Eigenvalue estimates for a class of elliptic differential operators on compact manifolds

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Abstract. The motivation of this paper is to study a second order elliptic operator which appears naturally in Riemannian geometry, for instance in the study of hypersurfaces with constant r-mean curvature. We prove a generalized Bochner-type formula for such a kind of operators and as applications we obtain some sharp estimates for the first nonzero eigenvalues in two special cases. These results can be considered as generalizations of the Lichnerowicz-Obata Theorem.

Keywords: Riemannian manifolds, first eigenvalue, elliptic operator, Bochner formula.

Mathematical subject classification: 53C42.

1 Introduction

Let $\{\omega_1, \ldots, \omega_n\}$ be a local coframe field defined on a Riemannian manifold (M, g). For a symmetric tensor $\phi = \sum_{i,j=1}^n \phi_{ij}\omega_i \otimes \omega_j$ on M, Cheng and Yau, see [11], define an operator associated to ϕ by

$$\Box f = \sum_{i,j=1}^{n} \phi_{ij} f_{ij}\,. \tag{1.1}$$

In this paper, we prove the following new Bochner type formula.

Theorem 1.1. *Let M^n be a Riemannian manifold and $\phi = \sum_{i,j=1}^n \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M. Then, for any smooth function $f : M \to \mathbb{R}$,*

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and for any $c \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{2}\Box(|\nabla f|^2) \;=\; & \langle \nabla f, \nabla(\Box f)\rangle + \langle \phi(\nabla f), \nabla(\Delta f)\rangle \\
& + 2\sum_{i,j,k=1}^{n} \phi_{ij} f_{jk} f_{ki} + 2\sum_{i,j,k,m=1}^{n} f_i f_j \phi_{im} R_{mkjk} \\
& + c\sum_{i,j=1}^{n} (\operatorname{tr}\phi)_{ij} f_i f_j - \sum_{i,j=1}^{n} f_i f_j \Delta\phi_{ij} \\
& + \sum_{i,j=1}^{n} f_i f_j \left(\sum_{k=1}^{n}\phi_{ikk} - c\sum_{k=1}^{n}\phi_{kki}\right)_j \\
& + \sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki})\right)_k \\
& - \sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_j \phi_{ij} f_{ik}\right)_k .
\end{aligned} \tag{1.2}$$

Remark 1.1. If ϕ is equal to the metric g, then $\sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_j \phi_{ij} f_{ik}\right)_k = \frac{1}{2}\Delta|\nabla f|^2$, and Theorem 1.1 is exactly the Bochner formula for the Laplacian

$$\Delta|\nabla f|^2 = 2\langle \nabla f, \nabla(\Delta f)\rangle + 2|\operatorname{Hess} f|^2 + 2\operatorname{Ric}(\nabla f, \nabla f).$$

Remark 1.2. Notice that the last two terms in (1.2) are in divergent form and thus their integrals vanish when the manifold M is compact. In applications we have some natural examples of ϕ satisfying $\sum_{k=1}^{n}\phi_{ikk} - c\sum_{k=1}^{n}\phi_{kki} = 0$ for some constant c (see Appendix).

Of course, an application of the new Bochner formula is to recover the well-known Lich- nerowicz-Obata Theorem about the first eigenvalue for the Laplacian [16] and [18].

Theorem. *Let M be an n-dimensional compact Riemannian manifold with Ricci curvature bounded below by $(n-1)a^2$. Then the first nonzero eigenvalue λ of the Laplacian acting on functions of M satisfies $\lambda \geq na^2$ and the equality holds if and only if M is isometric to the round sphere.*

Before we state two more applications for second order differential operators, we discuss some known properties of $\Box$.

Associated to tensor ϕ we have the (1, 1)-tensor, still denoted by ϕ, defined by

$$\phi(X, Y) = \langle \phi(X), Y\rangle, \forall X, Y \in TM.$$

Here are two basic properties of the operator $\square$.

1) It follows from Cheng and Yau (Proposition 1 in [11]) that

$$\square f = \operatorname{div}(\phi(\nabla f)) - \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\phi_{ijj}\right) f_i.$$

2) We say that ϕ is *divergence free* if $\operatorname{div}\phi \equiv 0$ or, equivalently, $\sum_{j=1}^{n}\phi_{ijj} \equiv 0$, for all $1 \le j \le n$.

If M is compact, we know that $\square$ is self-adjoint if and only if ϕ is divergence free (see also [11], Proposition 1). If ϕ is symmetric and positive definite, then $\square$ is strictly elliptic. Therefore, if ϕ is divergence free, symmetric and positive definite, then $\square$ is strictly elliptic and self-adjoint. Furthermore, the spectrum of $\square$ is discrete and it makes sense to consider eigenvalues, see for example [14], p. 214.

Now let us explain the applications of Theorem 1.1 to get estimates for the first eigenvalue for two types of operators $\square$ which arise naturally in Riemannian geometry and submanifold theory.

a) Let us denote by Ric the *Ricci tensor* of M. Namely

$$\operatorname{Ric}(X, Y) = \sum_{i=1}^{n}\langle \operatorname{Rm}(X, e_i)Y, e_i\rangle,$$

where $\operatorname{Rm}(U, V)W = \nabla_V\nabla_U W - \nabla_U\nabla_V W + \nabla_{[U,V]}W$ is the curvature tensor of M. The *scalar curvature* R of M is defined by the trace of Ricci tensor. We will also denote by Ric the linear operator associated with the Ricci tensor, i.e., $\operatorname{Ric}(X, Y) = \langle \operatorname{Ric}(X), Y\rangle$, as well as its coordinates will be denoted by Ric_{ij}. If $\{e_1, \ldots, e_n\}$ is an orthonormal frame, the components of the curvature tensor of M can be written by (see [8] p. 48)

$$\begin{aligned} R_{ijkl} &= \frac{1}{n-2}\left(\operatorname{Ric}_{ik} g_{jl} - \operatorname{Ric}_{il} g_{jk} + \operatorname{Ric}_{jl} g_{ik} - \operatorname{Ric}_{jk} g_{il}\right) \\ &\quad - \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) + W_{ijkl}. \end{aligned}$$

where W_{ijkl} are the components of the *Weyl tensor* W.

When $n \geq 3$, the components of *Schouten operator* S of M are defined by

$$S_{ij} = \mathrm{Ric}_{ij} - \frac{R}{2(n-1)} g_{ij}.$$

In this case, one can rewrite the components of the curvature tensor by

$$R_{ijkl} = \frac{1}{n-2}\left(S_{ik}g_{jl} - S_{il}g_{jk} + S_{jl}g_{ik} - S_{jk}g_{il}\right) + W_{ijkl}.$$

The operator $\Box_S$ is defined by

$$\Box_S f = \sum_{i,j=1}^{n} S_{ij} f_{ij} = \sum_{i,j=1}^{n} \left(\mathrm{Ric}_{ij} - \frac{R}{2(n-1)} g_{ij}\right) f_{ij}.$$

We prove (see equation (5.1), p. 512) that S is divergence free (or equivalently, $\Box_S$ is self-adjoint) if and only if M has constant scalar curvature.

Definition 1.1. *A Riemannian manifold is called to have* harmonic Weyl tensor *if* $\operatorname{div} W \equiv 0$.

In this case, the Schouten operator is a Codazzi operator, i.e., $S_{ijk} = S_{ikj}$. Our first application of Theorem 1.1 is the following

Theorem 1.2. *Let $M^n, n \geq 4$ be a compact Riemannian manifold which has harmonic Weyl tensor. If M has constant scalar curvature R and the Schouten tensor is positive definite, then the first nonzero eigenvalue $\mu_1(\Box_S, M)$ satisfies*

$$\mu_1(\Box_S, M) \geq \frac{n-2}{2(n-1)}\left(\frac{R}{R-2L_0}\right)\left[L_0^2 - \left(\frac{R}{2(n-1)} + K_0\right)L_0 + \frac{1}{2}K_0 R\right], \tag{1.3}$$

where K_0 and L_0 are the lower bounds of the sectional curvature and Ricci curvature of M, respectively.

Furthermore, the equality holds if and only if M is the round sphere $\mathbb{S}^n(K_0)$.

Remark 1.3. Recall that a Riemannian manifold (M, g) is said *locally conformally flat* if, for any $p \in M$, there exists a neighborhood V of p and a smooth function f defined on V such that $(V, e^{2f} g)$ is flat. It is well known (cf. [8], p. 60) that M^n, $n \geq 4$, is locally conformally flat if and only if the Weyl tensor vanishes. In [9], Q.M. Cheng has proved that the only compact, connected, oriented, locally conformally flat, $n-$dimensional Riemannian manifold with constant scalar curvature and non-negative Ricci curvature are those which are quotients of a space form or a Riemannian product $\mathbb{S}^1 \times \mathbb{S}^n(\kappa)$. On the other hand, there are many examples of compact manifolds with harmonic Weyl tensor, see, for example, [8], p. 440–443.

b) Our second application is about isometric immersions.

Let M^n be a n-dimensional Riemannian manifold and $x : M^n \to \overline{M}^{n+1}$ be an isometric immersion of M to $(n+1)$-dimensional Riemannian manifold. Denote by A and H the *shape operator* and the mean curvature of the immersion. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, i.e. the principal curvatures of the immersion, then $H = \sum_{i=1}^n \lambda_i$.

The *first Newton transformation* $P_1 : TM \to TM$, associated with the second fundamental form A, is defined by

$$P_1 = HI - A.$$

Let us define the differential operator L_1 by

$$L_1 f = \sum_{i,j=1}^{n} (P_1)_{ij} f_{ij} = \sum_{i,j=1}^{n} (Hg_{ij} - h_{ij}) f_{ij}, \tag{1.4}$$

where h_{ij} are the components of second fundamental form. Note that P_1 is a symmetric linear operator. The operator L_1 was first introduced by Voss in [24] and appears naturally in the study of variation theory for curvature functional $\mathcal{A}_1 = \int_M H dM$, which is called $1-$area of M. See for example [20] and [7] for more details.

It has been shown by Reilly, [20], that P_1 is divergence free when $\overline{M}$ is a space form of constant sectional curvature. Therefore, under these assumptions, L_1 is self-adjoint.

The eigenvalues of L_1 plays an important role in the study of stability for hypersurfaces with constant r-mean curvature (see, for examples, [1, 2, 3, 4, 6]). In the case that $A > 0$, we have P_1 positive definite. Therefore, L_1 is an elliptic operator. We have the following first eigenvalue estimate.

Theorem 1.3. *Let $x : M^n \to \overline{M}^{n+1}(\kappa)$ be an isometric immersion of a compact Riemannian manifold into a space form of constant sectional curvature κ. Suppose that shape operator A satisfies*

$$0 < \alpha I \leq A \leq a\alpha I,$$

where $\alpha > 0$ and $a > 1$ are constants. Then

1) *when* $\kappa > 0$*, the first nonzero eigenvalue* $\mu(L_1, M)$ *of operator* L_1 *satisfies*

$$\mu(L_1, M) \geq \frac{1}{2}\left(\frac{na}{na-1}\right)\left[2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2\alpha - \sigma\right]$$

where $\sigma = \max_{(p,v)\in TM} \operatorname{tr}(\operatorname{Hess} H|_{v^\perp})(p)$ *and* $v^\perp = \{u \in T_pM; \langle u, v\rangle = 0\}$*;*

2) *when* $\kappa \leq 0$*, the first nonzero eigenvalue* $\mu(L_1, M)$ *of operator* L_1 *satisfies*

$$\mu(L_1, M) \geq \frac{1}{2}\left(\frac{na}{na-1}\right)\left[2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2 a\alpha - \sigma\right].$$

Furthermore, the equalities hold if and only if M *is a geodesic sphere with the canonical immersion.*

Remark 1.4. If $A \geq \alpha I > 0$ then, by using Gauss equation,

$$\operatorname{Ric} \geq (n-1)[\kappa + \alpha^2] = \operatorname{Ric}_{\mathbb{S}^n(\alpha)} > 0,$$

for $\alpha^2 > -\kappa$. That is: $\operatorname{Ric} \geq (n-1)\Lambda > 0$ for some constant $\Lambda > 0$. Conversely, if we assume the Lichnerowicz condition $\operatorname{Ric} \geq (n-1)\Lambda > 0$, then by using Gauss equation again, we have $\langle A \circ P_1(X), X\rangle \geq (n-1)[\Lambda - \kappa]|X|^2$. If we assume in addition that $P_1 > 0$ and $\Lambda > \kappa$, then A is positive definite.

Remark 1.5. If the mean curvature H is constant and $A \geq \kappa I$, then $x(M^n)$ is a geodesic sphere. In fact,

(1) if $\kappa = 0$, by Hadamard theorem, cf. [15], [12], the immersion $x : M^n \to \mathbb{R}^{n+1}$ is an embedding and $x(M^n)$ is a boundary of a convex domain of $\mathbb{R}^{n+1}$. Thus by using the Alexandrov Theorem, cf. [5], $x(M^n)$ is a round sphere;

(2) if $\kappa > 0$, by do Carmo-Warner Theorem, cf. [12], then $x : M^n \to \mathbb{S}^{n+1}(\kappa)$ is an embedding and $x(M^n)$ is either totally geodesic or contained in a open hemisphere. In the last case, $x(M^n)$ is a boundary of a convex domain in $\mathbb{S}^{n+1}(\kappa)$. Since $A \geq \kappa I > 0$, $x(M)$ cannot be totally geodesic. Thus $x(M)$ is contained in an open hemisphere. On the other hand, in [17], S. Montiel and A. Ros proved that if $x : M^n \to \mathbb{S}^{n+1}(\kappa)$ is an embedding such that the r-mean curvature S_r is constant for some r and $x(M^n)$ is contained in a open hemisphere, then M^n is a geodesic sphere;

(3) if $\kappa < 0$, by do Carmo-Warner Theorem, cf. [12], then $x : M^n \to \mathbb{H}^{n+1}(\kappa)$ is an embedding and $x(M^n)$ is a boundary of a convex domain in $\mathbb{H}^{n+1}(\kappa)$. On the other hand, in [17], S. Montiel and A. Ros proved that if $x : M^n \to \mathbb{H}^{n+1}(\kappa)$ is an embedding such that S_r is constant for some r then M^n is a geodesic sphere.

The rest of the paper is organized as follows: In Section 2, we give the proof of Theorem 1.1, in Section 3 we prove Theorem 1.2, and in Section 4 we prove Theorem 1.3. Eventually, in the Appendix, we prove Proposition 5.1, which collects some properties of the Newton and Shouten tensors that we use throughout the paper.

2 A Bochner-type formula

In this section we will prove a Bochner type formula for the differential operator $\square$ mentioned in the introduction.

Proof of Theorem 1.1. For a point $p \in M$, for any orthonormal frame $\{e_1, \ldots, e_n\}$ near p, we have $|\nabla f|^2 = \sum_{i=1}^n (f_i)^2$ and

$$\begin{aligned}\frac{1}{2}\square(|\nabla f|^2) &= \frac{1}{2}\sum_{i=1}^n \sum_{j,k=1}^n \phi_{jk}(f_i^2)_{jk} \\ &= \sum_{i=1}^n \sum_{j,k=1}^n f_i \phi_{jk}(f_i)_{jk} + \sum_{i=1}^n \sum_{j,k=1}^n \phi_{jk}(f_i)_j(f_i)_k.\end{aligned}$$

Now we choose an orthonormal frame $\{e_1, \ldots, e_n\}$ such that ϕ is diagonalized at p, i.e. $\phi_{jk} = \mu_j g_{jk}$, where μ_j are eigenvalues of (ϕ_{jk}). Then we choose an orthonormal frame in a neighborhood of $p \in M$ by parallel translating frame $\{e_1, \ldots, e_n\}$ at p. Here at p, we have $\nabla_{e_i} e_j = 0$ at p. Moreover, $\nabla_{e_i} e_j = 0$ along the geodesic tangent to e_i which implies $\nabla_{e_i}\nabla_{e_i} e_j = 0$ at p for all i, j. Thus we have

$$\frac{1}{2}\square(|\nabla f|^2) = \sum_{i,j=1}^n f_i \mu_j (f_i)_{jj} + \sum_{i,j=1}^n \phi_{jj}(f_i)_j(f_i)_j.$$

Since the terms $(f_i)_j$ and $(f_i)_{jj}$ denote differentiation of the function f_i, in general they are not equal to the covariant derivatives f_{ij} and f_{ijj} of f. However, by the special choice of our frame, we have $(f_i)_j = f_{ij}$ and $(f_i)_{jj} = f_{ijj}$ at p.

Therefore, at p

$$\sum_{i,j=1}^{n} f_i \mu_j (f_i)_{jj} = \sum_{i,j,k=1}^{n} f_i f_{jki} \phi_{jk} + \sum_{i,j,k,m=1}^{n} f_i f_m R_{mjik} \phi_{jk}$$
$$= \sum_{i,j,k=1}^{n} f_i (f_{jk}\phi_{jk})_i - \sum_{i,j,k=1}^{n} f_i f_{jk} \phi_{jki} + \sum_{i,j,k,m=1}^{n} f_i f_m R_{mjik} \phi_{jk}.$$

Hence

$$\begin{aligned} \frac{1}{2}\Box(|\nabla f|^2) &= \langle \nabla f, \nabla(\Box f)\rangle + \sum_{i,j,k=1}^{n} \phi_{ij} f_{jk} f_{ki} \\ &\quad + \sum_{i,j,k,m=1}^{n} f_i f_m R_{mjik} \phi_{jk} - \sum_{i,j,k=1}^{n} f_i f_{jk} \phi_{jki}. \end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned} -\sum_{i,j,k=1}^{n} f_i f_{jk} \phi_{jki} &= \sum_{i,j,k=1}^{n} f_i f_{jk} (\phi_{jik} - \phi_{jki}) - \sum_{i,j,k=1}^{n} f_i f_{jk} \phi_{jik} \\ &= \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki}) \right)_k - \sum_{i,j,k=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki})_k \\ &\quad - \sum_{i,j,k=1}^{n} f_{ik} f_j (\phi_{jik} - \phi_{jki}) - \sum_{i,j,k=1}^{n} f_i f_{kj} \phi_{jik} \\ &= \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki}) \right)_k - \sum_{i,j,k=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki})_k \\ &\quad - \sum_{i,j,k=1}^{n} f_j f_{ik} \phi_{jik}, \end{aligned}$$

when we used in the last equality that $\sum_{i,j,k=1}^{n} f_j f_{ik} \phi_{jki} = \sum_{i,j,k=1}^{n} f_i f_{kj} \phi_{jik}$. Then

$$\begin{aligned} \frac{1}{2}\Box\left(|\nabla f|^2\right) =& \langle \nabla f, \nabla(\Box f)\rangle + \sum_{i,j,k=1}^{n} \phi_{ij} f_{jk} f_{ki} + \sum_{i,j,k,m=1}^{n} f_i f_m R_{mjik} \phi_{jk} \\ & + \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki}) \right)_k \\ & - \sum_{i,j,k=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki})_k - \sum_{i,j,k=1}^{n} f_j f_{ik} \phi_{jik}. \end{aligned} \tag{2.2}$$

In order to find a more suited expression for the term $-\sum_{i,j,k=1}^{n} f_i f_j (\phi_{jik} - \phi_{jki})_k$ above, we use the following computation (see [11], Eq. (2.4), p. 198):

$$\Delta\phi_{ij} = \sum_{k=1}^{n}(\phi_{ijkk} - \phi_{ikjk}) + \sum_{k=1}^{n}(\phi_{ikkj} - c\phi_{kkij}) + c\left(\sum_{k=1}^{n}\phi_{kk}\right)_{ij} - \sum_{m,k=1}^{n}\phi_{mk}R_{mikj} - \sum_{m,k=1}^{n}\phi_{im}R_{mkkj},$$

which implies

$$\begin{aligned} -\sum_{i,j,k=1}^{n} f_i f_j (\phi_{ijk} - \phi_{ikj})_k = &\sum_{i,j=1}^{n} f_i f_j \left(\sum_{k=1}^{n}\phi_{ikk} - c\sum_{k=1}^{n}\phi_{kki}\right)_j \\ &+ c\sum_{i,j=1}^{n}(\operatorname{tr}\phi)_{ij} f_i f_j \\ &- \sum_{i,j,k,m=1}^{n} f_i f_j \phi_{mk} R_{mikj} \\ &+ \sum_{i,j,k,m=1}^{n} f_i f_j \phi_{im} R_{mkik} \\ &- \sum_{i,j=1}^{n} f_i f_j \Delta\phi_{ij}. \end{aligned} \tag{2.3}$$

Rewriting the last term in right hand side of (2.2) we have

$$\begin{aligned} -\sum_{i,j,k=1}^{n} f_j f_{ik}\phi_{jik} &= -\sum_{i,j,k=1}^{n}(f_j f_{ik}\phi_{ji})_k + \sum_{i,j,k=1}^{n} f_{jk} f_{ik}\phi_{ji} + \sum_{i,j,k=1}^{n} f_j f_{ikk}\phi_{ij} \\ &= -\sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_j\phi_{ij} f_{ik}\right)_k + \sum_{i,j,k=1}^{n}\phi_{ij} f_{jk} f_{ki} \\ &\quad \sum_{i,j,k=1}^{n} f_j f_{kki}\phi_{ij} + \sum_{i,j,k,m=1}^{n} f_j f_m R_{mkik}\phi_{ij}. \end{aligned} \tag{2.4}$$

Equation (1.2) follows by replacing the expressions of (2.3) and (2.4) in (2.2), and noting that

$$\sum_{i,j,k,m=1}^{n} f_i f_j \phi_{mk} R_{mikj} = \sum_{i,j,k,m=1}^{n} f_i f_m \phi_{jk} R_{mjik}.$$

□

3 Estimate of first eigenvalue of the Shouten operator

The main purpose of this section is to prove Theorem 1.2. We start with proving two lemmas.

Lemma 3.1 (Generalized Newton inequality). *Let A and B be two $n \times n$ symmetric matrices. If B is positive definite, then*

$$\operatorname{tr}(A^2 B) \geq \frac{[\operatorname{tr}(AB)]^2}{\operatorname{tr} B}, \tag{3.1}$$

and the equality holds if and only if $A = \alpha I$ for some $\alpha \in \mathbb{R}$.

Proof. Let C be a positive definite matrix. By using the Cauchy-Schwarz inequality with $A\sqrt{C}$ and $(\sqrt{C})^{-1}B$, and the fact $\operatorname{tr}[(AB)^2] \leq \operatorname{tr}(A^2B^2)$, which holds for symmetric matrices, we have

$$[\operatorname{tr}(AB)]^2 = \operatorname{tr}(A\sqrt{C}(\sqrt{C})^{-1}B)^2 \leq \operatorname{tr}(A^2C)\operatorname{tr}(B^2C^{-1}).$$

In particular, since B is positive definite, we can choose $C = B$ to obtain

$$[\operatorname{tr}(AB)]^2 \leq \operatorname{tr}(A^2B)\operatorname{tr} B,$$

i.e.,

$$\operatorname{tr}(A^2 B) \geq \frac{[\operatorname{tr}(AB)]^2}{\operatorname{tr} B}.$$

The equality holds if and only if

$$\begin{aligned} A\sqrt{B} &= \alpha(\sqrt{B})^{-1}B \Leftrightarrow (A\sqrt{B})\sqrt{B} \\ &= \alpha(\sqrt{B})^{-1}B\sqrt{B} \Leftrightarrow AB = \alpha B \Leftrightarrow A = \alpha I. \end{aligned}$$

□

Remark 3.1. When $B = I$, the inequality (3.1) becomes

$$\|A\|^2 \geq \frac{1}{n}(\operatorname{tr} A)^2,$$

which is known as the *(classical) Newton inequality*.

When $A = [f_{ij}]_{n\times n}$ and $B = [\phi_{ij}]_{n\times n}$, (3.1) implies

Lemma 3.2. *If ϕ is positive definite, then*

$$\sum_{i,j,k=1}^{n} \phi_{ij} f_{jk} f_{ki} \geq \frac{(\Box f)^2}{\operatorname{tr}\phi},$$

and the equality holds if and only if, $f_{ij} = \alpha g_{ij}$*, i.e.,* $\operatorname{Hess} f(X, Y) = \alpha\langle X, Y\rangle$.

In Proposition 5.1, item (4), in the Appendix, we have

$$\operatorname{div} S = \nabla(\operatorname{tr} S).$$

Therefore S is divergence free if and only if M has constant scalar curvature. Now let us prove Theorem 1.2.

Proof of Theorem 1.2. Since $S_{jik} = S_{jki}$ and $\operatorname{div} S = 0$, the Bochner formula of Theorem 1.1 in this case becomes

$$\begin{aligned}\frac{1}{2}\Box_S|\nabla f|^2 &= \langle \nabla f, \nabla(\Box_S f)\rangle + 2\sum_{i,j,k=1}^{n} S_{ij} f_{jk} f_{ki} + \langle S(\nabla f), \nabla(\Delta f)\rangle \\ &\quad + 2\operatorname{Ric}(\nabla f, S(\nabla f)) - \sum_{i,j=1}^{n} f_i f_j \Delta S_{ij} - \sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_j S_{ij} f_{ik}\right)_k .\end{aligned} \tag{3.2}$$

Let us integrate and estimate each terms in the equation (3.2). We will complete our proof after proving two claims.

Claim 1. *Let* $\mu > 0$ *and a smooth function* $f : M \to \mathbb{R}$ *such that* $\Box_S f = -\mu f$*, then*

$$\int_M \langle S(\nabla f), \nabla(\Delta f)\rangle dM = -\mu \int_M |\nabla f|^2 dM.$$

In fact, by using Divergence Theorem, we have

$$\begin{aligned}\int_M \langle S(\Delta f), \nabla(\Delta f)\rangle dM &= \int_M \operatorname{div}(\Delta f \cdot S(\nabla f)) dM - \int_M \Delta f \cdot \operatorname{div}(S(\nabla f)) dM \\ &= -\int_M \Delta f \cdot \Box_S f dM \\ &= \mu \int_M f \Delta f dM \\ &= -\mu \int_M |\nabla f|^2 dM.\end{aligned}$$

Claim 2.

$$\int_M \left[2\operatorname{Ric}(\nabla f, S(\nabla f)) - \sum_{i,j=1}^{n} f_i f_j \Delta S_{ij}\right] dM \geq \Gamma \int_M |\nabla f|^2 dM,$$

where

$$\Gamma = L_0^2 - \left(\frac{R}{2(n-1)} + K_0\right) L_0 + \frac{1}{2} K_0 R.$$

To prove this claim, first note that

$$\begin{aligned}
\mathrm{Ric}(\nabla f, S(\nabla f)) &= \langle \mathrm{Ric}(\nabla f), S(\nabla f)\rangle \\
&= \left\langle \left(S + \frac{R}{2(n-1)} I\right)(\nabla f), S(\nabla f)\right\rangle \\
&= |S(\nabla f)|^2 + \frac{R}{2(n-1)}\langle S(\nabla f), \nabla f\rangle.
\end{aligned}$$

Since S is a Codazzi tensor,

$$(\Delta S)_{ij} := \sum_{k=1}^{n} S_{ijkk} = \sum_{k=1}^{n} S_{ikjk} = \sum_{k=1}^{n} S_{kijk}.$$

By using Ricci identity

$$S_{kijk} = S_{kikj} + \sum_{m=1}^{n} S_{mk} R_{mijk} + \sum_{m=1}^{n} S_{mi} R_{mkjk},$$

and following a computation of Cheng and Yau, cf. [11], we have

$$\begin{aligned}
(\Delta S)_{ij} &= \sum_{k=1}^{n} S_{kkij} + \sum_{m,k=1}^{n} S_{mk} R_{mijk} + \sum_{m,k=1}^{n} S_{mi} R_{mkjk} \\
&= (\mathrm{tr}\, S)_{ij} + \sum_{m=1}^{n} S_{mi}\, \mathrm{Ric}_{mj} + \sum_{m,k=1}^{n} S_{mk} R_{mijk} \\
&= \sum_{m=1}^{n} S_{mi}\left(S_{mj} + \frac{R}{2(n-1)} g_{mj}\right) + \sum_{m,k=1}^{n} S_{mk} R_{mijk} \\
&= \sum_{m=1}^{n} S_{mi} S_{mj} + \frac{R}{2(n-1)} \sum_{m=1}^{n} S_{mi} g_{mj} + \sum_{m,k=1}^{n} S_{mk} R_{mijk} \\
&= (S^2)_{ij} + \frac{\mathrm{tr}\, S}{n-2} S_{ij} + \sum_{m,k=1}^{n} S_{mk} R_{mijk}.
\end{aligned}$$

We now choose an orthonormal frame such that $S_{ij} = \lambda_i g_{ij}$ at a point $p \in M$.

Let $K(u, v)$ denote the sectional curvature of the plane generated by u, v. Then

$$\begin{aligned}
&2\,\mathrm{Ric}(\nabla f, S(\nabla f)) - \sum_{i,j=1}^{n} f_i f_j \Delta S_{ij} \\
&= 2\sum_{i=1}^{n}\left(\lambda_i + \frac{R}{2(n-1)}\right)\lambda_i f_i^2 - \sum_{i=1}^{n}\lambda_i^2 f_i^2 \\
&\quad - \frac{R}{2(n-1)}\sum_{i=1}^{n}\lambda_i f_i^2 - \sum_{i,j,k=1}^{n}\lambda_k R_{kijk} f_i f_j \\
&= \sum_{i=1}^{n}\lambda_i^2 f_i^2 + \frac{R}{2(n-1)}\sum_{i=1}^{n}\lambda_i f_i^2 + \sum_{k=1}^{n}\lambda_k K(e_k, \nabla f)[|\nabla f|^2 - \langle e_k, \nabla f\rangle^2] \\
&\geq \sum_{i=1}^{n}\lambda_i^2 f_i^2 + \frac{R}{2(n-1)}\sum_{i=1}^{n}\lambda_i f_i^2 + K_0\sum_{k=1}^{n}\lambda_k[|\nabla f|^2 - \langle e_k, \nabla f\rangle^2] \\
&= \sum_{i=1}^{n}\lambda_i^2 f_i^2 + \left(\frac{R}{2(n-1)} - K_0\right)\sum_{i=1}^{n}\lambda_i f_i^2 + \frac{n-2}{2(n-1)}K_0 R|\nabla f|^2.
\end{aligned}$$

Note that, if $K_0 < 0$, then $\frac{R}{2(n-1)} - K_0 > 0$, and if $K_0 > 0$, then

$$\begin{aligned}
\frac{R}{2(n-1)} - K_0 &= \frac{1}{2(n-1)}\sum_{i,j=1}^{n} K(e_i, e_j) - K_0 \\
&\geq \frac{n(n-1)}{2(n-1)}K_0 - K_0 \\
&= \left(\frac{n}{2} - 1\right)K_0 > 0.
\end{aligned}$$

It implies,

$$\begin{aligned}
&2\,\mathrm{Ric}(\nabla f, S(\nabla f)) - \sum_{i,j=1}^{n} f_i f_j \Delta S_{ij} \\
&\geq \sum_{i=1}^{n}\lambda_i^2 f_i^2 + \left(\frac{R}{2(n-1)} - K_0\right)\sum_{i=1}^{n}\lambda_i f_i^2 + \frac{n-2}{2(n-1)}K_0 R|\nabla f|^2 \\
&\geq \left[\lambda_0^2 + \left(\frac{R}{2(n-1)} - K_0\right)\lambda_0 + \frac{n-2}{2(n-1)}K_0 R\right]|\nabla f|^2,
\end{aligned}$$

where $\lambda_0 = \min_{p\in M}\left\{\min_{1\le i\le n}\lambda_i(p)\right\}$. Since $\lambda_0 = L_0 - \dfrac{R}{2(n-1)}$, where L_0 is the minimum of the Ricci curvature, then the claim follows from the definition of Γ.

Now we are ready to complete the proof of Theorem 1.2. Since $\Box_S f = -\mu f$, we have

$$\int_M \langle \nabla f, \nabla(\Box_S f)\rangle = -\mu \int_M |\nabla f|^2 dM \tag{3.3}$$

and, by using the Lemma 3.1,

$$2\int_M \left(\sum_{i,j,k=1}^{n} S_{ij} f_{jk} f_{ki}\right) dM \ge 2\int_M \frac{(\Box_S f)^2}{\operatorname{tr} S} dM \ge \frac{2\mu^2}{\operatorname{tr} S}\int_M f^2 dM. \tag{3.4}$$

Since, by using Divergence Theorem, $\int_M \left[\sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_j \phi_{ij} f_{ik}\right)_k\right] dM = 0$, then replacing these estimates in the equation (3.2), p. 501, we have

$$0 \ge -2\mu \int_M |\nabla f|^2 dM + \frac{2\mu^2}{\operatorname{tr} S}\int_M f^2 dM + \Gamma \int_M |\nabla f|^2 dM. \tag{3.5}$$

Since

$$\begin{aligned}\int_M |\nabla f|^2 dM \;&\le\; \frac{1}{\lambda_0}\int_M \langle S(\nabla f), \nabla f\rangle dM\\ &=\; \frac{\mu}{\lambda_0}\int_M f^2 dM,\end{aligned}$$

we obtain

$$\begin{aligned}0 \;&\ge\; (\Gamma - 2\mu)\int_M |\nabla f|^2 dM + \frac{2\mu\lambda_0}{\operatorname{tr} S}\int_M |\nabla f|^2 dM\\ &=\; \left(\Gamma - 2\mu\left(1 - \frac{\lambda_0}{\operatorname{tr} S}\right)\right)\int_M |\nabla f|^2 dM.\end{aligned}$$

Thus

$$\begin{aligned}\mu \;&\ge\; \frac{\Gamma}{2}\left(\frac{\operatorname{tr} S}{\operatorname{tr} S - \lambda_0}\right)\\ &=\; \frac{n-2}{2(n-1)}\left(\frac{R}{R-2L_0}\right)\Gamma\\ &=\; \frac{n-2}{2(n-1)}\left(\frac{R}{R-2L_0}\right)\left[L_0^2 - \left(\frac{R}{2(n-1)} + K_0\right)L_0 + \frac{1}{2}K_0 R\right].\end{aligned}$$

To prove the equality case, we suppose $K_0 = 1$ and $M^n = \mathbb{S}^n$. In this case $S = \frac{n-2}{2} I$, $\square_S f = \frac{n-2}{2}\Delta f$ and $\Gamma = \frac{(n-2)(n-1)}{2}$. Then the estimate becomes equality. Conversely, if the equality holds, Lemma 3.1, p. 500, gives us that $\mathrm{Hess}(f) = \alpha I$. Following the proof of Obata Theorem step-by-step, cf. [18], we can see that M is a sphere. $\square$

4 The estimate of the first eigenvalue of L_1

This section will give the proof of Theorem 1.3. We start with the following lemma.

Lemma 4.1. *Let $x : M^n \to \overline{M}^{n+1}(\kappa)$ be an isometric immersion of a n-dimensional Riemannian manifold M into a $(n+1)$-dimensional space form $\overline{M}$ of constant sectional curvature κ. Then*

$$\begin{aligned} & 2\,\mathrm{Ric}(\nabla f, P_1(\nabla f)) - \langle (\Delta P_1)(\nabla f), \nabla f\rangle \\ = \ & \mathrm{Hess}(H)(\nabla f, \nabla f) - (\Delta H)|\nabla f|^2 + \langle Q(A)(\nabla f), \nabla f\rangle, \end{aligned} \tag{4.1}$$

where $\langle (\Delta P_1)(\nabla f), \nabla f\rangle = \sum_{i,j=1}^n f_i f_j \Delta(H g_{ij} - h_{ij})$ and $Q(A) = 2A^3 - 3HA^2 + (2H^2 - |A|^2 - \kappa(n-2))A + \kappa(2n-3)HI$.

Proof. By following Schoen-Simon-Yau's computations, see [22], eq. (1.20), p. 278, we have

$$\Delta h_{ij} = H_{ij} + (\kappa n - |A|^2)h_{ij} - \kappa H g_{ij} + H\sum_{k=1}^n h_{ik}h_{kj},$$

equivalently,

$$(\Delta A)(X) = \nabla_X \nabla H + (\kappa n - |A|^2)A(X) - \kappa H X + HA^2(X).$$

It implies

$$\begin{aligned} -\langle (\Delta P_1)(\nabla f), \nabla f\rangle &= \langle (\Delta A - (\Delta H)I)(\nabla f), \nabla f\rangle \\ &= \langle (\Delta A)(\nabla f), \nabla f\rangle - \Delta H|\nabla f|^2 \\ &= \mathrm{Hess}(H)(\nabla f, \nabla f) + (\kappa n - |A|^2)\langle A(\nabla f), \nabla f\rangle \\ &\quad + H\langle A^2(\nabla f), \nabla f\rangle - \kappa H|\nabla f|^2 - \Delta H|\nabla f|^2. \end{aligned}$$

By using Gauss equation

$$\begin{aligned} \langle \mathrm{Rm}(X, Y)Z, T\rangle &= \kappa(\langle X, Z\rangle\langle Y, T\rangle - \langle Y, Z\rangle\langle X, T\rangle) \\ &\quad + \langle A(X), Z\rangle\langle A(Y), T\rangle - \langle A(Y), Z\rangle\langle A(X), T\rangle, \end{aligned}$$

we have

$$\begin{aligned}\langle \mathrm{Rm}(\nabla f, e_i)P_1(\nabla f), e_i\rangle &= \kappa(\langle \nabla f, P_1(\nabla f)\rangle\langle e_i, e_i\rangle - \langle e_i, P_1(\nabla f)\rangle\langle \nabla f, e_i\rangle) \\ &\quad + \langle A(\nabla f), P_1(\nabla f)\rangle\langle A(e_i), e_i\rangle \\ &\quad -\langle A(e_i), P_1(\nabla f)\rangle\langle A(\nabla f), e_i\rangle.\end{aligned}$$

After tracing, we obtain

$$\begin{aligned}2\,\mathrm{Ric}(\nabla f, P_1(\nabla f)) &= 2\kappa(n-1)\langle \nabla f, P_1(\nabla f)\rangle + 2H\langle A(\nabla f), P_1(\nabla f)\rangle \\ &\quad -2\langle A^2(\nabla f), P_1(\nabla f)\rangle \\ &= 2\kappa(n-1)\langle \nabla f, (HI - A)(\nabla f)\rangle + 2H\langle A(\nabla f), \\ &\quad (HI - A)(\nabla f)\rangle - 2\langle A^2(\nabla f), (HI - A)(\nabla f)\rangle \\ &= 2\kappa(n-1)H|\nabla f|^2 + (2H^2 - 2\kappa(n-1))\langle A(\nabla f), \nabla f\rangle \\ &\quad - 4H\langle A^2(\nabla f), \nabla f\rangle + 2\langle A^3(\nabla f), \nabla f\rangle.\end{aligned}$$

Then

$$\begin{gathered}2\,\mathrm{Ric}(\nabla f, P_1(\nabla f)) - \langle(\Delta P_1)(\nabla f), \nabla f\rangle = \\ \mathrm{Hess}(H)(\nabla f, \nabla f) - (\Delta H)|\nabla f|^2 + \langle Q(A)(\nabla f), \nabla f\rangle.\end{gathered}$$ □

Next lemma is a local estimate for $Q(A)$.

Lemma 4.2. *If* $0 < \alpha I \leq A \leq a\alpha I$ *then,*

(i) if $\kappa > 0$,

$$\langle Q(A)(X), X\rangle \geq \left[2(n-1)\alpha^3(n - a^2) + 2\kappa(n-1)^2\alpha\right]|X|^2,$$

(ii) if $\kappa \leq 0$,

$$\langle Q(A)(X), X\rangle \geq \left[2(n-1)\alpha^3(n - a^2) + 2\kappa(n-1)^2 a\alpha\right]|X|^2,$$

for any $X \in TM$.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal base of eigenvectors of the shape operator A, and $h_1, h_2, \ldots, h_n$ be its eigenvalues. Denote by

$$S_2 = \sum_{i<j} h_i h_j = \frac{1}{2}(H^2 - |A|^2).$$

For $\kappa \geq 0$, we have

$$
\begin{aligned}
\langle Q(A)(e_i), e_i\rangle &= 2h_i^3 - 3Hh_i^2 + (2H^2 - |A|^2)h_i - \kappa(n-2)h_i + \kappa(2n-3)H \\
&= 2h_i^3 - 3h_i^2(h_i + H - h_i) + 2(H^2 - |A|^2)h_i + |A|^2 h_i \\
&\quad + \kappa[(n-2)(H - h_i) + (n-1)H] \\
&= (|A|^2 - h_i^2)h_i + 2S_2 h_i - 3h_i^2(H - h_i) \\
&\quad + \kappa[(n-2)(H - h_i) + (n-1)H] \\
&= h_i\left[(|A|^2 - h_i^2) + 2S_2 - 3h_i(H - h_i)\right] \\
&\quad + \kappa[(n-2)(H - h_i) + (n-1)H] \\
&= h_i\left[\left(\sum_{j=1}^n h_j^2 - h_i^2\right) + 2S_2 - 3\sum_{j=1}^n h_i h_j + 3h_i^2\right] \\
&\quad + \kappa[(n-2)(H - h_i) + (n-1)H] \\
&= h_i\left[2S_2 + 2h_i^2 + \sum_{j=1}^n h_j^2 - 2\sum_{j=1}^n h_i h_j - \sum_{j=1}^n h_i h_j\right] \\
&\quad + \kappa[(n-2)(H - h_i) + (n-1)H] \\
&\geq h_i\left[2S_2 + 2h_i^2 + \sum_{j=1}^n h_j^2 - \sum_{j=1}^n (h_i^2 + h_j^2) - \sum_{j=1}^n h_i h_j\right] \\
&\quad + \kappa[(n-2)(H - h_i) + (n-1)H] \\
&= h_i\left[2S_2 - (n-1)h_i^2 - h_i(H - h_i)\right] \\
&\quad + \kappa[(n-2)(H - h_i) + (n-1)H] \\
&\geq 2\alpha[S_2 - (n-1)a^2\alpha^2] + \kappa[(n-2)(n-1)\alpha + (n-1)n\alpha] \\
&\geq 2\alpha[S_2 - (n-1)a^2\alpha^2] + 2\kappa(n-1)^2\alpha \\
&\geq 2\alpha(n(n-1)\alpha^2 - (n-1)a^2\alpha^2) + 2\kappa(n-1)^2\alpha \\
&= 2(n-1)\alpha^3(n - a^2) + 2\kappa(n-1)^2\alpha
\end{aligned}
\tag{4.2}
$$

and if $\kappa < 0$,

$$
\begin{aligned}
\langle Q(A)(e_i), e_i\rangle &\geq 2\alpha[S_2 - (n-1)a^2\alpha^2] + 2\kappa(n-1)^2 a\alpha \\
&\geq 2(n-1)\alpha^3(n - a^2) + 2\kappa(n-1)^2 a\alpha.
\end{aligned}
\tag{4.3}
$$

□

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Applying the formula of Theorem 1.1 to the operator L_1, and by using Codazzi Equation $h_{jik} = h_{jki}$, we have

$$\begin{aligned}\frac{1}{2}L_1|\nabla f|^2 &= \langle \nabla f, \nabla(L_1 f)\rangle + \langle P_1(\nabla f), \nabla(\Delta f)\rangle \\ &\quad + 2\sum_{i,j,k=1}^{n}(Hg_{ij} - h_{ij})f_{jk}f_{ki} \\ &\quad + 2\,\mathrm{Ric}(\nabla f, P_1(\nabla f)) - \langle(\Delta P_1)(\nabla f), \nabla f\rangle \\ &\quad - \sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_j(Hg_{ij} - h_{ij})f_{ik}\right)_k \\ &\quad + \sum_{k=1}^{n}\left(|\nabla f|^2 H_k - \langle\nabla H, \nabla f\rangle f_k\right)_k.\end{aligned} \tag{4.4}$$

Integrating this formula and, by using the divergence theorem and the fact that L_1 is divergence free, we have

$$\begin{aligned}0 &= \int_M \langle\nabla f, \nabla(L_1 f)\rangle dM + \int_M \langle P_1(\nabla f), \nabla(\Delta f)\rangle dM \\ &\quad + 2\int_M\left(\sum_{i,j,k=1}^{n}(Hg_{ij} - h_{ij})f_{jk}f_{ki}\right)dM \\ &\quad + \int_M [2\,\mathrm{Ric}(\nabla f, P_1(\nabla f))dM - \langle(\Delta P_1)(\nabla f), \nabla f\rangle]dM.\end{aligned} \tag{4.5}$$

Let us estimate each of these integrals. The three first integrals in expression (4.5) have canonical estimates, as follows. Since $L_1 f = -\mu f$ we have

$$\int_M \langle\nabla f, \nabla(L_1 f)\rangle dM = -\mu\int_M |\nabla f|^2 dM.$$

By using Divergence Theorem in the expression

$$\begin{aligned}\mathrm{div}(\Delta f P_1(\nabla f)) &= \Delta f\,\mathrm{div}(P_1(\nabla f)) + \langle P_1(\nabla f), \nabla(\Delta f)\rangle \\ &= \Delta f \cdot L_1 f + \langle P_1(\nabla f), \nabla(\Delta f)\rangle,\end{aligned}$$

we obtain

$$\begin{aligned}\int_M \langle P_1(\nabla f), \nabla(\Delta f)\rangle dM &= -\int_M \Delta f \cdot L_1 f dM\\ &= \mu \int_M f \Delta f dM\\ &= -\mu \int_M |\nabla f|^2 dM.\end{aligned}$$

Applying Lemma 3.1, p. 500, we obtain

$$\begin{aligned}2\int_M \left(\sum_{i,j,k=1}^{n} (Hg_{ij} - h_{ij}) f_{jk} f_{ki}\right) dM &\geq 2\int_M \frac{(L_1 f)^2}{(n-1)H} dM\\ &\geq \frac{2\mu^2}{n(n-1)a\alpha}\int_M f^2 dM,\end{aligned}$$

where we have used that tr $P_1 = (n-1)H$. To estimate the last integral, we claim that, for $\kappa \geq 0$,

$$\begin{aligned}&\int_M [2\,\mathrm{Ric}(\nabla f, P_1(\nabla f))dM - \langle(\Delta P_1)(\nabla f), \nabla f\rangle] dM\\ &\geq [2\alpha^3(n-1)(n-a^2) + 2\kappa\alpha(n-1)^2 - \sigma]\int_M |\nabla f|^2 dM\end{aligned}$$

and for $\kappa < 0$,

$$\begin{aligned}&\int_M [2\,\mathrm{Ric}(\nabla f, P_1(\nabla f))dM - \langle(\Delta P_1)(\nabla f), \nabla f\rangle] dM\\ &\geq [2\alpha^3(n-1)(n-a^2) + 2\kappa a\alpha(n-1)^2 - \sigma]\int_M |\nabla f|^2 dM,\end{aligned}$$

where $\sigma = \max_{(p,v)\in TM}(\mathrm{tr}(\mathrm{Hess}\, H)|_{v^\perp}(p))$, $v^\perp = \{u \in T_pM; \langle u, v\rangle = 0\}$. In fact, by using Lemma 4.1, we have

$$\begin{aligned}&\int_M [2\,\mathrm{Ric}(\nabla f, P_1(\nabla f)) - \langle(\Delta P_1)(\nabla f), \nabla f\rangle] dM\\ &= \int_M \left[\mathrm{Hess}(H)(\nabla f, \nabla f) - (\Delta H)|\nabla f|^2\right] dM\\ &\quad + \int_M \langle Q(A)(\nabla f), \nabla f\rangle dM.\end{aligned}$$

By using Lemma 4.2, we have

$$\int_M \langle Q(A)(\nabla f), \nabla f\rangle dM \geq [2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2\alpha] \int_M |\nabla f|^2 dM$$

for $\kappa > 0$, and

$$\int_M \langle Q(A)(\nabla f), \nabla f\rangle dM \geq [2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2 a\alpha] \int_M |\nabla f|^2 dM$$

for $\kappa \leq 0$. On the other hand,

$$\begin{aligned}
&\int_M (\text{Hess}(H)(\nabla f, \nabla f) - (\Delta H)|\nabla f|^2) dM \\
&= \int_M \left[\text{Hess}(H)\left(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}\right) - (\Delta H)\right] |\nabla f|^2 dM \\
&\geq -\sigma \int_M |\nabla f|^2 dM.
\end{aligned}$$

Replacing these estimates in expression (4.5), we obtain, for $\kappa > 0$,

$$\begin{aligned}
0 \geq &-2\mu \int_M |\nabla f|^2 dM - \sigma \int_M |\nabla f|^2 dM + \frac{2\mu^2}{n(n-1)a\alpha} \int_M f^2 dM \\
&+ [2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2\alpha] \int_M |\nabla f|^2 dM,
\end{aligned}$$

and an analogous expression for $\kappa \leq 0$. Note that

$$(n-1)\alpha|\nabla f|^2 \leq \langle P_1(\nabla f), \nabla f\rangle \leq (n-1)a\alpha|\nabla f|^2.$$

Since $\frac{1}{2}L_1(f^2) = fL_1 f + \langle \nabla f, P_1(\nabla f)\rangle$, by using Divergence Theorem, we have

$$\int_M \langle \nabla f, P_1(\nabla f)\rangle dM = -\int_M f L_1 f dM = \mu \int_M f^2 dM.$$

It implies

$$\int_M f^2 dM \geq \frac{(n-1)\alpha}{\mu} \int_M |\nabla f|^2 dM.$$

Denoting by $C = 2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2\alpha$, we have

$$0 \geq (-2\mu - \sigma + C) \int_M |\nabla f|^2 dM + \frac{2\mu}{na} \int_M |\nabla f|^2 dM,$$

i.e.,

$$-2\mu + \frac{2\mu}{na} - \sigma + C \leq 0.$$

Therefore,

$$\mu \geq \frac{1}{2}\left(\frac{na}{na-1}\right)\left[2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2\alpha - \sigma\right],$$

for $\kappa > 0$, and

$$\mu \geq \frac{1}{2}\left(\frac{na}{na-1}\right)\left[2(n-1)\alpha^3(n-a^2) + 2\kappa(n-1)^2 a\alpha - \sigma\right],$$

for $\kappa \leq 0$. Now, consider the case of the canonical immersion of a geodesic sphere $x : \mathbb{S}^n(\alpha) \to \overline{M}^{n+1}(\kappa)$. In this case we have $A = \alpha I$, $a = 1$ and $L_1 f = n(n-1)\alpha\Delta f$. Since $\mu(L_1, M) = n(n-1)\alpha[\alpha^2 + \kappa]$ then, replacing these data in the estimate, the inequality becomes equality and the estimate is sharp. On the other hand, if the equality holds, the equality case of Lemma 3.1, p. 500 implies that $f_{ij} = cg_{ij}$, for some real constant c, and following the proof of Obata Theorem, cf. [18], we can conclude that M is a geodesic sphere. □

5 Appendix

In this appendix we include the Proposition mentioned in the introduction which gives examples of tensor ϕ, we refer to [10] for more related discussions.

Proposition 5.1. *Let M be a Riemannian manifold.*

(1) *If M has constant scalar curvature and c is a real constant, then the linear operator* $S_c := \mathrm{Ric} - cI$ *satisfies* $\mathrm{div}\, S_c \equiv 0$*;*

(2) *The* Einstein operator $E := \frac{1}{2}RI - \mathrm{Ric}$ *satisfies* $\mathrm{div}\, E = 0$*;*

(3) *If M is an immersed hypersurface in an Einstein manifold, then the Newton transformation* P_1 *satisfies* $\mathrm{div}\, P_1 \equiv 0$*;*

(4) *If M is an immersed hypersurface in a space form of constant sectional curvature, then the Newton transformation* P_r *satisfies* $\mathrm{div}\, P_r \equiv 0$*;*

(5) *The Shouten operator S satisfies* $\mathrm{div}\, S = \nabla(\mathrm{tr}\, S)$*;*

(6) *If M is locally conformally flat, then the Newton transformations* $T_k(S) = T_k$ *associated with S,* $1 \leq k \leq n$*, satisfies* $\mathrm{div}\, T_k(S) \equiv 0$.

Proof. It is well known, cf. [19], p. 39, and [11], p. 197, that

$$\operatorname{div}(\mathrm{Ric}) = \frac{1}{2} dR.$$

If M has constant scalar curvature, then $\operatorname{div}(\mathrm{Ric}) = 0$, which implies that $\operatorname{div}(S_c) = 0$, since c is constant. Claim (2) follows from

$$\operatorname{div} E = \operatorname{div}(\mathrm{Ric}) - \frac{1}{2}\operatorname{div}(RI) = \frac{1}{2} dR - \frac{1}{2} dR = 0.$$

The proof of claim (3) is simple and follows from well known identity

$$\operatorname{div}(A) = dH,$$

which holds for hypersurfaces immersed in an Einstein manifold, (see [13], for a proof). We have

$$\operatorname{div}(P_1) = \operatorname{div}(HI) - \operatorname{div}(A) = dH - \operatorname{div}(A) = 0.$$

The proof of claim (4) can be found in [20] or [21]. To prove Claim (5), we can use the identity $\sum_{j=1}^{n} \mathrm{Ric}_{ijj} = \frac{1}{2} R_i$, to obtain

$$\begin{aligned}
\sum_{j=1}^{n} S_{ijj} &= \sum_{j=1}^{n} \left(\mathrm{Ric}_{ij} - \frac{R}{2(n-1)} g_{ij} \right)_j \\
&= \sum_{j=1}^{n} \mathrm{Ric}_{ijj} - \sum_{j=1}^{n} \frac{R_j}{2(n-1)} g_{ij} \\
&= \frac{1}{2} R_i - \frac{R_i}{2(n-1)} \\
&= \frac{n-2}{2(n-1)} R_i \\
&= (\operatorname{tr} S)_i,
\end{aligned} \tag{5.1}$$

i.e.,

$$\operatorname{div} S = \nabla(\operatorname{tr} S).$$

Claim (6) was proved by Viaclovsky, and can be found in [23]. □

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References

[1] Hilário Alencar, Manfredo do Carmo and Maria Fernanda Elbert. *Stability of hypersurfaces with vanishing r-mean curvatures in Euclidean spaces.* J. Reine Angew. Math., **554** (2003), 201–216, DOI 10.1515/crll.2003.006. MR1952173 (2003k:53061).

[2] Hilário Alencar, Manfredo do Carmo and Walcy Santos. *A gap theorem for hypersurfaces of the sphere with constant scalar curvature one.* Comment. Math. Helv., **77**(3) (2002), 549–562, DOI 10.1007/s00014-002-8351-1. MR1933789 (2003m:53098).

[3] Hilário Alencar, Harold Rosenberg and Walcy Santos. *On the Gauss map of hypersurfaces with constant scalar curvature in spheres*. Proc. Amer. Math. Soc., **132**(12) (2004), 3731–3739 (electronic), DOI 10.1090/S0002-9939-04-07493-3. MR2084098 (2005h:53099).

[4] Hilário Alencar, Walcy Santos and Detang Zhou. *Stable hypersurfaces with constant scalar curvature.* Proc. Amer. Math. Soc., **138**(9) (2010), 3301–3312, DOI 10.1090/S0002-9939-10-10388-8. MR2653960 (2011m:53095)

[5] Alexander Alexandrov. Die innere Geometrie der konvexen Flachen, in Russian. German translation: Akademie, Berlin (1955).

[6] Luis J. Alías and J. Miguel Malacarne. *On the first eigenvalue of the linearized operator of the higher order mean curvature for closed hypersurfaces in space forms.* Illinois J. Math., **48**(1) (2004), 219–240. MR2048223 (2005b:53093)

[7] João Lucas Marques Barbosa and Antônio Gervásio Colares. *Stability of hypersurfaces with constant r-mean curvature.* Ann. Global Anal. Geom., **15**(3) (1997), 277–297, DOI 10.1023/A:1006514303828. MR1456513 (98h:53091).

[8] Arthur L. Besse. *Einstein manifolds, Classics in Mathematics.* Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition. MR2371700 (2008k:53084).

[9] Qing-Ming Cheng. *Compact locally conformally flat Riemannian manifolds.* Bull. London Math. Soc., **33**(4) (2001), 459–465, DOI 10.1017/S0024609301008074. MR1832558 (2002g:53045).

[10] Xu Cheng. *An almost-Schur type lemma for symmetric* (2, 0) *tensors and applications.* Pacific J. Math., **267**(2) (2014), 325–340, DOI 10.2140/pjm.2014.267.325, MR3207587.

[11] Shiu Yuen Cheng and Shing Tung Yau. *Hypersurfaces with constant scalar curvature.* Math. Ann., **225**(3) (1977), 195–204. MR0431043 (55 #4045).

[12] M.P. do Carmo and F.W. Warner. *Rigidity and convexity of hypersurfaces in spheres.* J. Differential Geometry, **4** (1970), 133–144. MR0266105 (42 #1014).

[13] Maria Fernanda Elbert. *Constant positive* 2*-mean curvature hypersurfaces.* Illinois J. Math., **46**(1) (2002), 247–267. MR1936088 (2003g:53103).

[14] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004).

[15] Jaques Hadamard. *Sur certaines proprits des trajectories en dynamique.* J. Math. Pures Appl., **3** (1897), 331–387.

[16] André Lichnerowicz. *Géométrie des groupes de transformations.* Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958 (French). MR0124009 (23 #A1329).

[17] Sebastián Montiel and Antonio Ros. *Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures.* Differential geometry, Pitman Monogr. Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 279–296. MR1173047 (93h:53062).

[18] Morio Obata. *Certain conditions for a Riemannian manifold to be isometric with a sphere.* J. Math. Soc. Japan, **14** (1962), 333–340. MR0142086 (25 #5479).

[19] Peter Petersen. *Riemannian geometry.* 2nd ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR2243772 (2007a:53001).

[20] Robert C. Reilly. *Variational properties of functions of the mean curvatures for hypersurfaces in space forms.* J. Differential Geometry, **8** (1973), 465–477. MR0341351 (49 #6102).

[21] Harold Rosenberg. *Hypersurfaces of constant curvature in space forms.* Bull. Sci. Math., **117**(2) (1993), 211–239. MR1216008 (94b:53097).

[22] R. Schoen, L. Simon and S.T. Yau. *Curvature estimates for minimal hypersurfaces.* Acta Math., **134**(3-4) (1975), 275–288. MR0423263 (54 #11243).

[23] Jeff A. Viaclovsky. *Conformal geometry, contact geometry, and the calculus of variations.* Duke Math. J., **101**(2) (2000), 283–316, DOI 10.1215/S0012-7094-00-10127-5. MR1738176 (2001b:53038).

[24] K. Voss. *Einige differentialgeometrische Kongruenzsätze für geschlossene Flächen und Hyperflächen.* Math. Ann., **131** (1956), 180–218 (German). MR0080327 (18,229f).

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