CURVATURE INTEGRAL ESTIMATES FOR COMPLETE HYPERSURFACES

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday.

ABSTRACT. We consider the integrals of the r-mean curvatures S_r of a complete hypersurface M in the space form $\mathcal{Q}_c^{n+1}$. Among other results, we prove that $\int_M S_r\, dM = \infty$ for a complete properly immersed hypersurfaces in a space form with $S_r \geq 0$, $S_r \not\equiv 0$ and $S_{r+1} \equiv 0$ for some $r \leq n-1$.

1. Introduction

Let M^n be a complete orientable hypersurface immersed in the space form $\mathcal{Q}_c^{n+1}$ of constant sectional curvature c. We denote by A and $\lambda_1, \ldots, \lambda_n$ the second fundamental operator and the eigenvalues of A, respectively. It is well known that the *r-mean curvature* at a point p is defined by

$$H_r(p) = \frac{1}{\binom{n}{r}} \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} = \frac{1}{\binom{n}{r}} S_r(p),$$

where S_r is the r-symmetric function of $\lambda_1, \ldots, \lambda_n$, for $1 \leq r \leq n$, and H_0 is defined to be zero and $H_r = 0$, for all $r \geq n+1$. In particular, for $r = 1$, $H_1 = H$ is the mean curvature.

We define the *r-area* of a domain $D \subset M$ by

$$\mathcal{A}_r(D) = \int_D S_r(p)\, dM.$$

Then, when $r = 0$, $\mathcal{A}_0$ is the volume of D.

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In this paper, we are interested in r-areas estimates. When $r = 0$, it is well known that a complete properly immersed minimal hypersurface in $\mathbb{R}^{n+1}$ has, at least, polynomial volume growth. In fact, infinity volume results hold for more general ambient spaces. Precisely, we have the following result of K. Frensel [9].

Theorem ([9], Theorem 1). *Let M^m be a complete, noncompact manifold and let $x : M^m \to N^n$ be an isometric immersion with mean curvature vector field bounded in norm. If N^n has sectional curvature bounded from above and injective radius bounded from below by a positive constant, then the volume of M^m is infinite.*

It is also true that each end of M has infinite volume under the same conditions (see [4]). These estimates have been used in studying the topology and geometric properties of minimal hypersurfaces and hypersurfaces with constant mean curvature (see for example [4], [9], [7]). It is natural to ask the following.

Question. *Let M^n be a complete noncompact manifold and let $x : M^n \to N^{n+1}$ be an isometric immersion such that there is a positive constant C satisfying*

$$|S_{r+1}| \leq C S_r$$

for some $r = 0, 1, \ldots, n-1$. Is the r-area of M^n infinite?

When $r = n$, $S_{r+1} = 0$, one can find a negative answer to this question. For example, if M is a complete noncompact surface in $\mathbb{R}^3$ with positive Gaussian curvature, then the total curvature is finite by a theorem of Cohn-Vossen. When $r < n$ we obtain a r-area estimate and give positive answers to this question in some interesting cases.

In order to state our results we need the rth Newton transformation, $P_r : T_pM \to T_pM$, which is defined inductively by

$$\begin{aligned} &P_0 = I, \\ &P_r = S_r I - A \circ P_{r-1}, \quad r > 1. \end{aligned}$$

Theorem A (Theorem 2.8). *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature c and let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that there exists a nonnegative constant α such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r$$

for some $r \leq n-1$. If P_r is positive semidefinite, then for any $q \in M$ such that $S_r(q) \neq 0$ and any $\mu_0 > 0$ there exists a positive constant C depending on μ_0, q and M such that for every $\mu > \mu_0$,

$$\mathcal{A}_r\big(\overline{B}_\mu(q) \cap M\big) = \int_{\overline{B}_\mu(q)\cap M} S_r \, dM \geq \int_{\mu_0}^{\mu} C\mathrm{e}^{-\alpha\tau} \, d\tau,$$

where $\overline{B}_\mu(q)$ *is the ball of radius* μ *and center* q *in* $\mathcal{Q}_c^{n+1}$. *For the case* $c>0$, *we assume* $\mu \leq \frac{\pi}{2\sqrt{c}}$.

As a consequence of this result we obtain the following.

THEOREM B (Corollary 2.9). *Let* $\mathcal{Q}_c^{n+1}$ *be a Riemannian manifold with constant sectional curvature* $c \leq 0$ *and let* M^n *be a complete noncompact properly immersed hypersurface of* $\mathcal{Q}_c^{n+1}$. *Assume that* $S_r \geq 0$, $S_r \not\equiv 0$ *and* $S_{r+1} \equiv 0$ *for some* $r \leq n-1$. *Then* $\int_M S_r\, dM = \infty$.

REMARK 1.1. The cases when r is even and r is odd are different. If r is odd and $S_r \leq 0$, we can change the orientation so that $S_r \geq 0$. But when r is even, S_r is independent of the choice of orientation. It has been proved by Gromov and Lawson that the existence of a complete metric with nonpositive scalar curvature ($r=2$) implies some topological obstructions, which is called enlargeable (see Corollary A in [11]). Enlargeable manifolds cannot carry metrics of positive scalar curvature.

Topping [18] used Sobolev inequality to get a diameter estimate in terms of the mean curvature integral. In Section 4, using his estimate we get a global estimate of the mean curvature integral.

THEOREM C (Theorem 4.1). *Let* M^m *be an* m*-dimensional complete noncompact Riemannian manifold isometrically immersed in* $\mathbb{R}^n$. *Then there exists a positive constant* δ *depending on* m *such that if*

$$\lim_{r\to+\infty} \sup \frac{V(x,r)}{r^m} < \delta,$$

where $V(x,r)$ *denotes the volume of the geodesic ball* $B_r(x)$, *then*

$$\lim_{R\to+\infty} \sup \frac{\int_{B_R(x)} |H|^{m-1}\, dM}{R} > 0.$$

In particular, $\int_M |H|^{m-1}\, dM = +\infty$.

For a complete noncompact surface M with finite total curvature, Cohn-Vossen theorem says that (see Theorem 6 in [6])

$$\int_M K\, dM \leq 2\pi\chi(M).$$

A special case of Corollary 4.3 says that if $\int_M K\, dM = 2\pi\chi(M)$, then $\int_M |H|\, dM = +\infty$.

The rest of the paper is organized as follows. In Section 2, we obtain the formulas relating the distance function and the r-mean curvature. The estimate obtained in Section 2 can be improved when $r=0$ and this is proved in Section 3. In Section 4, we give the proof of Theorem C.

2. r-area estimate

Let $x : M^n \to N^{n+1}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N.

In [15], Reilly showed that P_r satisfies the following

Proposition 2.1 ([15], p. 224, see also [2], Lemma 2.1). *Let $x : M^n \to N^{n+1}$ be an isometric immersion between two Riemannian manifolds and let A be the second fundamental form of x. The rth Newton transformation P_r associated to A satisfies*:

$$\operatorname{trace}(P_r) = (n-r)S_r, \tag{2.1}$$

$$\operatorname{trace}(A \circ P_r) = (r+1)S_{r+1}. \tag{2.2}$$

For hypersurfaces with bounded mean curvature, the Laplacian of the intrinsic distance to a fixed point of M played an important role in the proof of Frensel's estimate of the volume of M. In the case of H_r bounded, with $r > 1$, we used another second order differential operator defined on M, which seems to be natural for this problem. Associated to each Newton transformation P_r, if $f : M \to \mathbb{R}$ is a smooth function, we define

$$L_r(f) = \operatorname{trace}(P_r \circ \operatorname{Hess} f).$$

These operators are, in a certain sense, generalizations of the Laplace operator since $L_0(f) = \operatorname{trace}(\operatorname{Hess} f) = \Delta f$. They were introduced by Voss [19] in connection with variational arguments. In general, these operators are not elliptic and some conditions are necessary to ensure the ellipticity. For completeness, we include here some useful facts.

Proposition 2.2 ([8], Lemma 3.10). *Let N^{n+1} be an $(n+1)$-dimensional oriented Riemannian manifold and let M^n be a connected n-dimensional orientable Riemannian manifold. Suppose $x : M \to N$ is an isometric immersion. If $H_2 > 0$, then the operator L_1 is elliptic.*

Proposition 2.3 ([5], Proposition 3.2). *Let N^{n+1} be an $(n+1)$-dimensional oriented Riemannian manifold and let M^n be a connected n-dimensional orientable Riemannian manifold (with or without boundary). Suppose $x : M \to N$ is an isometric immersion with $H_r > 0$ for some $1 \le r \le n$. If there exists an interior point p of M such that all the principal curvatures at p are nonnegative, then for all $1 \le j \le r-1$, the operator L_j is elliptic, and the j-mean curvature H_j is positive.*

We need the following proposition which is essentially the content of Lemma 1.1 and equation (1.3) of [12]. We include here with a direct proof.

Proposition 2.4. *Let $M^n \to N^{n+1}$ be an isometric immersion. Suppose that $S_{r+1}(p) = 0$, for some r, $0 \le r < n$. Then P_r is semidefinite at p.*

Proof. Consider $S_r = S_r(\lambda_1, \dots, \lambda_n)$. Then $\frac{\partial S_r}{\partial \lambda_i}$ are the eigenvalues of P_r. Let $(\lambda_1^0, \dots, \lambda_n^0)$ be the principal curvatures of M at p. Hence

$$S_{r+1}\big(\lambda_1^0, \dots, \lambda_n^0\big) = 0.$$

We choose $\epsilon = \min_{\lambda_i^0 \neq 0}\{1, |\lambda_i^0|\}$. Then, for all $(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_i \in (0, \epsilon)$, $S_{r+1}(\lambda_1^0 + \varepsilon_1, \dots, \lambda_n^0 + \varepsilon_n)$ does not change sign. This implies that $\frac{\partial S_r}{\partial \lambda_i} \geq 0$ for all $i = 1, \dots, n$ or $\frac{\partial S_r}{\partial \lambda_i} \leq 0$ for all $i = 1, \dots, n$. Thus P_r is semidefinite at p. $\square$

Let M^n and N^{n+1} be Riemannian manifolds and let $x : M^n \to N^{n+1}$ be an isometric immersion. Henceforth, we shall tacitly make the usual identification of $X \in T_pM$ with $dx_p(X)$. In particular, if $F : N \to \mathbb{R}$ is smooth and we consider the composition $f = F \circ x$, then we have at $p \in M$, for every $X \in T_pM$:

$$\langle \operatorname{grad}_M f, X\rangle = df(X) = dF(X) = \langle \operatorname{grad}_N F, X\rangle,$$

where grad_M and grad_N denote the gradient on M and the gradient on N, respectively. So that

$$\operatorname{grad}_N F = \operatorname{grad}_M f + (\operatorname{grad}_N F)^{\perp}, \tag{2.3}$$

where $(\operatorname{grad} F)^{\perp}$ is perpendicular to T_pM. Let $F : N \to \mathbb{R}$ be a C^2 function and denote $f : M \to \mathbb{R}$ the function induced by F by restriction, that is $f = F \circ x$. We have the following.

LEMMA 2.5. *Let $x : M^n \to N^{n+1}$ be an isometric immersion. Let $F : N \to \mathbb{R}$ a smooth function and consider $f = F \circ x : M \to \mathbb{R}$. For an orthonormal frame $\{e_i\}$ on M, we have*

$$L_r f = \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + (r+1)S_{r+1}\langle \operatorname{grad}_N F, \eta\rangle, \tag{2.4}$$

where η denotes the normal vector field of the immersion and grad_N *is the gradient of N.*

Proof. Let ∇ and $\overline{\nabla}$ be the connections of M and N, respectively. If α denotes the second fundamental form of the immersion, Gauss' equation and equations (2.2) and (2.3) imply that

$$\begin{aligned} L_r f &= \sum_{i=1}^{n} \big\langle \nabla_{e_i}(\operatorname{grad}_M f), P_r(e_i)\big\rangle \\ &= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}(\operatorname{grad}_M f) - \big[\overline{\nabla}_{e_i}(\operatorname{grad}_M f) - \nabla_{e_i}(\operatorname{grad}_M f)\big], P_r(e_i)\big\rangle \\ &= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}(\operatorname{grad}_M f) - \alpha(e_i, \operatorname{grad}_M f), P_r(e_i)\big\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}(\operatorname{grad}_M f), P_r(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}\big(\operatorname{grad}_N F - (\operatorname{grad}_N F)^{\perp}\big), P_r(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} \operatorname{grad}_N F, P_r(e_i)\big\rangle - \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}(\operatorname{grad}_N F)^{\perp}, P_r(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) - \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}\big(\langle \operatorname{grad}_N F, \eta\rangle \eta\big), P_r(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) - \sum_{i=1}^{n} \big\langle \langle \operatorname{grad}_N F, \eta\rangle \overline{\nabla}_{e_i}\eta, P_r(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) - \langle \operatorname{grad}_N F, \eta\rangle \sum_{i=1}^{n} \big\langle -A(e_i), P_r(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + \langle \operatorname{grad}_N F, \eta\rangle \sum_{i=1}^{n} \big\langle e_i, AP_r(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + \langle \operatorname{grad}_N F, \eta\rangle \operatorname{trace}(AP_r) \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + (r+1)S_{r+1}\langle \operatorname{grad}_N F, \eta\rangle.
\end{aligned}
$$

□

Let $c \in \mathbb{R}$. Define the function:

$$\theta_c(t) := \int_0^t s_c(u)\, du,$$

where

$$
s_c(t) = \begin{cases} \frac{\sin \sqrt{c}t}{\sqrt{c}}, & \text{if } c > 0; \\ t, & \text{if } c = 0; \\ \frac{\sinh \sqrt{|c|}t}{\sqrt{|c|}}, & \text{if } c < 0. \end{cases} \tag{2.5}
$$

Let ρ denotes the distance function to the point Q in N^{n+1}, and $F : N^{n+1} \to \mathbb{R}$ given by $F(p) = \theta_c(\rho(p))$. In this case, Lemma 2.5 with $f = F \circ x$ and $F = \theta_c \circ \rho$ implies the following corollary.

Corollary 2.6. *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature c. Let M be an immersed hypersurface in $\mathcal{Q}_c^{n+1}$. Then, for* *all* $p \in M$,

$$(2.6)\quad \begin{aligned} &L_r\big(\theta_c\big(\rho(p)\big)\big) \\ &\quad = (n-r)s_c'\big(\rho(p)\big)S_r(p) + (r+1)S_{r+1}(p)s_c\big(\rho(p)\big)\big\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(p), \eta\big\rangle. \end{aligned}$$

In particular, when $c = 0$,

$$\frac{1}{2}L_r\big(\rho^2(p)\big) = (n-r)S_r(p) + (r+1)S_{r+1}(p)\rho(p)\big\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(p), \eta\big\rangle.$$

Proof. First observe that

$$(2.7)\qquad \operatorname{Hess}_{\mathcal{Q}_c^{n+1}} F(X,Y) = s_c'(\rho)\langle X, Y\rangle,$$

where $X, Y \in T_{x(p)}\mathcal{Q}_c^{n+1}$. In fact,

$$\begin{aligned} \operatorname{Hess}_{\mathcal{Q}_c^{n+1}} F(X,Y) &= \operatorname{Hess}_{\mathcal{Q}_c^{n+1}} \big(\theta_c(\rho)\big) \\ &= \big\langle \overline{\nabla}_X \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \big(\theta_c(\rho)\big), Y\big\rangle \\ &= \big\langle \overline{\nabla}_X s_c(\rho) \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\big\rangle \\ &= s_c(\rho) \operatorname{Hess}_{\mathcal{Q}_c^{n+1}} \rho(X,Y) \\ &\qquad + s_c'(\rho)\big\langle \langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, X\rangle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\big\rangle. \end{aligned}$$

On the other hand, see [1], p. 6,

$$\begin{aligned} \operatorname{Hess}_{\mathcal{Q}_c^{n+1}} \rho(X,Y) &= \langle \overline{\nabla}_X \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\rangle \\ &= \frac{s_c'(\rho)}{s_c(\rho)}\big[\langle X, Y\rangle - \langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, X\rangle\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\rangle\big]. \end{aligned}$$

This concludes the proof of (2.7). Now, by using equation (2.4), we have

$$\begin{aligned} L_r f &= \sum_{i=1}^{n} s_c'(\rho)\big\langle e_i, P_r(e_i)\big\rangle + (r+1)S_{r+1}\big\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} (\theta_c \circ \rho), \eta\big\rangle \\ &= s_c'(\rho) \operatorname{trace} P_r + (r+1)S_{r+1}s_c(\rho)\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, \eta\rangle. \end{aligned}$$

Finally, by using equation (2.1), we conclude the proof of equation (2.6). The case $c = 0$ follows immediately. $\square$

It follows from Codazzi equation (see Rosenberg [16], p. 225) that L_r is a divergent form operator, that is,

$$L_r(f) = \operatorname{div}_M(P_r \nabla f)$$

for all smooth functions $f : M \to \mathbb{R}$. Denote by $B_r(Q)$ the geodesic ball of $\mathcal{Q}_c^{n+1}$ with radius r centered at $Q \in \mathcal{Q}_c^{n+1}$, and by $\overline{B}_r(Q)$ its closure. We will use the following proposition.

Proposition 2.7. *Let* $\mathcal{Q}_c^{n+1}$ *be a Riemannian manifold with constant sectional curvature* c *and let* $x : M^n \to \mathcal{Q}_c^{n+1}$ *be an isometric immersion. For*

$Q \in \mathcal{Q}_c^{n+1}$, we denote by $\rho(x)$ the distance to the point $Q \in \mathcal{Q}_c^{n+1}$ and $\rho(x(p))$, $p \in M$, its restriction to M. If for some $r \leq n-1$, $S_r \geq 0$, then

$$
\begin{aligned}
(2.8)\qquad &\int_{\partial D} s_c\big(\rho(q)\big)\big\langle P_r\big(\operatorname{grad}_M \rho(q)\big), \nu\big\rangle\, dA \\
&\quad \geq (n-r)\int_D \left(s_c'\big(\rho(q)\big) S_r(p) - \frac{r+1}{n-r}\big|S_{r+1}(p)\big| s_c\big(\rho(q)\big)\right) dM,
\end{aligned}
$$

where $q = x(p)$, $D \subset M$ is a bounded domain with nonempty boundary ∂D and ν is the conormal vector along ∂D. In the case $c > 0$, we assume that $D \subset \overline{B}_{\frac{\pi}{2\sqrt{c}}}(Q)$.

Proof. Since $|\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(x(p))| \leq 1$ and $s_c'(\rho(x(p))) \geq 0$, from (2.6) we have

$$
L_r\big(\theta_c\big(\rho(x)\big)\big) \geq (n-r)\left[s_c'(\rho) S_r - \frac{r+1}{n-r}|S_{r+1}| s_c(\rho)\right].
$$

Integrating this inequality, we get

$$
\begin{aligned}
(2.9)\qquad &\int_D L_r\big(\theta_c\big(\rho(x)\big)\big)\, dM \\
&\quad \geq (n-r)\int_D \left[s_c'\big(\rho(x)\big) S_r - \frac{r+1}{n-r}|S_{r+1}| s_c\big(\rho(x)\big)\right] dM.
\end{aligned}
$$

On the other hand, we have that

$$
\begin{aligned}
\int_D L_r\big(\theta_c\big(\rho(x)\big)\big)\, dM &= \int_D \operatorname{div} P_r\big(\operatorname{grad}_M \big(\theta_c\big(\rho\big(x(p)\big)\big)\big)\big)\, dM \\
&= \int_D \operatorname{div}\big(s_c \rho\big(x(p)\big) P_r(\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho)^\top\big)\, dM \\
&= \int_{\partial D} s_c\big(\rho(x)\big)\big\langle P_r\big((\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho)^\top\big), \nu\big\rangle\, dA,
\end{aligned}
$$

where ν denotes the outward unit normal vector field on ∂D. Therefore, if $q = x(p)$,

$$
\begin{aligned}
&\int_{\partial D} s_c\big(\rho(q)\big)\big\langle P_r\big(\big(\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(q)\big)^\top\big), \nu\big\rangle\, dA \\
&\quad \geq (n-r)\int_D \left[s_c'\big(\rho(x)\big) S_r - \frac{r+1}{n-r}|S_{r+1}| s_c\big(\rho(x)\big)\right] dM,
\end{aligned}
$$

and the proposition is proved. □

We would like to point out that the above proposition is valid for a more general class of domains. For instance, it is valid in the setting of Gauss–Green Theorem (see [10], p. 478). In particular, if we take D to be the intersection of the extrinsic ball with M i.e. $D = \overline{B}_\mu \cap M$ in Proposition 2.7, we have the following

Theorem 2.8. *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature c and let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that there exists a nonnegative constant α such that*

$$(r+1)|S_{r+1}| \le (n-r)\alpha S_r \tag{2.10}$$

for some $r \le n-1$. If P_r is positive semidefinite, then for any $q \in M$ such that $S_r(q) \neq 0$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, q and M such that for every $\mu > \mu_0$,

$$\mathcal{A}_r\big(\overline{B}_\mu(q) \cap M\big) = \int_{\overline{B}_\mu(q)\cap M} S_r\, dM \ge \int_{\mu_0}^{\mu} C\mathrm{e}^{-\alpha\tau}\, d\tau,$$

where $\overline{B}_\mu(q)$ is the ball of radius μ and center q in $\mathcal{Q}_c^{n+1}$. For the case $c > 0$, we assume $\mu \le \frac{\pi}{2\sqrt{c}}$.

Proof. We use the notation introduced in Proposition 2.7. Take $D_\tau = \overline{B}_\tau(q) \cap M$, $\mu \le 2\pi/\sqrt{c}$, if $c > 0$. Since the immersion is proper, we have that $\partial D_\tau \neq \emptyset$, for all $0 < \tau < \mu$. Thus, by using (2.10) in equation (2.8), we obtain that

$$\begin{aligned}\int_{\partial D_\mu} & s_c\big(\rho(x)\big)\big\langle P_r(\operatorname{grad}_M \rho), \nu\big\rangle\, dA \\ &\ge (n-r)\int_{D_\mu} \big(s_c'\big(\rho(x)\big) - \alpha s_c\big(\rho(x)\big)\big) S_r\, dM \\ &= (n-r)\int_0^\mu \int_{\partial D_\tau} \frac{s_c'(\rho(x)) - \alpha s_c(\rho(x))}{s_c(\rho(x))} \\ &\quad \times s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho|^{-1} S_r\, dA\, d\tau,\end{aligned} \tag{2.11}$$

where we have used the co-area formula (see [3], p. 80). Observe that the conormal vector ν to ∂D_τ is parallel to $\operatorname{grad}_M \rho$. This fact together with the fact that P_r is positive semidefinite, imply the following:

$$\big\langle P_r(\operatorname{grad}_M \rho), \nu\big\rangle \le \operatorname{trace}(P_r)|\operatorname{grad}_M \rho| = (n-r)S_r|\operatorname{grad}_M \rho|.$$

Using the above equation and the fact that along ∂D_τ, $\rho(x) = \tau$, we get

$$\begin{aligned}\int_{\partial D_\mu} & s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho| S_r\, dA \\ &\ge \int_0^\mu \frac{s_c'(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \int_{\partial D_\tau} s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho|^{-1} S_r\, dA\, d\tau.\end{aligned} \tag{2.12}$$

Now we define

$$\varphi(\tau) = \int_{\partial D_\tau} s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho|^{-1} S_r\, dA.$$

Since $|\operatorname{grad}_M \rho| \le 1$, equation (2.12) implies

$$\varphi(\mu) \ge \int_0^\mu \frac{s_c'(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau)\, d\tau.$$

By writing

$$\phi(\mu) = \int_0^\mu \frac{s_c'(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau)\, d\tau,$$

one finds

$$\phi'(\mu) \ge \frac{s_c'(\mu) - \alpha s_c(\mu)}{s_c(\mu)} \phi(\mu).$$

Thus, by integrating from $\mu_0 > 0$ to μ, the above differential inequality arises

$$\ln \frac{\phi(\mu)}{\phi(\mu_0)} \ge \ln\left(\frac{s_c(\mu)}{s_c(\mu_0)}\right) - \alpha(\mu - \varepsilon) = \ln\left(\left(\frac{s_c(\mu)}{s_c(\mu_0)}\right) \mathrm{e}^{-\alpha(\mu-\mu_0)}\right).$$

Hence,

$$\phi(\mu) \ge \frac{\phi(\mu_0)}{s_c(\mu_0)} s_c(\mu) \mathrm{e}^{-\alpha\mu}.$$

Define

$$f(\mu) = \int_{D_\mu} S_r\, dM.$$

Again, by the co-area formula, it follows that

$$f(\mu) = \int_0^\mu \left(\int_{\partial D_\tau} |\operatorname{grad}_M \rho|^{-1} S_r\, dA\right) d\tau.$$

Since

$$f'(\mu) = \int_{\partial D_\mu} |\operatorname{grad}_M \rho|^{-1} S_r\, dA = \frac{1}{s_c(\mu)} \varphi(\mu) \ge \frac{\phi(\mu_0)}{s_c(\mu_0)} \mathrm{e}^{-\alpha\mu},$$

then for $\mu > \mu_0$,

$$f(\mu) \ge \int_{\mu_0}^\mu \frac{\phi(\mu_0)}{s_c(\mu_0)} \mathrm{e}^{-\alpha\tau}\, d\tau. \qquad \square$$

Corollary 2.9. *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature $c \le 0$ and let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that $S_r \ge 0$, $S_r \not\equiv 0$ and $S_{r+1} \equiv 0$ for some $r \le n-1$. Then $\int_M S_r\, dM = \infty$.*

Proof. Since the immersion is proper, we have $\partial(M \cap \overline{B}_\mu(q))$ is nonempty for all $\mu > \mu_0$. By using Proposition 2.4, since $S_{r+1} = 0$, we have that P_r is semidefinite. Now, the condition $S_r \ge 0$ implies that P_r is positive semidefinite. Therefore, using Theorem 2.8, with $\alpha = 0$, for all $\mu > \mu_0$,

$$\int_{\overline{B}_\mu \cap M} S_r\, dM \ge \int_{\mu_0}^\mu C\mathrm{e}^{-\alpha\tau}\, d\tau = C(\mu - \mu_0).$$

Then

$$\int_M S_r\, dM = \infty. \qquad \square$$

REMARK 2.10. When r is odd, the condition $S_r \geq 0$ can be obtained by choosing the right orientation.

The condition of semi-positiveness of P_2 is satisfied when M is a hypersurface immersed in $\mathbb{R}^{n+1}$ with $S_3 = 0$ (which is called 2-minimal) and $S_2 > 0$. In fact, in this case P_2 is positive definite, since L_2 is elliptic (see Proposition 2.2). So we have

COROLLARY 2.11. *Let M^n be a complete 2-minimal noncompact properly immersed hypersurface of $\mathbb{R}^{n+1}$ with nonnegative scalar curvature. Then either the scalar curvature is zero or the total scalar curvature is infinite.*

REMARK 2.12. When $n = 3$ the corollary can be proved by using Theorem III in [13] without the assumption that the immersion is proper. In this case, M^n has to be a cylinder and the conclusion of the above corollary follows immediately.

REMARK 2.13. The condition of semi-positiveness of P_r is also satisfied when M is a hypersurface in $\mathbb{R}^{n+1}$ with nonnegative sectional or positive Ricci curvature, $\mathrm{Ric}_M > 0$. Indeed when $\mathrm{Ric}_M > 0$, for each point in M, the principal curvatures can be arranged as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i < 0 < \lambda_{i+1} \leq \cdots \leq \lambda_n$. The positivity of the Ricci curvature implies

$$\mathrm{Ric}_M(e_j) = \lambda_j \left(\sum_{k \neq j} \lambda_k \right) > 0 \quad \forall j = 1, \ldots, n.$$

If $i \in \{1, \ldots, n-1\}$, it follows from the above equation that

$$\sum_{k \neq j} \lambda_k < 0, \quad \text{when } j \leq i, \tag{2.13}$$

and

$$\sum_{k \neq j} \lambda_k > 0, \quad \text{when } j > i. \tag{2.14}$$

From (2.13), we have for $j_1 \leq i$,

$$\sum_{k \neq j_1} \lambda_k = \left(\sum_{k=1}^{i} \lambda_k - \lambda_{j_1} \right) + \sum_{k=i+1}^{n} \lambda_k < 0.$$

Thus

$$-\sum_{k=1}^{i} \lambda_k > \sum_{k=1}^{i} \lambda_k + \lambda_{j_1} > \sum_{k=i+1}^{n} \lambda_k.$$

On the other hand, using (2.14), for $j_2 > i$, we find

$$\sum_{k\neq j_2}\lambda_k = \left(\sum_{k=1}^{i}\lambda_k - \lambda_{j_2}\right) + \sum_{k=i+1}^{n}\lambda_k > 0,$$

hence

$$-\sum_{k=1}^{i}\lambda_k < \sum_{k=1}^{i}\lambda_k + \lambda_{j_1} < \sum_{k=i+1}^{n}\lambda_k,$$

which is a contradiction. Thus, all λ_i has the same sign (we are indebted to F. Fontenele for this observation). So we can choose an orientation such that P_r is positive definite and $S_r > 0$.

Thus we have the following consequence.

Corollary 2.14. *Let M^n be a complete noncompact properly immersed hypersurface of $\mathbb{R}^{n+1}$ with positive Ricci curvature. Assume that there exists a positive constant α such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r$$

for some $r \leq n-1$. Then, for any $q \in M$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, Q and M such that

$$\int_{\overline{B}_\mu(q)\cap M} S_r\, dM \geq \int_{\mu_0}^{\mu} C\mathrm{e}^{-\alpha\tau}\, d\tau,$$

where $\overline{B}_\mu(q)$ is the geodesic ball in $\mathbb{R}^{n+1}$ centered at q.

The following is a direct consequence of Theorem 2.8 and Proposition 2.3.

Corollary 2.15. *Let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that S_r is positive and there exists a positive constant α such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r$$

for some $r \leq n-1$. If there exists a point such that all principal curvatures are nonnegative, then, for any $q \in M$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, q and M such that

$$\int_{\overline{B}_\mu(q)\cap M} S_r\, dM \geq \int_{\mu_0}^{\mu} C\mathrm{e}^{-\alpha\tau}\, d\tau,$$

where $\overline{B}_\mu(q)$ is the geodesic ball in $\mathcal{Q}_c^{n+1}$ centered at q. For the case $c > 0$, we assume $\mu \leq \frac{\pi}{2\sqrt{c}}$.

3. Volume estimates in general manifolds

In this section we consider N^{n+p} a Riemannian manifold with sectional curvature bounded from above by a constant c. Let M^n be a submanifold isometrically immersed in $N = N^{n+p}$.

Let $F : N \to \mathbb{R}$ be a C^2 function and denote $f : M \to \mathbb{R}$ the function induced by F by restriction. Essentially, following the steps involved in the proof of Lemma 2.5, we obtain

$$\Delta f = \sum_{i=1}^{n} \operatorname{Hess}_N F(e_i, e_i) + n\langle \operatorname{grad}_N F, \mathbf{H}\rangle,$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal frame along M and $\mathbf{H}$ is the mean curvature vector. Similar to Proposition 2.7, we have

Proposition 3.1. *Let N be a Riemannian manifold with sectional curvature bounded from above by a constant c and M^n an immersed connected submanifold of N. We denote by $\overline{\rho}(x)$ the distance in N between x and $Q \in N^{n+p}$ and $\rho(x)$ the induced function of $\overline{\rho}$ by restriction. Then*

$$\int_{\partial D} s_c\big(\rho(x)\big)\langle \operatorname{grad}_M \rho, \nu\rangle \, dA \geq n \int_D \big(s_c'\big(\rho(x)\big) - |\mathbf{H}| s_c\big(\rho(x)\big)\big)\, dM, \tag{3.1}$$

where $q = x(p)$, $D \subset M$ is a bounded domain with nonempty boundary ∂D and $D \cap C_N(Q) = \emptyset$, where $C_N(Q)$ is the cut locus of the point Q in N, and ν is the conormal vector along ∂D.

Proof. Let $V = s_c(\overline{\rho}) \operatorname{grad}_N \overline{\rho}$ and $V^\top$ the orthogonal projection of V into the tangent space of M. Then we have $V^\top = s_c(\rho) \operatorname{grad}_M \rho$, where $\rho(x)$ is the induced function of $\overline{\rho}$ to M by restriction. Thus, Lemma 2.5 of [14], p. 713, implies, when $\overline{\rho} < \operatorname{inj}_N(Q)$,

$$\operatorname{Hess}_N F(X, X) \geq s_c'(\overline{\rho})\langle X, X\rangle. \tag{3.2}$$

Then

$$\langle \overline{\nabla}_{e_i} V, e_i\rangle \geq s_c'(\overline{\rho})$$

for all $\overline{\rho}$ when $c \leq 0$, and $\rho \leq \frac{\pi}{\sqrt{c}}$, when $c > 0$. We find that

$$\Delta\big(\theta_c\big(\rho(x)\big)\big) \geq n\big[s_c'(\rho) - s_c(\rho)|\mathbf{H}|\big].$$

Integrating this inequality and applying Stokes' formula, we get

$$\int_{\partial D} s_c\big\langle (\operatorname{grad}_N \overline{\rho})^\top, \nu\big\rangle \, dA \geq n \int_D \big[s_c'\big(\rho(x)\big) - s_c\big(\rho(x)\big)|\mathbf{H}|\big]\, dM,$$

and the proposition follows. $\square$

Similar to Proposition 2.7, the above result is valid in a more general setting, such as extrinsic geodesic balls. Using this fact, we arrive at

Theorem 3.2. *Let M be a Riemannian manifold isometrically immersed in a geodesic ball $\overline{B}(O,\rho_0)\subset N^{n+p}$ with codimension p. Assume that the sectional curvature of N in $\overline{B}(O,\rho_0)$ is bounded from above by c and moreover that there exists a positive constant α such that*

$$|\mathbf{H}|\leq\alpha.$$

Then

$$\operatorname{vol}\bigl(B_\mu(q)\bigr)\geq n\omega_n\int_0^\mu s_c(t)^{n-1}\mathrm{e}^{-n\alpha s}\,dt,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $B_\mu(q)$ is the intrinsic geodesic ball in M with center $q\in M$ and radius $\mu<\operatorname{inj}_N(q)$.

Proof. By taking $D=B_\tau(q)$ in Proposition 3.1, we obtain

$$\langle\operatorname{grad}_M\rho,\nu\rangle\leq|\operatorname{grad}_M\rho|.$$

Thus,

$$\begin{aligned}(3.3)\qquad &\int_{\partial B_\tau(q)}\frac{s_c(\rho(x))}{n}|\operatorname{grad}_M\rho|\,dA\\ &\geq\int_{B_\tau(q)}\bigl(s_c'\bigl(\rho(x)\bigr)-\alpha s_c\bigl(\rho(x)\bigr)\bigr)\,dM\\ &=\int_0^\mu\int_{\partial B_\tau(q)}\frac{s_c'(\rho(x))-\alpha s_c(\rho(x))}{s_c(\rho(x))}s_c\bigl(\rho(x)\bigr)|\operatorname{grad}_M\rho|^{-1}\,dA\,d\tau,\end{aligned}$$

where we have used the co-area formula (see [3], p. 80). Since the intrinsic distance is not less than the extrinsic one and

$$\left(\frac{s_c'}{s_c}\right)'\leq 0,$$

then

$$\begin{aligned}(3.4)\qquad &\frac{1}{n}\int_{\partial B_\mu(q)}s_c\bigl(\rho(x)\bigr)|\operatorname{grad}_M\rho|\,dA\\ &\geq\int_0^\mu\frac{s_c'(\tau)-\alpha s_c(\tau)}{s_c(\tau)}\int_{\partial B_\tau(q)}s_c\bigl(\rho(x)\bigr)|\operatorname{grad}_M\rho|^{-1}\,dA\,d\tau.\end{aligned}$$

Now we define

$$\varphi(\tau)=\int_{\partial B_\tau(q)}s_c\bigl(\rho(x)\bigr)|\operatorname{grad}_M\rho|^{-1}\,dA.$$

Equation (3.4) implies that

$$\frac{1}{n}\varphi(\mu)\geq\int_0^\mu\frac{s_c'(\tau)-\alpha s_c(\tau)}{s_c(\tau)}\varphi(\tau)\,d\tau.$$

By writing

$$\phi(\mu)=\int_0^\mu\frac{s_c'(\tau)-\alpha s_c(\tau)}{s_c(\tau)}\varphi(\tau)\,d\tau,$$

we have

$$\phi'(\mu) \geq \frac{n(s_c'(\mu) - \alpha s_c(\mu))}{s_c(\mu)} \phi(\mu).$$

Thus, by integrating from $\varepsilon > 0$ to μ, with $\mu \leq \min\{\mathrm{inj}_N(q), \frac{\pi}{2\sqrt{c}}\}$ when $c > 0$, the above differential inequality arises

$$\frac{1}{n} \ln \frac{\phi(\mu)}{\phi(\varepsilon)} \geq \ln\left(\frac{s_c(\mu)}{\varepsilon}\right) - \alpha(\mu - \varepsilon) = \ln\left[\left(\frac{s_c(\mu)}{\varepsilon}\right) \mathrm{e}^{-\alpha(\mu-\varepsilon)}\right].$$

Hence,

$$\frac{\phi(\mu)}{\phi(\varepsilon)} \geq \left[\left(\frac{s_c(\mu)}{\varepsilon}\right) \mathrm{e}^{-\alpha(\mu-\varepsilon)}\right]^n. \tag{3.5}$$

Observe that by the mean value theorem,

$$\lim_{\varepsilon \to 0} \frac{\phi(\varepsilon)}{\varepsilon^n} = \omega_n.$$

Then

$$\phi(\mu) \geq \omega_n s_c(\mu)^n \mathrm{e}^{-n\alpha\mu}.$$

Now, define

$$f(\mu) = \int_{B_\mu(q)} dM = \mathrm{vol}\big(B_\mu(q)\big).$$

Again, by the co-area formula, we can write $f(\mu)$ as

$$f(\mu) = \int_0^\mu \left(\int_{\partial B_\tau(q)} |\operatorname{grad}_M \rho|^{-1} \, dA\right) d\tau.$$

Hence

$$f'(\mu) = \int_{\partial B_\mu(q)} |\operatorname{grad}_M \rho|^{-1} \, dA.$$

This equality together with $|\operatorname{grad}_M \rho| \leq 1$, and equation (3.3) imply that

$$\frac{s_c(\mu)}{n} f'(\mu) \geq \int_{\partial B_\mu(q)} \frac{s_c(\rho(x))}{n} |\operatorname{grad}_M \rho| \, dA \geq \int_0^\mu \big(s_c'(\tau) - \alpha s_c(\tau)\big) f'(\tau) \, d\tau.$$

Since

$$f'(\mu) \geq \frac{\varphi(\mu)}{s_c(\mu)},$$

then

$$f(\mu) \geq \int_0^\mu \omega_n n s_c(\tau)^{n-1} \mathrm{e}^{-n\alpha\tau} \, d\tau,$$

which concludes the proof. □

The following corollary follows immediately.

COROLLARY 3.3. (i) *Let M^n be an immersed minimal hypersurface of the Euclidean space $\mathbb{R}^{n+p}$. Then*

$$\operatorname{vol}\bigl(B_\mu(q)\bigr) \geq \omega_n \mu^n,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $B_\mu(q)$ is the intrinsic geodesic ball in M centered at $q \in M$.

(ii) *Let M^n be an immersed hypersurface of the hyperbolic space $\mathbb{H}^{n+p}(-1)$. Assume that there exists a positive constant α such that*

$$|H| \leq \alpha < \frac{n-1}{n}.$$

Then, there exists a constant $C > 0$ so that, for $\mu \geq 1$,

$$\operatorname{vol}\bigl(B_\mu(q)\bigr) \geq C e^{(n-1-n\alpha)\mu},$$

where $B_\mu(q)$ is the intrinsic geodesic ball in M with center $q \in M$.

4. Mean curvature integral

In this section, inspired by a recent work of Topping [18], we prove a type of mean curvature integral estimate for complete submanifold in a Euclidean space $\mathbb{R}^n$ and we apply it to surfaces in $\mathbb{R}^n$.

THEOREM 4.1. *Let M^m be a m-dimensional complete noncompact Riemannian manifold isometrically immersed in $\mathbb{R}^n$. Then there exists a positive constant δ depending on m such that if*

$$\limsup_{r\to+\infty} \frac{V(x,r)}{r^m} < \delta, \tag{4.1}$$

where $V(x,r)$ denotes the volume of the geodesic ball $B_r(x)$, then

$$\limsup_{R\to+\infty} \frac{\int_{B_R(x)} |H|^{m-1}\, dM}{R} > 0. \tag{4.2}$$

In particular, $\int_M |H|^{m-1}\, dM = +\infty$.

We need the following lemma of Topping [18].

LEMMA 4.2 ([18], Lemma 1.2). *Let M^m be a m-dimensional complete Riemannian manifold isometrically immersed in $\mathbb{R}^n$. Then a positive constant δ depending on m exists, such that for any $x \in M$ and $R > 0$, at least one of the following statements is true*:

(i) $\sup_{r\in(0,R]} r^{-\frac{1}{m-1}} [V(x,r)]^{-\frac{m-2}{m-1}} \int_{B(x,r)} |H|^{m-1}\, dM > \delta$,

(ii) $\inf_{r\in(0,R]} \frac{V(x,r)}{r^m} > \delta$.

Proof of Theorem 4.1. We can choose L large enough so that $V(z,L) \leq \delta L^m$ for all $z \in M$. Then, from Lemma 4.2, we have

$$\sup_{r\in(0,L]} r^{-\frac{1}{m-1}} \big[V(z,r)\big]^{-\frac{m-2}{m-1}} \int_{B_r(z)} |H|^{m-1}\, dM > \delta.$$

Since

$$\int_{B_r(z)} |H|\, dM \leq \left(\int_{B_r(z)} |H|^{m-1}\, dM\right)^{\frac{1}{m-1}} \cdot \big(V(z,r)\big)^{\frac{m-2}{m-1}}$$

for any $z \in M$, there exists a $r(z) \in (0,R]$ such that

$$\int_{B_r(z)} |H|^{m-1}\, dM > \delta^{m-1} r(z).$$

Fix a point $o \in M$, and let $\gamma : [0,+\infty) \to M$ be a ray parametrized by an arclength with $\gamma(0) = o$. For any fixed $R > 0$,

$$\gamma\big([0,R]\big) \subset \bigcup_{t\in[0,R]} B_{r(\gamma(t))}\big(\gamma(t)\big).$$

From a covering argument used in Theorem 1.1 of [18], we can find an at most countable sequence $t_1, t_2, \dots, t_q, \dots \in [0,R]$ such that $\sum_i r(\gamma(t_i)) \geq \frac{1}{4}R$. Thus, when $i \neq j$,

$$B_{r(\gamma(t_i))}\big(\gamma(t_i)\big) \cap B_{r(\gamma(t_j))}\big(\gamma(t_j)\big) = \emptyset.$$

Then

$$\begin{aligned}\int_{B_{2R}(o)} |H|^{m-1}\, dM &\geq \sum_i \int_{B_{r(\gamma(t_i))}(\gamma(t_i))} |H|^{m-1}\, dM\\ &\geq \delta^{m-1} \sum_i r\big(\gamma(t_i)\big)\\ &\geq \delta^{m-1}\frac{1}{4}R.\end{aligned}$$

And the result is proved. $\square$

For complete surfaces in $\mathbb{R}^n$ that satisfy the Gauss–Bonnet relation, we obtain the following result.

Corollary 4.3. *Let δ be as in Theorem 4.1. If M is a complete noncompact surface in $\mathbb{R}^n$ satisfying*

$$2\pi\chi(M) - \int_M K\, dM < 2\delta, \tag{4.3}$$

where $\chi(M)$ is the Euler characteristic of M, then

$$\int_M |H|\, dM = +\infty.$$

Proof. From Theorem A of Shiohama [17], for any $q \in M$, we find that

$$\lim_{r\to\infty} \frac{2V(B_r(q))}{r^2} = 2\pi\chi(M) - \int_M K\,dM.$$

It should be noted here that there is a misprint in the denominator of this expression in Shiohama's paper. So,

$$\lim_{r\to\infty} \frac{V(B_r(q))}{\pi r^2} < \delta.$$

Thus, Theorem 4.1 implies the result. □

REMARK 4.4. The flat plane embedded in $\mathbb{R}^n$ shows that the condition (4.3) is necessary.

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