm{d}\sigma)$ be the space of square integrable functions on M with respect to the measure $e^{-f}\mathrm{d}\sigma$ and with the norm

$$|u|_{L^2_f} = \left(\int_M u^2 e^{-f}\,\mathrm{d}\sigma\right)^{\frac{1}{2}}.$$

It is known that the *weighted Laplacian operator* Δ_f, defined by

$$\Delta_f u := \Delta u - \langle \nabla f, \nabla u\rangle,$$

is associated with $e^{-f}\mathrm{d}\sigma$ as well as Δ is associated with $\mathrm{d}\sigma$. Moreover, Δ_f is a self-adjoint operator on $L^2(e^{-f}\mathrm{d}\sigma)$, and therefore, the $L^2(e^{-f}\mathrm{d}\sigma)$ spectrum of Δ_f on M, denoted by $\sigma(-\Delta_f)$, is a subset of $[0, +\infty)$.

Next, let $F\colon (-\varepsilon, \varepsilon) \times M \to \overline{M}_f$, $F_f(p) = F(t, p)$ for all $t \in (-\varepsilon, \varepsilon)$ and $p \in M$ be a variation of the immersion x associated with the normal vector field $u\eta$, where $u \in C_c^\infty(M)$. The corresponding variation of the *weighted area functional* $\mathscr{A}_f(t) = \mathrm{Vol}_f(F_t(M))$ satisfies

$$\mathscr{A}_f'(0) = \int_M H_f u e^{-f}\,\mathrm{d}\sigma, \tag{3}$$

where H_f is such that $\mathbf{H}_f = -H_f\eta$. The expression (3) is known as *first variation formula*.

The f-minimal hypersurfaces are critical points of the weighted area functional. Yet, the CWMC hypersurfaces can be viewed as critical points of the weighted area functional restricted to variations which preserve the *enclosed weighted volume,* i.e., for functions $u \in C_c^\infty(M)$ which satisfy the additional condition

$$\int_M ue^{-f} \mathrm{d}\sigma = 0.$$

For such critical points, the *second variation* of the weighted area functional is given by

$$\mathscr{A}_f''(0) = -\int_M \left(u\Delta_f u + \left(|A|^2 + \overline{\mathrm{Ric}}_f(\eta, \eta)\right) u^2\right) \mathrm{d}\sigma,$$

where $\overline{\mathrm{Ric}}_f$ is the Bakry-Émery Ricci curvature and A is the second fundamental form. For more details, see [1], [6] and [17].

Remark 1 When f is a constant function, the first and second variation formula were given by Barbosa and do Carmo [2] and Barbosa, do Carmo, and Eschenburg [3].

The operator

$$L_f = \Delta_f + |A|^2 + \overline{\mathrm{Ric}}_f(\eta, \eta)$$

is called the f*-stability operator* of the immersion x. In the f-minimal case, the f-stability operator is viewed as acting on $\mathscr{F} = C_c^\infty(M)$; in the case of the CWMC hypersurfaces, the f-stability operator is viewed as acting on

$$\mathscr{F} = \left\{u \in C_c^\infty(M);\ \int_M ue^{-f} \mathrm{d}\sigma = 0\right\}.$$

Associated with L_f is the quadratic form

$$I_f(u, u) = -\int_M uL_f ue^{-f} \mathrm{d}\sigma.$$

For each compact domain $\Omega \subset M$, define the index, $\mathrm{Ind}_f(\Omega)$, of L_f in Ω as the maximal dimension of a subspace of $\mathscr{F}$ where I_f is a negative definite. The *index*, $\mathrm{Ind}_f(M)$, of L_f in M (or simply, the L_f-index of M) is then defined by

$$\mathrm{Ind}_f(M) = \sup_{\Omega \subset M} \mathrm{Ind}_f \Omega,$$

where the supreme is taken over all compact domains $\Omega \subset M$. We highlight that $\mathrm{Ind}_f(M)$ is the Morse index of the operator L_f in f-minimal hypersurfaces. For more details, see [8] and [11].

Let $M \subset \mathbb{R}^{n+1}$ be a proper, non-planar, two-sided hypersurface satisfying

$$\mathrm{Vol}_f(M) < \infty, \quad H = \frac{1}{2}\langle x, \eta\rangle + C \ \text{ and } \ \mathrm{Ind}_f(M) \leq n,$$

where H is the mean curvature, x is the position vector of $\mathbb{R}^{n+1}$, η is the unit normal field of the hypersurface, $\mathrm{Ind}_f(M)$ is the L_f-index and C is a real constant. McGonagle and Ross ([15], Theorem 5.6) showed that there exists a natural number i such that $n+1-\mathrm{Ind}_f(M) \leq i \leq n$ and $M = M_0 \times \mathbb{R}^i$. In addition, they obtained $\mathrm{Ind}_f(M) \geq 2$.

It is important to mention that the properness hypothesis can be removed of Theorem 5.6 of [15]. In fact, by Theorem 1, part (i), the finite weighted volume implies in the properness of its immersion.

Next, we obtain an estimate for L_f-index of a CWMC hypersurface with finite weighted volume and isometrically immersed in a gradient Ricci soliton that admits at least one parallel field globally defined.

Theorem 2 *Let M^n be a CWMC hypersurface isometrically immersed in a shrinking gradient Ricci soliton $\overline{M}^{n+1}_f$ with $\mathrm{Vol}_f(M) < \infty$. Denote by $\mathscr{P}_{\overline{M}_f}$ the set of parallel fields globally defined on $\overline{M}^{n+1}_f$ and η the unit normal field to M^n.*

(i) If the unit function $1 \notin \{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\}$,

$$\mathrm{Ind}_f(M) \geq \dim \mathscr{P}_{\overline{M}_f} - \dim\{X \in P_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\}. \tag{4}$$

(ii) If the unit function $1 \in \{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\}$, M^n is totally geodesic.

As a consequence of Theorem 2, we have

Corollary 1 *Let M^n be a CWMC hypersurface with finite weighted volume and isometrically immersed in a shrinking gradient Ricci soliton $\overline{M}^{n+1}_f$. If the unit function $1 \notin \{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\}$ and there exists a parallel field X_0 such that $\langle X_0, \eta\rangle \not\equiv 0$, then*

$$\mathrm{Ind}_f(M) \geq 1.$$

Moreover,

$$\dim\{X \in \mathscr{P}_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\} = \dim \mathscr{P}_{\overline{M}_f} - 1$$

whenever $\mathrm{Ind}_f(M) = 1$.

A necessary condition for equality to be achieved in the estimate (4) of Theorem 2 is given by

Theorem 3 *Let M^n be a CWMC hypersurface isometrically immersed in a shrinking gradient Ricci soliton $\overline{M}^{n+1}_f$ that satisfies $\overline{\mathrm{Ric}_f} = kg$. Denote by $\mathscr{P}_{\overline{M}_f}$ the set of parallel fields globally defined on $\overline{M}^{n+1}_f$ and η the unit normal field to M. Suppose that $\mathrm{Vol}_f(M) < \infty$, $\mathrm{Ind}_f(M) < \infty$, $\dim \mathscr{P}_{\overline{M}_f} > 0$, and*

$$\mathrm{Ind}_f(M) = \dim \mathscr{P}_{\overline{M}_f} - \dim\{X \in P_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\}.$$

(i) If $\mathrm{Ind}_f(M) = \dim \mathscr{P}_{\overline{M}_f}$, M is totally geodesic and the bottom $\mu_1(M)$ of the $L^2(e^{-f}\mathrm{d}\sigma)$ spectrum of the f-stability operator satisfies $\mu_1(M) = -k$.
(ii) If $\mathrm{Ind}_f(M) \neq \dim \mathscr{P}_{\overline{M}_f}$, either M^n is diffeomorphic to the product of a Euclidean space with some other manifold or there is a circle action on M whose orbits are not real homologous to zero.

It is important to highlight that for f-minimal, stability properties were studied by Colding and Minicozzi [10] and Hussey [14]. In fact, in [14], Hussey founded the spectrum and the eigenfunctions of the f-stability operator on the f-minimal hypersurfaces of the form $\mathbb{S}^k \times \mathbb{R}^{n-k}$ isometrically immersed in shrinking Gaussian soliton $\mathbb{R}^{n+1}$. He also used [10] to prove that, for any complete embedded non-planar f-minimal hypersurface with polynomial volume growth, the L_f-index is at least $n+2$. When the ambient space is a shrinking cylinder soliton $\mathbb{S}^n_{\sqrt{2(n-1)}} \times \mathbb{R}$, it was proved by Cheng et al. [5], and Cheng and Zhou [8] that the complete oriented proper f-minimal hypersurfaces have L_f-index at least one. They still classified those with L_f-index one. Moreover, in [18], Vieira and Zhou founded a domain in the shrinking cylinder soliton $\mathbb{S}^k_{\sqrt{2(k-1)}} \times \mathbb{R}^{n+1-k}$ that cannot contain proper f-minimal hypersurfaces.

The ambient spaces $\mathbb{R}^{n+1}$, $\mathbb{S}^n_{\sqrt{2(n-1)}} \times \mathbb{R}$ and $\mathbb{S}^k_{\sqrt{2(k-1)}} \times \mathbb{R}^{n+1-k}$ are examples of gradient Ricci solitons that admit parallel field globally defined.

2 Properness and finite weighted volume of CWMC hypersurfaces

We will begin by proving Theorem 1 which gives a relationship between the properness and extrinsic volume growth. For this, we highlight the following:

Remark 2 Let $\overline{M}_f$ be a complete gradient Ricci soliton satisfying $\overline{\mathrm{Ric}}_f = \frac{1}{2}g$. Cao and Zhou [4] showed that, by translating f,

$$\overline{R} + |\overline{\nabla} f|^2 - f = 0, \qquad \overline{R} + \overline{\Delta} f = \frac{m}{2}, \qquad \text{and} \qquad \overline{R} \geq 0. \tag{5}$$

Thus it follows from the expressions in (5) that

$$|\overline{\nabla} f|^2 \leq f.$$

In addition, there exists constants $c_1, c_2 \in \mathbb{R}$ such that

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2, \tag{6}$$

where $r(x) = \mathrm{dist}_{\overline{M}}(x, o)$ is the distance from $x \in \overline{M}$ to a fixed point $o \in \overline{M}$. The constant c_2 depends only on the dimension of the manifold and c_1 depends on the geometry of g on unit ball center in o (see [4], Theorem 1.1). In [16], Munteanu and Wang showed that the inequalities in (6) are true only assuming that $\overline{\mathrm{Ric}}_f \geq \frac{1}{2}g$ and $|\overline{\nabla} f|^2 \leq f$.

Proof of Theorem 1 Part (i): We suppose that M is not proper. Thus there exists a positive real number R such that $\overline{B}^{\overline{M}}_R(o) \cap M$ is not compact in M, where $\overline{B}^{\overline{M}}_R(o)$ denotes the closure of $B^{\overline{M}}_R(o)$. Then, for any $a > 0$ sufficiently small with $a < 2R$, there exists a sequence $\{p_k\}$ of the points in $B^{\overline{M}}_R(o) \cap M$ with $\mathrm{dist}_M(p_k, p_j) \geq a > 0$ for any different k and j. Since $B^M_{a/2}(p_k) \cap B^M_{a/2}(p_j) = \emptyset$ for any $k \neq j$, we obtain $B^M_{a/2}(p_j) \subset B^{\overline{M}}_{2R}(o)$, where $B^M_{a/2}(p_k)$ and $B^M_{a/2}(p_j)$ denote the intrinsic balls of M of radius $a/2$, center in p_k and p_j, respectively. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $T_x M$. If $x \in B^M_{a/2}(p_j)$, then the function extrinsic distance to p_j, denoted by $r_j(x) = \mathrm{dist}_{\overline{M}}(x, p_j)$, satisfies

$$\begin{aligned}
\sum_{i=1}^n \overline{\nabla}^2 r_j(e_i, e_i) &= \sum_{i=1}^n \langle \overline{\nabla}_{e_i} \overline{\nabla} r_j, e_i \rangle = \sum_{i=1}^n \Big(\langle \overline{\nabla}_{e_i} \nabla r_j, e_i \rangle + \langle \overline{\nabla}_{e_i} (\overline{\nabla} r_j)^\perp, e_i \rangle \Big) \\
&= \sum_{i=1}^n \langle \nabla_{e_i} \nabla r_j, e_i \rangle - \sum_{i=1}^n \langle A(e_i, e_i), \overline{\nabla} r_j \rangle \\
&= \Delta r_j - \langle \mathbf{H}, \overline{\nabla} r_j \rangle.
\end{aligned}$$

Observe that $\overline{M}$ has bounded locally geometry, i.e., there exists positive real numbers k and i_0 so that the sectional curvature of $\overline{M}$ is bounded above by k and the injectivity radius of $\overline{M}$ is bounded below by i_0 in a neighborhood of a point $o \in \overline{M}$. Choosing $R > 0$ such that $2R < \min\{i_0, 1/\sqrt{k}\}$, it follows from Hessian comparison theorem (see, for example, Lemma 7.1 in [9]), that

$$\overline{\nabla}^2 r_j(e_i, e_i) \geq -\sqrt{k} + \frac{1}{r_j}\left|e_i - \langle e_i, \overline{\nabla} r_j\rangle \overline{\nabla} r_j\right|^2$$

in $\overline{B}^{\overline{M}}_{2R}(o)$. Hence, in $\overline{B}^{\overline{M}}_{2R}(o) \cap M$,

$$\begin{aligned}
\Delta r_j &= \sum_{i=1}^{n} \overline{\nabla}^2 r_j(e_i, e_i) + \langle \mathbf{H}, \overline{\nabla} r_j\rangle \\
&\geq \sum_{i=1}^{n}\left(-\sqrt{k} + \frac{1}{r_j}\left|e_i - \langle e_i, \overline{\nabla} r_j\rangle \overline{\nabla} r_j\right|^2\right) \\
&\qquad + \langle \mathbf{H}, \overline{\nabla} r_j\rangle + \langle (\overline{\nabla} f)^{\perp}, \overline{\nabla} r_j\rangle - \langle (\overline{\nabla} f)^{\perp}, \overline{\nabla} r_j\rangle \\
&= -n\sqrt{k} + \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} + \langle \mathbf{H}_f, \overline{\nabla} r_j\rangle - \langle (\overline{\nabla} f)^{\perp}, \overline{\nabla} r_j\rangle \\
&\geq -n\sqrt{k} + \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - |\mathbf{H}_f| - |\overline{\nabla} f|.
\end{aligned}$$

By hypothesis, the norm of $\mathbf{H}_f$ is bounded above. Thus

$$\begin{aligned}
\Delta r_j &\geq -n\sqrt{k} + \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - \sup_{p \in B^{\overline{M}}_{2R}(o)\cap M} |\mathbf{H}_f(p)| - \sup_{p \in B^{\overline{M}}_{2R}(o)\cap M} |\overline{\nabla} f(p)| \\
&\geq \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - C,
\end{aligned}$$

where

$$C = n\sqrt{k} + \sup_{p \in B^{\overline{M}}_{2R}(o)\cap M} |\mathbf{H}_f(p)| + \sup_{p \in \overline{B}^{\overline{M}}_{2R}(o)} |\overline{\nabla} f(p)|.$$

Therefore, in $B^{\overline{M}}_{2R}(o) \cap M$,

$$\Delta r_j^2 = 2r_j \Delta r_j + 2|\nabla r_j|^2 \geq 2r_j\left(\frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - C\right) + 2|\nabla r_j|^2 = 2n - 2Cr_j.$$

Choosing $a < \min\{2n/C, 2R\}$, we have for $0 < \zeta \leq a/2$,

$$\begin{aligned}
\int_{B^M_\zeta(p_j)} (2n - 2Cr_j)\,\mathrm{d}\sigma &\leq \int_{B^M_\zeta(p_j)} \Delta r_j^2\,\mathrm{d}\sigma = \int_{\partial B^M_\zeta(p_j)} \langle \nabla r_j^2, \nu\rangle\,\mathrm{d}A \\
&\leq \int_{\partial B^M_\zeta(p_j)} 2r_j|\nabla r_j||\nu|\mathrm{d}A \leq \int_{\partial B^M_\zeta(p_j)} 2r_j\,\mathrm{d}A \\
&\leq 2\zeta A_j(\zeta), \qquad (7)
\end{aligned}$$

where ν denotes the unit normal vector field pointing out of $\partial B^M_\zeta(p_j)$ and $A_j(\zeta)$ denotes the area of $\partial B^M_\zeta(p_j)$. Using co-area formula, we have

$$\int_{B^M_\zeta(p_j)} (n - Cr_j)\,\mathrm{d}\sigma = \int_0^\zeta \int_{\partial B^M_t(p_j)} (n - Cr_j)\mathrm{d}A_t\,\mathrm{d}t$$

$$
\begin{aligned}
&\geq (n - C\zeta)\ {}_{0}\ \ {}_{\partial B_t^M(p_j)} \mathrm{d}A_t\, \mathrm{d}t \\
&= (n - C\zeta) V_j(\zeta),
\end{aligned}
$$

where $V_j(\zeta)$ denotes the volume of $B_\zeta^M(p_j)$. Therefore, it follows from (7) and previous inequality that

$$
(n - C\zeta) V_j(\zeta) \leq \zeta A_j(\zeta). \tag{8}
$$

Since

$$
V_j'(\zeta) = \frac{\mathrm{d}}{\mathrm{d}\zeta} \int_{B_\zeta^M(p_j)} \mathrm{d}\sigma = \frac{\mathrm{d}}{\mathrm{d}\zeta} \int_0^\zeta \int_{\partial B_t^M(p_j)} \mathrm{d}A_t\, \mathrm{d}t = \int_{\partial B_\zeta^M(p_j)} \mathrm{d}A = A_j(\zeta),
$$

then

$$
(n - C\zeta) V_j(\zeta) \leq \zeta V_j'(\zeta).
$$

Thus

$$
\frac{\mathrm{d}}{\mathrm{d}\zeta} \log(V_j(\zeta)) = \frac{V_j'(\zeta)}{V_j(\zeta)} \geq \frac{n}{\zeta} - C. \tag{9}
$$

Integrating (9) from $\varepsilon > 0$ to ζ, we obtain

$$
\log(V_j(\zeta)) - \log(V_j(\varepsilon)) \geq n \log \zeta - n \log \varepsilon - C(\zeta - \varepsilon),
$$

that is,

$$
\frac{V_j(\zeta)}{V_j(\varepsilon)} \geq \frac{\zeta^n}{\varepsilon^n} e^{-C(\zeta - \varepsilon)}.
$$

Now, observing that

$$
\lim_{\varepsilon \to 0^+} \frac{V_j(\varepsilon)}{\varepsilon^n} = \omega_n,
$$

we obtain

$$
V_j(\zeta) \geq \omega_n \zeta^n e^{-C\zeta}
$$

for $0 < \zeta \leq a/2$. Thus we conclude that

$$
\begin{aligned}
\mathrm{Vol}_f(M) = \int_M e^{-f} \mathrm{d}\sigma &\geq \sum_{j=1}^{\infty} \int_{B_{a/2}^M(p_j)} e^{-f} \mathrm{d}\sigma \\
&\geq \left(\inf_{\overline{B}_{2R}^{\overline{M}}(o)} e^{-f} \right) \sum_{j=1}^{\infty} V_j(a/2) = \infty.
\end{aligned}
$$

This contradicts the assumption of the finite weighted volume of M. Therefore M^n is a proper hypersurface of $\overline{M}_f^m$.

Part (ii): By hypothesis, $\overline{\mathrm{Ric}}_f = \frac{1}{2} g$. It follows from Remark 2 that

$$
\overline{\Delta} f - |\overline{\nabla} f|^2 + f = \frac{m}{2} \quad \text{and} \quad |\overline{\nabla} f|^2 \leq f.
$$

Since f is a convex function, i.e., $\overline{\nabla}^2 f \geq 0$, we have

$$\begin{aligned}\Delta_f f + f &= \Delta f - |\nabla f|^2 + f = \overline{\nabla}^2 f(e_i, e_i) + \langle \mathbf{H}, \overline{\nabla} f^{\perp}\rangle - |\nabla f|^2 + f \\ &= \overline{\Delta} f - \sum_{i=n+1}^{m} \overline{\nabla}^2 f(\eta_i, \eta_i) + \langle \mathbf{H}_f, \overline{\nabla} f^{\perp}\rangle - |\overline{\nabla} f^{\perp}|^2 - |\nabla f|^2 + f \\ &= \overline{\Delta} f - \sum_{i=n+1}^{m} \overline{\nabla}^2 f(\eta_i, \eta_i) + \langle \mathbf{H}_f, \overline{\nabla} f^{\perp}\rangle - |\overline{\nabla} f|^2 + f \\ &\leq \frac{m}{2} + C,\end{aligned}$$

with $C = \sup_{x \in M} \langle \mathbf{H}_f, \overline{\nabla} f^{\perp}\rangle < \infty$. Observe that

$$\frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2, \tag{10}$$

where c is a constant (see Remark 2, inequalities in (6)). Hence we can conclude that f is proper on $\overline{M}$. Since, by hypothesis, $x\colon M \to \overline{M}_f$ is a proper immersion, then $f|_M\colon M \to \mathbb{R}$ is a proper function. Therefore, it follows from Theorem 1.1 of [7] that M has finite weighted volume and polynomial volume growth of the sub-level set of the potential function f. $\square$

3 The L_f-index of CWMC hypersurfaces

In this section, we will prove Theorem 2, Corollary 1, and Theorem 3, which are results about the L_f-index of CWMC hypersurfaces isometrically immersed in gradient Ricci solitons that admit at least one parallel field globally defined. For this, we are going to give some definitions and state known results.

Proposition 1 *Let $\overline{M}_f^{n+1}$ be a gradient Ricci soliton satisfying $\overline{\mathrm{Ric}}_f = kg$ and X a parallel vector field on $\overline{M}_f^{n+1}$. If M^n is a CWMC hypersurface isometrically immersed in $\overline{M}_f^{n+1}$, then*

$$L_f\langle X, \eta\rangle = k\langle X, \eta\rangle$$

and

$$\Delta_f \langle X, \eta\rangle^2 = -2|A|^2\langle X, \eta\rangle^2 + 2|AX^{\top}|^2,$$

where η is the unit normal vector field to M.

Proof Let $\{e_1, e_2, \ldots, e_n\}$ be a geodesic orthonormal frame on M. By hypothesis, $H_f = C$, this is, $H = \langle \overline{\nabla} f, \eta\rangle + C$, where C is a real constant. Thus

$$\begin{aligned}\nabla H &= \sum_{i=1}^{n} e_i(H)e_i = \sum_{i=1}^{n} e_i(\langle \overline{\nabla} f, \eta\rangle)e_i = \sum_{i=1}^{n}\langle \overline{\nabla}_{e_i}\overline{\nabla} f, \eta\rangle e_i + \sum_{i=1}^{n}\langle \overline{\nabla} f, \overline{\nabla}_{e_i}\eta\rangle e_i \\ &= \sum_{i=1}^{n}\langle \overline{\nabla}_{e_i}\overline{\nabla} f, \eta\rangle e_i - \sum_{i=1}^{n}\langle A(e_i, e_j), \eta\rangle\langle \overline{\nabla} f, e_j\rangle e_i.\end{aligned} \tag{11}$$

For $u = \langle X, \eta\rangle$ and $a_{ij} = \langle Ae_i, e_j\rangle$, we have

$$\nabla u = \sum_{j=1}^{n} e_j(u)e_j = \sum_{j=1}^{n}\langle \overline{\nabla}_{e_j}\eta, X\rangle e_j = -\sum_{i,j=1}^{n} a_{ji}\langle e_i, X\rangle e_j. \tag{12}$$

It follows from (11) and (12),

$$\begin{aligned}\langle \nabla H, X\rangle &= \sum_{i=1}^{n}\langle \overline{\nabla}_{e_i}\overline{\nabla} f, \eta\rangle\langle e_i, X\rangle - \sum_{i,j=1}^{n} a_{ij}\langle \overline{\nabla} f, e_j\rangle\langle e_i, X\rangle \\ &= \langle \overline{\nabla}_{X^\top}\overline{\nabla} f, \eta\rangle + \langle \overline{\nabla} f, \nabla u\rangle. \end{aligned} \tag{13}$$

Moreover,

$$e_i(u) = \langle \overline{\nabla}_{e_i}\eta, X\rangle = -\sum_{j=1}^{n} a_{ij}\langle e_j, X\rangle.$$

Deriving the previous expression and observing that $\nabla_{e_k} e_j = 0$, we obtain

$$\begin{aligned} e_k(e_i(u)) &= -\sum_{j=1}^{n}\left(a_{ij,k}\langle e_j, X\rangle + a_{ij}\langle X, \overline{\nabla}_{e_k} e_j\rangle\right) \\ &= -\sum_{j=1}^{n} a_{ij,k}\langle e_j, X\rangle - \sum_{j=1}^{n} a_{ij}a_{kj}\langle X, \eta\rangle. \end{aligned} \tag{14}$$

It follows from Codazzi equation that

$$\overline{R}(e_j, e_k)e_i^{\perp} = \left(a_{ki,j} - a_{ji,k}\right)\eta,$$

that is,

$$\langle \overline{R}(e_j, e_k)e_i, \eta\rangle = a_{ki,j} - a_{ji,k}. \tag{15}$$

Replacing (15) in (14), we get

$$e_k(e_i(u)) = -\sum_{j=1}^{n} a_{ki,j}\langle e_j, X\rangle + \sum_{j=1}^{n}\langle \overline{R}(e_j, e_k)e_i, \eta\rangle\langle e_j, X\rangle - \sum_{j=1}^{n} a_{ij}a_{kj}\langle X, \eta\rangle$$

and using (13),

$$\begin{aligned} \Delta u &= \sum_{i=1}^{n} e_i(e_i(u)) = -\sum_{i,j=1}^{n} a_{ii,j}\langle e_j, X\rangle + \sum_{i,j=1}^{n}\langle \overline{R}(e_j, e_i)e_i, \eta\rangle\langle e_j, X\rangle - \sum_{i,j=1}^{n} a_{ij}a_{ij}\langle X, \eta\rangle \\ &= \langle \nabla H, X\rangle + \sum_{i=1}^{n}\langle \overline{R}(X^\top, e_i)e_i, \eta\rangle - |A|^2\langle X, \eta\rangle \\ &= \langle \overline{\nabla}_{X^\top}\overline{\nabla} f, \eta\rangle + \langle \overline{\nabla} f, \nabla u\rangle + \overline{\mathrm{Ric}}(X^\top, \eta) - |A|^2 u. \end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned} 0 = k\langle X^\top, \eta\rangle &= \overline{\mathrm{Ric}}_f(X^\top, \eta) = \overline{\mathrm{Ric}}(X^\top, \eta) + \overline{\nabla}^2 f(X^\top, \eta) \\ &= \overline{\mathrm{Ric}}(X^\top, \eta) + \langle \overline{\nabla}_{X^\top}\overline{\nabla} f, \eta\rangle. \end{aligned} \tag{17}$$

Therefore, it follows from (16) and (17),

$$\begin{aligned} \Delta_f u &= \Delta u - \langle \nabla f, \nabla u\rangle \\ &= \langle \overline{\nabla}_{X^\top}\overline{\nabla} f, \eta\rangle + \langle \overline{\nabla} f, \nabla u\rangle - \langle \overline{\nabla}_{X^\top}\overline{\nabla} f, \eta\rangle - |A|^2 u - \langle \nabla f, \nabla u\rangle \\ &= -|A|^2 u, \end{aligned} \tag{18}$$

implying that

$$L_f u = \Delta_f u + |A|^2 u + ku = ku.$$

Moreover, using the equality (18), we have that

$$\Delta_f u^2 = 2u\Delta_f u + 2|\nabla u|^2 = -2|A|^2 u^2 + 2|AX^\top|^2.$$

$\square$

Remark 3 In ([5], Proposition 2), Cheng et al. obtained expressions for $\Delta_f \langle X, \eta\rangle$ and $L_f\langle X, \eta\rangle$ of a f-minimal hypersurface isometrically immersed in a smooth metric measure space. However, the demonstration of Proposition 2 [5] obtained by them extends to CWMC hypersurfaces. We did the above demonstration of Proposition 1 only to the sake of completeness of the results.

The vector subspace of $C^\infty(M)$ *generated* by $E \subset C^\infty(M)$, denoted by Span E, is the set of all the linear combinations of the elements of E.

Let $\mathscr{P}_{\overline{M}}$ be the set of all tangent vector fields to $\overline{M}$ which are parallel and globally defined.

Lemma 1 *Let $\overline{M}^{n+1}_f$ be a shrinking gradient Ricci soliton satisfying $\overline{\mathrm{Ric}}_f = kg$ and M^n be a CWMC hypersurface isometrically immersed in $\overline{M}^{n+1}_f$. If M is compact, then I_f is negative defined in the*

$$\mathrm{Span}\left\{1, \langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\right\}.$$

Moreover,

$$\int_M |A|^2\langle X, \eta\rangle e^{-f}\mathrm{d}\sigma = 0.$$

Proof It follows from Proposition 1 that the function $u = \langle X, \eta\rangle$, with $X \in \mathscr{P}_{\overline{M}_f}$, satisfies $L_f u = ku$. Since M is compact,

$$\begin{aligned}\int_M kue^{-f}\mathrm{d}\sigma &= \int_M L_f ue^{-f}\mathrm{d}\sigma = \int_M \left(\Delta_f u + |A|^2 u + ku\right) e^{-f}\mathrm{d}\sigma \\ &= \int_M |A|^2 ue^{-f}\mathrm{d}\sigma + \int_M kue^{-f}\mathrm{d}\sigma.\end{aligned}$$

Thus

$$\int_M |A|^2 ue^{-f}\mathrm{d}\sigma = 0. \tag{19}$$

Observe that

$$I_f(1, 1) = -\int_M 1L_f 1e^{-f}\mathrm{d}\sigma = -\int_M \left(|A|^2 + k\right) e^{-f}\mathrm{d}\sigma$$

and

$$I_f(u, u) = -\int_M uL_f ue^{-f}\,\mathrm{d}\sigma = -k\int_M u^2 e^{-f}\,\mathrm{d}\sigma.$$

Therefore

$$I_f(c_0 + u, c_0 + u) = I_f(c_0, c_0) + I_f(u, u) + 2I_f(c_0, u)$$

$$= -\int_M \left[c_0^2|A|^2 + kc_0^2 + ku^2 + 2c_0\left(\Delta_f u + |A|^2 u + ku\right)\right] e^{-f} \mathrm{d}\sigma$$
$$= -\int_M \left[c_0^2|A|^2 + kc_0^2 + ku^2 + 2c_0 ku\right] e^{-f} \mathrm{d}\sigma$$
$$= -c_0^2 \int_M |A|^2 e^{-f} d\sigma - k \int_M (c_0 + u)^2 e^{-f} \mathrm{d}\sigma < 0,$$

where $u = \langle X, \eta \rangle$. This shows that I_f is negative defined in

$$\mathrm{Span}\left\{1, \langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\right\}.$$

□

Remark 4 Supposing that M^n has finite weighted volume and putting

$$\alpha = -\frac{\displaystyle\int_M \langle X, \eta\rangle e^{-f} \mathrm{d}\sigma}{\mathrm{Vol}_f(M)},$$

we can conclude that

$$\int_M (\alpha + \langle X, \eta\rangle) e^{-f} \mathrm{d}\sigma = 0.$$

Then

$$\mathscr{F} \cap \mathrm{Span}\left\{1, \langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\right\} \neq \emptyset.$$

Now we turn to noncompact manifolds. For this, we will consider the functions that have compact support in M.

Proposition 2 *Let $\overline{M}_f^{n+1}$ be a smooth metric measure space and M^n be a noncompact hypersurface isometrically immersed in $\overline{M}_f^{n+1}$. Then*

$$I_f(\phi u, \phi u) = -\int_M \phi^2 u L_f u e^{-f} \mathrm{d}\sigma + \int_M |\nabla \phi|^2 u^2 e^{-f} \mathrm{d}\sigma,$$

where $\phi \in C_c^\infty(M)$, $u \in C^\infty(M)$, and η denotes the unit normal field on M. Moreover, if $\overline{\mathrm{Ric}}_f = kg$ and M is a CWMC hypersurface, then

$$\int_M \phi^2 |A|^2 \langle X, \eta\rangle e^{-f} \mathrm{d}\sigma = -2\int_M \phi \langle \nabla \phi, AX^\top\rangle e^{-f} \mathrm{d}\sigma, \tag{20}$$

where X is a parallel vector field on $\overline{M}_f^{n+1}$.

Proof Note that

$$I_f(\phi u, \phi u) = -\int_M (\phi u) L_f(\phi u) e^{-f} \mathrm{d}\sigma$$
$$= -\int_M \left[(\phi u)\Delta_f(\phi u) + \left(|A|^2 + \overline{\mathrm{Ric}}_f(\eta, \eta)\right)\phi^2 u^2\right] e^{-f} \mathrm{d}\sigma$$
$$= -\int_M \left[\phi^2 u \Delta_f u + \phi u^2 \Delta_f \phi + 2\phi u \langle \nabla\phi, \nabla u\rangle\right] e^{-f} \mathrm{d}\sigma$$
$$- \int_M \left(|A|^2 + \overline{\mathrm{Ric}}_f(\eta, \eta)\right)\phi^2 u^2 e^{-f} \mathrm{d}\sigma.$$

Since ϕ has compact support, then

$$0=\int_M \operatorname{div}(\phi u^2 e^{-f}\nabla\phi)\mathrm{d}\sigma=\int_M \left(\phi u^2\Delta_f\phi+2u\phi\langle\nabla u,\nabla\phi\rangle+u^2|\nabla\phi|^2\right)e^{-f}\mathrm{d}\sigma.$$

Using the last two expressions, we obtain

$$\begin{aligned}I_f(\phi u,\phi u)&=-\int_M\left[\phi^2u\Delta_f u-|\nabla\phi|^2u^2+\left(|A|^2+\overline{\mathrm{Ric}}_f(\eta,\eta)\right)\phi^2u^2\right]e^{-f}\mathrm{d}\sigma\\&=-\int_M\phi^2uL_fue^{-f}\mathrm{d}\sigma+\int_M|\nabla\phi|^2u^2e^{-f}\mathrm{d}\sigma.\end{aligned}$$

Moreover, supposing $\overline{\mathrm{Ric}}_f=kg$, H_f is constant, and X is a parallel field on $\overline{M}_f$, we get

$$\begin{aligned}\int_M\phi^2k\langle X,\eta\rangle e^{-f}\mathrm{d}\sigma&=\int_M\phi^2L_f\langle X,\eta\rangle e^{-f}\mathrm{d}\sigma\\&=-\int_M\langle\nabla\phi^2,\nabla\langle X,\eta\rangle\rangle e^{-f}\mathrm{d}\sigma+\int_M\left(|A|^2+k\right)\phi^2\langle X,\eta\rangle e^{-f}\mathrm{d}\sigma.\end{aligned}$$

Therefore

$$\int_M|A|^2\phi^2\langle X,\eta\rangle e^{-f}\mathrm{d}\sigma=\int_M\langle\nabla\phi^2,\nabla\langle X,\eta\rangle\rangle e^{-f}\mathrm{d}\sigma=-2\int_M\phi\langle\nabla\phi,AX^\top\rangle e^{-f}\mathrm{d}\sigma.$$

□

Lemma 2 *Let $\overline{M}^{n+1}_f$ be a shrinking gradient Ricci soliton satisfying $\overline{\mathrm{Ric}}_f=kg$ and M^n be a CWMC hypersurface isometrically immersed in $\overline{M}^{n+1}_f$. If M is noncompact and $\mathrm{Vol}_f(M)<\infty$, then there exists $\phi\in C_c^\infty(M)$ such that I_f is negative defined in ϕV and $\dim(\phi V)=\dim V$, where*

$$V=\mathrm{Span}\left\{1,\langle X,\eta\rangle:\ X\in\mathscr{P}_{\overline{M}_f}\right\}.$$

Moreover, assuming that $\int_M|A|^2e^{-f}d\sigma<\infty$, we have

$$\int_M|A|^2\langle X,\eta\rangle e^{-f}\mathrm{d}\sigma=0.$$

Proof Let $u=c_0+\langle X,\eta\rangle$, where c_0 is a real constant and X is a parallel field on $\overline{M}^{n+1}_f$. Since H_f is constant, then by Proposition 1, we have

$$L_fu=L_f\langle X,\eta\rangle+\left(|A|^2+k\right)c_0=k\langle X,\eta\rangle+\left(|A|^2+k\right)c_0.\tag{21}$$

It follows from Proposition 2 and equality (21) that

$$\begin{aligned}I_f(\phi u,\phi u)&=-\int_M\phi^2uL_fue^{-f}\mathrm{d}\sigma+\int_M|\nabla\phi|^2u^2e^{-f}\mathrm{d}\sigma\\&=-\int_M\phi^2u\left(k\langle X,\eta\rangle+\left(|A|^2+k\right)c_0\right)e^{-f}\mathrm{d}\sigma+\int_M|\nabla\phi|^2u^2e^{-f}\mathrm{d}\sigma\\&=-k\int_M\phi^2u^2e^{-f}\mathrm{d}\sigma-\int_M\phi^2|A|^2c_0^2e^{-f}\mathrm{d}\sigma-\int_M\phi^2|A|^2c_0\langle X,\eta\rangle e^{-f}\mathrm{d}\sigma\\&\quad+\int_M|\nabla\phi|^2u^2e^{-f}\mathrm{d}\sigma.\end{aligned}$$

Now, using once more Proposition 2 and Cauchy–Schwarz's inequality,

$$\begin{aligned}\left|\int_M \phi^2|A|^2c_0\langle X,\eta\rangle e^{-f}\,\mathrm{d}\sigma\right| &= 2\left|\int_M \phi\langle\nabla\phi, AX^\top\rangle c_0\, e^{-f}\,\mathrm{d}\sigma\right|\\ &\le 2\int_M |\phi|\,|\nabla\phi|\,|A|\,|X^\top|\,|c_0|\,e^{-f}\,\mathrm{d}\sigma\\ &\le \int_M \phi^2|A|^2c_0^2\,e^{-f}\,\mathrm{d}\sigma + \int_M |\nabla\phi|^2|X^\top|^2e^{-f}\,\mathrm{d}\sigma.\end{aligned}$$

Therefore

$$I_f(\phi u,\phi u) \le -k\int_M \phi^2u^2e^{-f}\,\mathrm{d}\sigma + \int_M |\nabla\phi|^2(u^2+|X^\top|^2)e^{-f}\,\mathrm{d}\sigma. \tag{22}$$

Let $r(x)$ be the extrinsic distance from $x\in M$ to a fixed point $o\in\overline{M}_f$. For $R>0$ sufficiently large, define the function $\phi_R\colon M\to\mathbb{R}$ such that

$$\phi_R(x) = \begin{cases} 1, & r(x)\le R,\\ \frac{2R-r(x)}{R}, & R\le r(x)\le 2R,\\ 0, & r(x)\ge 2R.\end{cases} \tag{23}$$

Observe that $\phi_R\in C_c^\infty(M)$ because M is proper (see Theorem 1, part (i)), and $|\nabla\phi_R|\le 1/R$. Now, substituting $\phi=\phi_R$ in the inequality (22), we obtain

$$I_f(\phi_R u,\phi_R u) \le -k\int_M \phi_R^2u^2e^{-f}\,\mathrm{d}\sigma + \frac{|X|^2+c_0^2+2|c_0|\,|X|}{R^2}\int_{M\cap(B_{2R}\setminus B_R)} e^{-f}\,\mathrm{d}\sigma,$$

(recall that X is parallel and therefore $|X|$ is constant). Since $\mathrm{Vol}_f(M)<\infty$ and $|X|$ is constant, for each $u\in V$ there exists R_u sufficiently large such that

$$I_f(\phi_{R_u}u,\phi_{R_u}u)<0.$$

Let us find a function $\phi\in C_c^\infty(M)$ that is not dependent of the function u and $I_f(\phi u,\phi u)<0$. In fact, we consider the subset

$$S=\left\{u\in V:\ \int_M u^2e^{-f}\,\mathrm{d}\sigma = 1\right\}.$$

Note that $V\subset L^2(e^{-f}\mathrm{d}\sigma)$ is a subspace of finite dimension smaller than or equal to $\dim\mathscr{P}_{\overline{M}_f}+1$. Thus S is a compact set in $L^2(e^{-f}\mathrm{d}\sigma)$ and there exists a positive real number R_0 such that any function $u\in S$ does not vanishes on $M\cap B_{R_0}^{\overline{M}_f}(o)$. Otherwise, we could get a sequence $R_j\to\infty$ of positive numbers so that for each j there exists $u_j\in S$ with $u_j\equiv 0$ on $M\cap B_{R_j}^{\overline{M}_f}(o)$. Hence we would have

$$u=\lim_{j\to\infty}u_j\ \in\ S$$

and $u\equiv 0$ on M. However, this is not possible because if $u\in S$, then

$$\int_M u^2e^{-f}\,\mathrm{d}\sigma = 1.$$

For R sufficiently large and $R\ge R_0$, and for any function $u\in S$, we have

$$I_f(\phi_R u,\phi_R u) \le -k\int_M \phi_R^2u^2e^{-f}\,\mathrm{d}\sigma + \frac{|X|^2+c_0^2+2|c_0||X|}{R^2}\int_{M\cap(B_{2R}\setminus B_R)} e^{-f}\,\mathrm{d}\sigma < 0.$$

In fact,

$$M(R) = \int_M \phi_R^2 u^2 e^{-f} \mathrm{d}\sigma > 0$$

is an increasing function on R and

$$\lim_{R\to\infty} \frac{|X|^2 + c_0^2 + 2|c_0||X|}{R^2} \int_{M\cap(B_{2R}\setminus B_R)} e^{-f} \mathrm{d}\sigma = 0,$$

since $\mathrm{Vol}_f(M) < \infty$. Here, $\phi = \phi_R$ independent of $u \in S$. Therefore, if $u \in V$, $\frac{1}{|u|_{L_f^2}} u \in S$, and thus, for $u \not\equiv 0$,

$$I_f(\phi u, \phi u) = |u|_{L_f^2}^2 I_f\left(\phi \frac{u}{|u|_{L_f^2}}, \phi \frac{u}{|u|_{L_f^2}}\right) < 0.$$

We will show that $\dim V = \dim(\phi V)$. In fact, let $\{u_1, u_2, \ldots, u_s\}$ be an orthonormal basis to the vector subspace $V \subset L^2(e^{-f} \mathrm{d}\sigma)$. For the function ϕ built here, we have that $u_i \not\equiv 0$ on $M \cap B_{R_0}^{\overline{M}_f}(o)$. Therefore $\{\phi u_1, \phi u_2, \ldots, \phi u_s\}$ is linearly independent and $\dim(\phi V) = \dim V$.

Finally, it follows from Proposition 2, equality (20) with $\phi = \sqrt{\phi}_j$, and Cauchy Schwarz's inequality that

$$\begin{aligned}
\left|\int_M \phi_j |A|^2 \langle X, \eta\rangle e^{-f} \mathrm{d}\sigma\right| &= \left|-\int_M \langle \nabla\phi_j, AX^\top\rangle e^{-f} \mathrm{d}\sigma\right| \\
&\le \left(\int_M |\nabla\phi_j|^2 e^{-f} \mathrm{d}\sigma\right)^{\frac{1}{2}} \left(\int_M |AX^\top|^2 e^{-f} \mathrm{d}\sigma\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, putting $R = j$ in the function defined in (23) and reviewing that $|X|$ is constant, we get

$$\left|\int_M \phi_j |A|^2 \langle X, \eta\rangle e^{-f} \mathrm{d}\sigma\right| \le \frac{|X|}{j} \left(\int_{M\cap(B_{2j}\setminus B_j)} e^{-f} \mathrm{d}\sigma\right)^{\frac{1}{2}} \left(\int_M |A|^2 e^{-f} \mathrm{d}\sigma\right)^{\frac{1}{2}}. \quad (24)$$

By hypothesis, M has finite weighted volume and $\int_M |A|^2 e^{-f} \mathrm{d}\sigma < \infty$, hence the right-hand side of (24) tends to zero as $j \to \infty$, and we can conclude that

$$\lim_{j\to\infty} \int_M \phi_j |A|^2 \langle X, \eta\rangle e^{-f} \mathrm{d}\sigma = 0. \quad (25)$$

Note that

$$\lim_{j\to\infty} (\phi_j |A|^2 \langle X, \eta\rangle)(x) = (|A|^2 \langle X, \eta\rangle)(x) \text{ for each } x \in M,$$
$$|\phi_j |A|^2 \langle X, \eta\rangle| \le |A|^2 |\langle X, \eta\rangle|,$$

and

$$0 \le \int_M |A|^2 |\langle X, \eta\rangle| e^{-f} \mathrm{d}\sigma \le |X|^2 \int_M |A|^2 e^{-f} \mathrm{d}\sigma < \infty.$$

Therefore, using the dominated convergence theorem and expression (25), we get

$$\int_M |A|^2 \langle X, \eta\rangle e^{-f} \mathrm{d}\sigma = 0.$$

□

Proof of Theorem 2 Part (i): Observe that if $1 \notin \{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\}$,

$$\dim V = 1 + \dim \left\{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\right\},$$

where $V = \mathrm{Span}\{1, \langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\}$. By Lemmas 1 and 2, there exists a function $\phi \in C_c^\infty(M)$ such that $\dim \phi V = \dim V$ and I_f is negative defined in ϕV. Recall that the $\mathrm{Ind}_f(M)$ is the maximal dimension of a subspace of $\mathscr{F}$ which I_f is negative defined, where $\mathscr{F} = C_c^\infty(M)$ if M is a f-minimal and

$$\mathscr{F} = \left\{u \in C_c^\infty(M);\ \int_M ue^{-f}\mathrm{d}\sigma = 0\right\}$$

if M is a CWMC hypersurface.

Now observe that

$$\dim\{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\} \leq \dim(\mathscr{F} \cap \phi V).$$

In fact, $|X|$ is constant because X is a parallel field. Since the weighted volume of M is finite, then $\int_M \phi\langle X, \eta\rangle e^{-f}\mathrm{d}\sigma \leq |X| \int_M \phi e^{-f}\mathrm{d}\sigma < \infty$. Thus there exists a real number c_0 satisfying

$$\int_M \phi(c_0 + \langle X, \eta\rangle)e^{-f}\mathrm{d}\sigma = 0,$$

and hence $\phi(c_0 + \langle X, \eta\rangle) \in \mathscr{F} \cap \phi V$. Therefore

$$\dim\{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\} \leq \dim(\mathscr{F} \cap \phi V) \leq \mathrm{Ind}_f(M).$$

Consider the linear transformation $T : \mathscr{P}_{\overline{M}_f} \to C^\infty(M)$ defined by $T(X) = \langle X, \eta\rangle$. Now, applying the kernel and image theorem, it turns

$$\begin{aligned} \dim \mathscr{P}_{\overline{M}_f} &= \dim \left\{X \in \mathscr{P}_{\overline{M}_f} : T(X) \equiv 0\} + \dim\{T(X) : X \in \mathscr{P}_{\overline{M}_f}\right\} \\ &\leq \dim\{X \in \mathscr{P}_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\} + \mathrm{Ind}_f(M). \end{aligned} \tag{26}$$

Part (ii): Otherwise, if $1 \in \{\langle X, \eta\rangle : X \in \mathscr{P}_{\overline{M}_f}\}$, $L_f 1 = k$ by Proposition 1, and $L_f 1 = |A|^2 + k$. Thus $|A|^2 \equiv 0$, i.e., M is totally geodesic. □

Proof of Corollary 1 Note that

$$\dim\{X \in \mathscr{P}_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\} \leq \dim \mathscr{P}_{\overline{M}_f} - 1 \tag{27}$$

as long as we assume that there exists a field $X_0 \in \mathscr{P}_{\overline{M}_f}$ such that $\langle X_0, \eta\rangle \not\equiv 0$. It follows from Theorem 2 and from the inequality (27) that

$$\dim \mathscr{P}_{\overline{M}_f} - \mathrm{Ind}_f M \leq \dim\{X \in \mathscr{P}_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\} \leq \dim \mathscr{P}_{\overline{M}_f} - 1.$$

Therefore

$$\mathrm{Ind}_f M \geq 1.$$

Supposing that $\mathrm{Ind}_f M = 1$, we have

$$\dim \mathscr{P}_{\overline{M}_f} - 1 \leq \dim\{X \in \mathscr{P}_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\} \leq \dim \mathscr{P}_{\overline{M}_f} - 1,$$

and consequently,

$$\dim\{X \in \mathscr{P}_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\} = \dim \mathscr{P}_{\overline{M}_f} - 1.$$

□

Now, let us obtain a necessary condition for a CWMC hypersurface M^n with finite weighted volume to satisfy the following equality:

$$\mathrm{Ind}_f(M) = \dim \mathscr{P}_{\overline{M}_f} - \dim\{X \in P_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\}.$$

For this, we will prove some lemmas. They are adaptations of known results.

Lemma 3 *Let $\Omega \subset M$ be a compact set and let $u, v \in C^\infty(\Omega)$, then*

$$\int_\Omega (u\Delta_f v)e^{-f}\,\mathrm{d}\sigma + \int_\Omega \langle \nabla u, \nabla v\rangle e^{-f}\,\mathrm{d}\sigma = \int_{\partial\Omega} u\nu(v)e^{-f}\,\mathrm{d}\partial\Omega, \quad \text{and} \tag{28}$$

$$\int_\Omega (u\Delta_f v - v\Delta_f u)e^{-f}\,\mathrm{d}\sigma = \int_{\partial\Omega} (u\nu(v) - v\nu(u))e^{-f}\,\mathrm{d}\partial\Omega, \tag{29}$$

where ν denotes the exterior unit normal to Ω along $\partial\Omega$.

Proof In fact,

$$\begin{aligned}\mathrm{div}(e^{-f}u\nabla v) &= e^{-f}u\,\mathrm{div}(\nabla v) + \langle \nabla(e^{-f}u), \nabla v\rangle \\ &= e^{-f}u\Delta v - e^{-f}u\langle \nabla f, \nabla v\rangle + e^{-f}\langle \nabla u, \nabla v\rangle \\ &= e^{-f}u\Delta_f v + e^{-f}\langle \nabla u, \nabla v\rangle.\end{aligned}$$

Integrating both sides from above identity and applying the divergent theorem to the field $X = e^{-f}u\nabla v$, it follows the equality (28). The equality (29) is obtained by integrating the difference $u\Delta_f v - v\Delta_f u$ and applying the equality (28). □

Lemma 4 *([10], Corollary 3.10) Suppose that M is a complete hypersurface without boundary. If u, v are C^2 functions with*

$$\int_M \left(|u\nabla v| + |\nabla u|\,|\nabla v| + |u\Delta_f v|\right) e^{-f}\mathrm{d}\sigma < \infty, \tag{30}$$

then

$$\int_M u(\Delta_f v)e^{-f}\mathrm{d}\sigma = -\int_M \langle \nabla v, \nabla u\rangle e^{-f}\mathrm{d}\sigma. \tag{31}$$

Let $W^{1,2}(e^{-f}\mathrm{d}\sigma)$ be the weighted Sobolev space, which is the space of the functions u on M satisfying

$$\int_M (u^2 + |\nabla u|^2)e^{-f}\mathrm{d}\sigma < \infty,$$

with the norm

$$|u|_{W_f^{1,2}} := \left(\int_M (u^2 + |\nabla u|^2)e^{-f}\mathrm{d}\sigma\right)^{\frac{1}{2}}.$$

Lemma 5 *Let M^n be a CWMC hypersurface isometrically immersed in a gradient Ricci soliton $\overline{M}_f$ that satisfies $\overline{\mathrm{Ric}}_f = kg$. Suppose that h is a C^2 function with $L_f h = -\mu h$ for $\mu \in \mathbb{R}$.*

(i) If $h \in W^{1,2}(e^{-f}\mathrm{d}\sigma)$, then $|A|h \in L^2(e^{-f}\mathrm{d}\sigma)$ and

$$\int_M |A|^2h^2e^{-f}\mathrm{d}\sigma \le \int_M \left((1-k-\mu)h^2 + 2|\nabla h|^2\right)e^{-f}\mathrm{d}\sigma. \tag{32}$$

(ii) If $h>0$ and $\phi \in W^{1,2}(e^{-f}\mathrm{d}\sigma)$, then

$$\int_M \phi^2\left(2|A|^2+|\nabla \log h|^2\right)e^{-f}\mathrm{d}\sigma \le \int_M \left[4|\nabla\phi|^2-2(\mu+k)\phi^2\right]e^{-f}\mathrm{d}\sigma. \quad (33)$$

Proof Part (i): Let $\phi \in C_c^\infty(M)$. Note that

$$\Delta_f h^2 = 2|\nabla h|^2 + 2h\Delta_f h$$

and

$$\Delta_f h = \left(L_f - |A|^2 - k\right)h = -\left(\mu + |A|^2 + k\right)h. \quad (34)$$

By Lemma 3 and equality (34),

$$\begin{aligned}\int_M \langle \nabla\phi^2, \nabla h^2\rangle e^{-f}\mathrm{d}\sigma &= -\int_M \phi^2 \Delta_f h^2 e^{-f}\mathrm{d}\sigma \\ &= -2\int_M \phi^2\left[|\nabla h|^2 - \left(\mu+|A|^2+k\right)h^2\right]e^{-f}\mathrm{d}\sigma. \quad (35)\end{aligned}$$

Assume now that $\phi \le 1$ and $|\nabla\phi| \le 1$. Rearranging the terms in (35) and using the inequality

$$0 \le |\phi\nabla h - h\nabla\phi|^2 = \phi^2|\nabla h|^2 + h^2|\nabla\phi|^2 - 2\phi h\langle\nabla\phi,\nabla h\rangle \le |\nabla h|^2 + h^2 - 2\phi h\langle\nabla\phi,\nabla h\rangle,$$

we have

$$\begin{aligned}\int_M \phi^2\left(2k+2\mu+2|A|^2\right)h^2 e^{-f}\mathrm{d}\sigma &= 4\int_M \phi h\langle\nabla\phi,\nabla h\rangle e^{-f}\mathrm{d}\sigma + 2\int_M \phi^2|\nabla h|^2 e^{-f}\mathrm{d}\sigma \\ &\le 2\int_M h^2 e^{-f}\mathrm{d}\sigma + 4\int_M |\nabla h|^2 e^{-f}\mathrm{d}\sigma. \quad (36)\end{aligned}$$

Finally, consider the intrinsic ball $B_j = B_j(p)$ in M of radius j and center at a fixed point $p \in M$. Applying (36) with $\phi = \phi_j$, where ϕ_j is one on B_j and cuts off linearly to zero from ∂B_j to ∂B_{j+1}, letting $j \to \infty$, and using the monotone convergence theorem we obtain

$$\int_M |A|^2 h^2 e^{-f}\mathrm{d}\sigma \le \int_M \left((1-k-\mu)h^2 + 2|\nabla h|^2\right)e^{-f}\mathrm{d}\sigma.$$

Part (ii): Now we will prove the inequality (33). In fact, $\log h$ is well-defined and

$$\begin{aligned}\Delta_f \log h &= \frac{1}{h}\Delta_f h - |\nabla\log h|^2 = \frac{1}{h}L_f h - |A|^2 - k - |\nabla \log h|^2 \\ &= -\mu - |A|^2 - k - |\nabla\log h|^2. \quad (37)\end{aligned}$$

Let $\psi \in C_c^\infty(M)$. It follows from Lemma 3 and expression (37) that

$$\begin{aligned}\int_M \langle\nabla\psi^2, \nabla\log h\rangle e^{-f}\mathrm{d}\sigma &= -\int_M \psi^2(\Delta_f \log h)e^{-f}\mathrm{d}\sigma \\ &= \int_M \psi^2\left(\mu+|A|^2+k+|\nabla\log h|^2\right)e^{-f}\mathrm{d}\sigma. \quad (38)\end{aligned}$$

Combining (38) with the following inequality

$$\langle\nabla\psi^2,\nabla\log h\rangle \le 2|\nabla\psi|^2 + \frac{1}{2}\psi^2|\nabla\log h|^2,$$

gives us this

$$\int_M \psi^2\left(2|A|^2+|\nabla\log h|^2\right)e^{-f}\mathrm{d}\sigma \le \int_M \left(4|\nabla\psi|^2 - 2\mu\psi^2 - 2k\psi^2\right)e^{-f}\mathrm{d}\sigma. \quad (39)$$

Let ψ_j be one on B_j and cut off linearly to zero from ∂B_j to ∂B_{j+1}. Since $\phi \in W^{1,2}(e^{-f}\mathrm{d}\sigma)$, applying (39) with $\psi = \psi_j\phi$, and letting $j \to \infty$, using the monotone convergence theorem, we get

$$\int_M \phi^2 \left(2|A|^2 + |\nabla \log h|^2\right) e^{-f}\mathrm{d}\sigma \le \int_M \left[4|\nabla\phi|^2 - 2(\mu + k)\phi^2\right] e^{-f}\mathrm{d}\sigma.$$

□

Remark 5 Lemma 5 was obtained by Colding and Minicozzi (see [10], Lemma 9.15) for a complete noncompact hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ without boundary that satisfies $H = \frac{\langle x,\eta\rangle}{2}$, where x is position vector.

Through use Lemmas 4 and 5, and ideas as the in proof of Lemma 9.25 of [10], we have

Lemma 6 *Let $\mu_1(M)$ be the bottom for $L^2(e^{-f}\mathrm{d}\sigma)$ spectrum of the f-stability operator L_f. If $\mu_1(M) \neq -\infty$, then there exists a positive C^2 function u on M with $L_f u = -\mu_1(M)u$. Moreover, if $w \in W^{1,2}(e^{-f}\mathrm{d}\sigma)$ and $L_f w = -\mu_1(M)w$, then $w = Cu$ for some $C \in \mathbb{R}$.*

Lemma 7 *Let M be a complete oriented CWMC hypersurface in gradient Ricci soliton $\overline{M}_f$. If $\mu_1(M) \neq -\infty$ and $\mathrm{Vol}_f(M) < \infty$, then*

$$\int_M |A|^2 e^{-f}\mathrm{d}\sigma < \infty. \tag{40}$$

Proof Since $\mu_1(M) \neq -\infty$, there exists a C^2 positive function h on M satisfying $L_f h = -\mu_1(M)h$ by Lemma 6. Let ϕ_j be the cut off function such that $|\nabla\phi_j| \le 1$ and $\phi_j \le 1$. So $\phi_j \in W^{1,2}(e^{-f}\mathrm{d}\sigma)$ because $\mathrm{Vol}_f(M) < \infty$. By Lemma 5, equality (33), we have

$$\begin{aligned}
2\int_M \phi_j^2|A|^2e^{-f}\mathrm{d}\sigma &\le \int_M \phi_j^2\left(2|A|^2 + |\nabla \log h|^2\right)e^{-f}\mathrm{d}\sigma\\
&\le \int_M \left(4|\nabla\phi_j|^2 - 2(\mu_1(M) + k)\phi_j^2\right)e^{-f}\mathrm{d}\sigma\\
&\le (4 + 2|\mu_1(M) + k|)\int_M e^{-f}\mathrm{d}\sigma < \infty.
\end{aligned}$$

Letting $j \to \infty$, we obtain the conclusion of this lemma by monotone convergence theorem.

□

The compact manifolds which admit a parallel vector field with respect to some metric were characterized by Welsh in [19]. Namely, they are the compact fiber bundles over tori with finite structural group. The dimension of the torus can be assumed to be the number of linearly independent parallel vector fields. The noncompact manifolds case has also been solved by Welsh in [20]. If fact,

Proposition 3 *([20], Proposition 2.1) If a Riemannian manifold M admits a complete parallel vector field, then either M is diffeomorphic to the product of a Euclidean space with some other manifold or there is a circle action on M whose orbits are not real homologous to zero.*

Proof of Theorem 3 Now suppose that $\dim \mathscr{P}_{\overline{M}_f} = l > 0$. Since, by hypothesis,

$$\mathrm{Ind}_f(M) = \dim \mathscr{P}_{\overline{M}_f} - \dim\{X \in P_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\},$$

we can have two cases: either $\mathrm{Ind}_f(M) = \dim \mathscr{P}_{\overline{M}_f}$ or $\mathrm{Ind}_f(M) \neq \dim \mathscr{P}_{\overline{M}_f}$.
Part (i): Initially, suppose that $\mathrm{Ind}_f(M) = \dim \mathscr{P}_{\overline{M}_f}$, i.e.,

$$\dim\{X \in \mathscr{P}_{\overline{M}_f} : \langle X, \eta\rangle \equiv 0\} = 0$$

and there exists l independent linearly parallel unit vector fields $X_1, X_2, \dots, X_l$ such that $\langle X_j, \eta\rangle \not\equiv 0$ for all $j = 1, 2, \dots, l$. Put

$$u_j = \langle X_j, \eta\rangle \quad \text{for all } j = 1, 2, \dots, l.$$

By Proposition 1,

$$L_f u_j = k u_j,$$

where $\overline{\mathrm{Ric}}_f = kg$. Note that

$$\int_M u_j^2 e^{-f} \mathrm{d}\sigma \leq \int_M e^{-f} \mathrm{d}\sigma < \infty \tag{41}$$

because

$$u_j = \langle X_j, \eta\rangle \leq |X_j||\eta| = 1.$$

Hence $u_1, u_2, \dots, u_l$ are $L^2(e^{-f}\mathrm{d}\sigma)$ eigenfunctions with negative eigenvalues $-k$. Since, by hypothesis,

$$\mathrm{Ind}_f(M) = l > 0,$$

then the bottom $\mu_1(M)$ of the spectrum of L_f satisfies $\mu_1(M) = -k$.
Now consider a local orthonormal frame $\{e_1, e_2, \dots, e_n\}$ on M. Observe that

$$|\nabla u_j|^2 = \sum_{i=1}^n |\nabla_{e_i} u_j|^2 \leq \sum_{i=1}^n \left(\sum_{k=1}^n a_{ik}^2\right)\left(\sum_{k=1}^n \langle e_k, X_j\rangle^2\right) \leq |A|^2. \tag{42}$$

Hence it follows from Lemma 7 and inequality (42) that

$$\int_M |\nabla u_j|^2 e^{-f} \mathrm{d}\sigma \leq \int_M |A|^2 e^{-f} \mathrm{d}\sigma < \infty$$

and using (41), we get $u_j \in W^{1,2}(e^{-f}\mathrm{d}\sigma)$. By Lemma 6, $u_j > 0$ on M without lost of generality. Therefore, it follows from Lemma 2 and the integrability of $|A|^2$, that

$$\int_M |A|^2 u_j e^{-f} \mathrm{d}\sigma = 0.$$

Thus $|A| \equiv 0$ on M.
Part (ii): In the case which $\mathrm{Ind}_f(M) \neq \dim \mathscr{P}_{\overline{M}_f}$, we have that there exists $X_0 \in \mathscr{P}_{\overline{M}_f}$ such that $\langle X_0, \eta\rangle \equiv 0$. Thus $X_0 \in TM$ and it is parallel with respect to Riemannian connection of (M, g). By Proposition 3, either M is diffeomorphic to the product of a Euclidean space with some other manifold, or there is a circle action on M whose orbits are not real homologous to zero.

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