CURVATURE INTEGRAL ESTIMATES FOR COMPLETE HYPERSURFACES

HILÁRIO ALENCAR, WALCY SANTOS AND DETANG ZHOU

Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday.

ABSTRACT. We consider the integrals of the r-mean curvatures S_r of a complete hypersurface M in the space form $\mathcal{Q}_c^{n+1}$. Among other results, we prove that $\int_M S_r\, dM = \infty$ for a complete properly immersed hypersurfaces in a space form with $S_r \geq 0$, $S_r \not\equiv 0$ and $S_{r+1} \equiv 0$ for some $r \leq n-1$.

1. Introduction

Let M^n be a complete orientable hypersurface immersed in the space form $\mathcal{Q}_c^{n+1}$ of constant sectional curvature c. We denote by A and $\lambda_1, \ldots, \lambda_n$ the second fundamental operator and the eigenvalues of A, respectively. It is well known that the *r-mean curvature* at a point p is defined by

$$H_r(p) = \frac{1}{\binom{n}{r}} \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} = \frac{1}{\binom{n}{r}} S_r(p),$$

where S_r is the r-symmetric function of $\lambda_1, \ldots, \lambda_n$, for $1 \leq r \leq n$, and H_0 is defined to be zero and $H_r = 0$, for all $r \geq n+1$. In particular, for $r = 1$, $H_1 = H$ is the mean curvature.

We define the *r-area* of a domain $D \subset M$ by

$$\mathcal{A}_r(D) = \int_D S_r(p)\, dM.$$

Then, when $r = 0$, $\mathcal{A}_0$ is the volume of D.

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In this paper, we are interested in r-areas estimates. When $r=0$, it is well known that a complete properly immersed minimal hypersurface in $\mathbb{R}^{n+1}$ has, at least, polynomial volume growth. In fact, infinity volume results hold for more general ambient spaces. Precisely, we have the following result of K. Frensel [9].

THEOREM ([9], Theorem 1). *Let M^m be a complete, noncompact manifold and let $x: M^m \to N^n$ be an isometric immersion with mean curvature vector field bounded in norm. If N^n has sectional curvature bounded from above and injective radius bounded from below by a positive constant, then the volume of M^m is infinite.*

It is also true that each end of M has infinite volume under the same conditions (see [4]). These estimates have been used in studying the topology and geometric properties of minimal hypersurfaces and hypersurfaces with constant mean curvature (see for example [4], [9], [7]). It is natural to ask the following.

QUESTION. *Let M^n be a complete noncompact manifold and let $x: M^n \to N^{n+1}$ be an isometric immersion such that there is a positive constant C satisfying*

$$|S_{r+1}| \leq C S_r$$

for some $r=0,1,\ldots,n-1$. Is the r-area of M^n infinite?

When $r=n$, $S_{r+1}=0$, one can find a negative answer to this question. For example, if M is a complete noncompact surface in $\mathbb{R}^3$ with positive Gaussian curvature, then the total curvature is finite by a theorem of Cohn-Vossen. When $r<n$ we obtain a r-area estimate and give positive answers to this question in some interesting cases.

In order to state our results we need the rth Newton transformation, $P_r: T_pM \to T_pM$, which is defined inductively by

$$\begin{aligned} &P_0 = I, \\ &P_r = S_r I - A \circ P_{r-1}, \quad r > 1. \end{aligned}$$

THEOREM A (Theorem 2.8). *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature c and let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that there exists a nonnegative constant α such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r$$

for some $r \leq n-1$. If P_r is positive semidefinite, then for any $q \in M$ such that $S_r(q) \neq 0$ and any $\mu_0 > 0$ there exists a positive constant C depending on μ_0, q and M such that for every $\mu > \mu_0$,

$$\mathcal{A}_r\big(\overline{B}_\mu(q) \cap M\big) = \int_{\overline{B}_\mu(q)\cap M} S_r \, dM \geq \int_{\mu_0}^{\mu} C \mathrm{e}^{-\alpha\tau} \, d\tau,$$

where $\overline{B}_\mu(q)$ *is the ball of radius* μ *and center* q *in* $\mathcal{Q}_c^{n+1}$. *For the case* $c>0$, *we assume* $\mu \leq \frac{\pi}{2\sqrt{c}}$.

As a consequence of this result we obtain the following.

THEOREM B (Corollary 2.9). *Let* $\mathcal{Q}_c^{n+1}$ *be a Riemannian manifold with constant sectional curvature* $c \leq 0$ *and let* M^n *be a complete noncompact properly immersed hypersurface of* $\mathcal{Q}_c^{n+1}$. *Assume that* $S_r \geq 0$, $S_r \not\equiv 0$ *and* $S_{r+1} \equiv 0$ *for some* $r \leq n-1$. *Then* $\int_M S_r\, dM = \infty$.

REMARK 1.1. The cases when r is even and r is odd are different. If r is odd and $S_r \leq 0$, we can change the orientation so that $S_r \geq 0$. But when r is even, S_r is independent of the choice of orientation. It has been proved by Gromov and Lawson that the existence of a complete metric with nonpositive scalar curvature ($r=2$) implies some topological obstructions, which is called enlargeable (see Corollary A in [11]). Enlargeable manifolds cannot carry metrics of positive scalar curvature.

Topping [18] used Sobolev inequality to get a diameter estimate in terms of the mean curvature integral. In Section 4, using his estimate we get a global estimate of the mean curvature integral.

THEOREM C (Theorem 4.1). *Let* M^m *be an* m*-dimensional complete noncompact Riemannian manifold isometrically immersed in* $\mathbb{R}^n$. *Then there exists a positive constant* δ *depending on* m *such that if*

$$\lim \sup_{r\to+\infty} \frac{V(x,r)}{r^m} < \delta,$$

where $V(x,r)$ *denotes the volume of the geodesic ball* $B_r(x)$, *then*

$$\lim \sup_{R\to+\infty} \frac{\int_{B_R(x)} |H|^{m-1}\, dM}{R} > 0.$$

In particular, $\int_M |H|^{m-1}\, dM = +\infty$.

For a complete noncompact surface M with finite total curvature, Cohn-Vossen theorem says that (see Theorem 6 in [6])

$$\int_M K\, dM \leq 2\pi\chi(M).$$

A special case of Corollary 4.3 says that if $\int_M K\, dM = 2\pi\chi(M)$, then $\int_M |H|\, dM = +\infty$.

The rest of the paper is organized as follows. In Section 2, we obtain the formulas relating the distance function and the r-mean curvature. The estimate obtained in Section 2 can be improved when $r=0$ and this is proved in Section 3. In Section 4, we give the proof of Theorem C.

2. r-area estimate

Let $x : M^n \to N^{n+1}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N.

In [15], Reilly showed that P_r satisfies the following

Proposition 2.1 ([15], p. 224, see also [2], Lemma 2.1). *Let $x : M^n \to N^{n+1}$ be an isometric immersion between two Riemannian manifolds and let A be the second fundamental form of x. The rth Newton transformation P_r associated to A satisfies*:

$$\operatorname{trace}(P_r) = (n-r)S_r, \tag{2.1}$$

$$\operatorname{trace}(A \circ P_r) = (r+1)S_{r+1}. \tag{2.2}$$

For hypersurfaces with bounded mean curvature, the Laplacian of the intrinsic distance to a fixed point of M played an important role in the proof of Frensel's estimate of the volume of M. In the case of H_r bounded, with $r > 1$, we used another second order differential operator defined on M, which seems to be natural for this problem. Associated to each Newton transformation P_r, if $f : M \to \mathbb{R}$ is a smooth function, we define

$$L_r(f) = \operatorname{trace}(P_r \circ \operatorname{Hess} f).$$

These operators are, in a certain sense, generalizations of the Laplace operator since $L_0(f) = \operatorname{trace}(\operatorname{Hess} f) = \Delta f$. They were introduced by Voss [19] in connection with variational arguments. In general, these operators are not elliptic and some conditions are necessary to ensure the ellipticity. For completeness, we include here some useful facts.

Proposition 2.2 ([8], Lemma 3.10). *Let N^{n+1} be an $(n+1)$-dimensional oriented Riemannian manifold and let M^n be a connected n-dimensional orientable Riemannian manifold. Suppose $x : M \to N$ is an isometric immersion. If $H_2 > 0$, then the operator L_1 is elliptic.*

Proposition 2.3 ([5], Proposition 3.2). *Let N^{n+1} be an $(n+1)$-dimensional oriented Riemannian manifold and let M^n be a connected n-dimensional orientable Riemannian manifold (with or without boundary). Suppose $x : M \to N$ is an isometric immersion with $H_r > 0$ for some $1 \le r \le n$. If there exists an interior point p of M such that all the principal curvatures at p are nonnegative, then for all $1 \le j \le r-1$, the operator L_j is elliptic, and the j-mean curvature H_j is positive.*

We need the following proposition which is essentially the content of Lemma 1.1 and equation (1.3) of [12]. We include here with a direct proof.

Proposition 2.4. *Let $M^n \to N^{n+1}$ be an isometric immersion. Suppose that $S_{r+1}(p) = 0$, for some r, $0 \le r < n$. Then P_r is semidefinite at p.*

Proof. Consider $S_r = S_r(\lambda_1,\dots,\lambda_n)$. Then $\frac{\partial S_r}{\partial \lambda_i}$ are the eigenvalues of P_r. Let $(\lambda_1^0,\dots,\lambda_n^0)$ be the principal curvatures of M at p. Hence

$$S_{r+1}\big(\lambda_1^0,\dots,\lambda_n^0\big) = 0.$$

We choose $\epsilon = \min_{\lambda_i^0 \neq 0}\{1, |\lambda_i^0|\}$. Then, for all $(\varepsilon_1,\dots,\varepsilon_n)$ with $\varepsilon_i \in (0,\epsilon)$, $S_{r+1}(\lambda_1^0 + \varepsilon_1,\dots,\lambda_n^0 + \varepsilon_n)$ does not change sign. This implies that $\frac{\partial S_r}{\partial \lambda_i} \geq 0$ for all $i = 1,\dots,n$ or $\frac{\partial S_r}{\partial \lambda_i} \leq 0$ for all $i = 1,\dots,n$. Thus P_r is semidefinite at p. $\square$

Let M^n and N^{n+1} be Riemannian manifolds and let $x : M^n \to N^{n+1}$ be an isometric immersion. Henceforth, we shall tacitly make the usual identification of $X \in T_pM$ with $dx_p(X)$. In particular, if $F : N \to \mathbb{R}$ is smooth and we consider the composition $f = F \circ x$, then we have at $p \in M$, for every $X \in T_pM$:

$$\langle \operatorname{grad}_M f, X\rangle = df(X) = dF(X) = \langle \operatorname{grad}_N F, X\rangle,$$

where grad_M and grad_N denote the gradient on M and the gradient on N, respectively. So that

$$\operatorname{grad}_N F = \operatorname{grad}_M f + (\operatorname{grad}_N F)^\perp, \tag{2.3}$$

where $(\operatorname{grad} F)^\perp$ is perpendicular to T_pM. Let $F : N \to \mathbb{R}$ be a C^2 function and denote $f : M \to \mathbb{R}$ the function induced by F by restriction, that is $f = F \circ x$. We have the following.

Lemma 2.5. *Let $x : M^n \to N^{n+1}$ be an isometric immersion. Let $F : N \to \mathbb{R}$ a smooth function and consider $f = F \circ x : M \to \mathbb{R}$. For an orthonormal frame $\{e_i\}$ on M, we have*

$$L_r f = \sum_{i=1}^n \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + (r+1)S_{r+1}\langle \operatorname{grad}_N F, \eta\rangle, \tag{2.4}$$

where η denotes the normal vector field of the immersion and grad_N *is the gradient of N.*

Proof. Let ∇ and $\overline{\nabla}$ be the connections of M and N, respectively. If α denotes the second fundamental form of the immersion, Gauss' equation and equations (2.2) and (2.3) imply that

$$\begin{aligned}
L_r f &= \sum_{i=1}^n \big\langle \nabla_{e_i}(\operatorname{grad}_M f), P_r(e_i)\big\rangle \\
&= \sum_{i=1}^n \big\langle \overline{\nabla}_{e_i}(\operatorname{grad}_M f) - \big[\overline{\nabla}_{e_i}(\operatorname{grad}_M f) - \nabla_{e_i}(\operatorname{grad}_M f)\big], P_r(e_i)\big\rangle \\
&= \sum_{i=1}^n \big\langle \overline{\nabla}_{e_i}(\operatorname{grad}_M f) - \alpha(e_i, \operatorname{grad}_M f), P_r(e_i)\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} (\operatorname{grad}_M f), P_r(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} \big(\operatorname{grad}_N F - (\operatorname{grad}_N F)^{\perp}\big), P_r(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} \operatorname{grad}_N F, P_r(e_i) \big\rangle - \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} (\operatorname{grad}_N F)^{\perp}, P_r(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) - \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} \big(\langle \operatorname{grad}_N F, \eta\rangle \eta\big), P_r(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) - \sum_{i=1}^{n} \big\langle \langle \operatorname{grad}_N F, \eta\rangle \overline{\nabla}_{e_i} \eta, P_r(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) - \langle \operatorname{grad}_N F, \eta\rangle \sum_{i=1}^{n} \big\langle -A(e_i), P_r(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + \langle \operatorname{grad}_N F, \eta\rangle \sum_{i=1}^{n} \big\langle e_i, AP_r(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + \langle \operatorname{grad}_N F, \eta\rangle \operatorname{trace}(AP_r) \\
&= \sum_{i=1}^{n} \operatorname{Hess}_N(F)\big(e_i, P_r(e_i)\big) + (r+1) S_{r+1} \langle \operatorname{grad}_N F, \eta\rangle.
\end{aligned}
$$

□

Let $c \in \mathbb{R}$. Define the function:

$$\theta_c(t) := \int_0^t s_c(u)\, du,$$

where

$$
s_c(t) = \begin{cases} \frac{\sin \sqrt{c} t}{\sqrt{c}}, & \text{if } c > 0; \\ t, & \text{if } c = 0; \\ \frac{\sinh \sqrt{|c|} t}{\sqrt{|c|}}, & \text{if } c < 0. \end{cases} \tag{2.5}
$$

Let ρ denotes the distance function to the point Q in N^{n+1}, and $F : N^{n+1} \to \mathbb{R}$ given by $F(p) = \theta_c(\rho(p))$. In this case, Lemma 2.5 with $f = F \circ x$ and $F = \theta_c \circ \rho$ implies the following corollary.

Corollary 2.6. *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature c. Let M be an immersed hypersurface in $\mathcal{Q}_c^{n+1}$. Then, for* *all* $p \in M$,

$$(2.6)\quad L_r\big(\theta_c\big(\rho(p)\big)\big) = (n-r)s_c'\big(\rho(p)\big)S_r(p) + (r+1)S_{r+1}(p)s_c\big(\rho(p)\big)\big\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(p), \eta\big\rangle.$$

In particular, when $c=0$,

$$\frac{1}{2}L_r\big(\rho^2(p)\big) = (n-r)S_r(p) + (r+1)S_{r+1}(p)\rho(p)\big\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(p), \eta\big\rangle.$$

Proof. First observe that

$$(2.7)\qquad \operatorname{Hess}_{\mathcal{Q}_c^{n+1}} F(X,Y) = s_c'(\rho)\langle X, Y\rangle,$$

where $X, Y \in T_{x(p)}\mathcal{Q}_c^{n+1}$. In fact,

$$\begin{aligned}
\operatorname{Hess}_{\mathcal{Q}_c^{n+1}} F(X,Y) &= \operatorname{Hess}_{\mathcal{Q}_c^{n+1}}\big(\theta_c(\rho)\big) \\
&= \big\langle \overline{\nabla}_X \operatorname{grad}_{\mathcal{Q}_c^{n+1}}\big(\theta_c(\rho)\big), Y\big\rangle \\
&= \big\langle \overline{\nabla}_X s_c(\rho) \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\big\rangle \\
&= s_c(\rho)\operatorname{Hess}_{\mathcal{Q}_c^{n+1}} \rho(X,Y) \\
&\qquad + s_c'(\rho)\big\langle \langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, X\rangle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\big\rangle.
\end{aligned}$$

On the other hand, see [1], p. 6,

$$\begin{aligned}
\operatorname{Hess}_{\mathcal{Q}_c^{n+1}} \rho(X,Y) &= \langle \overline{\nabla}_X \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\rangle \\
&= \frac{s_c'(\rho)}{s_c(\rho)}\big[\langle X, Y\rangle - \langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, X\rangle\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, Y\rangle\big].
\end{aligned}$$

This concludes the proof of (2.7). Now, by using equation (2.4), we have

$$\begin{aligned}
L_r f &= \sum_{i=1}^{n} s_c'(\rho)\big\langle e_i, P_r(e_i)\big\rangle + (r+1)S_{r+1}\big\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}}(\theta_c \circ \rho), \eta\big\rangle \\
&= s_c'(\rho)\operatorname{trace} P_r + (r+1)S_{r+1}s_c(\rho)\langle \operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho, \eta\rangle.
\end{aligned}$$

Finally, by using equation (2.1), we conclude the proof of equation (2.6). The case $c=0$ follows immediately. □

It follows from Codazzi equation (see Rosenberg [16], p. 225) that L_r is a divergent form operator, that is,

$$L_r(f) = \operatorname{div}_M(P_r\nabla f)$$

for all smooth functions $f : M \to \mathbb{R}$. Denote by $B_r(Q)$ the geodesic ball of $\mathcal{Q}_c^{n+1}$ with radius r centered at $Q \in \mathcal{Q}_c^{n+1}$, and by $\overline{B}_r(Q)$ its closure. We will use the following proposition.

Proposition 2.7. *Let* $\mathcal{Q}_c^{n+1}$ *be a Riemannian manifold with constant sectional curvature* c *and let* $x : M^n \to \mathcal{Q}_c^{n+1}$ *be an isometric immersion. For*

$Q \in \mathcal{Q}_c^{n+1}$, we denote by $\rho(x)$ the distance to the point $Q \in \mathcal{Q}_c^{n+1}$ and $\rho(x(p))$, $p \in M$, its restriction to M. If for some $r \le n-1$, $S_r \ge 0$, then

$$\begin{aligned}(2.8)\qquad &\int_{\partial D} s_c\big(\rho(q)\big)\big\langle P_r\big(\operatorname{grad}_M \rho(q)\big), \nu\big\rangle\, dA \\ &\quad \ge (n-r)\int_D \left(s_c'\big(\rho(q)\big) S_r(p) - \frac{r+1}{n-r}\big|S_{r+1}(p)\big| s_c\big(\rho(q)\big)\right) dM,\end{aligned}$$

where $q = x(p)$, $D \subset M$ is a bounded domain with nonempty boundary ∂D and ν is the conormal vector along ∂D. In the case $c > 0$, we assume that $D \subset \overline{B}_{\frac{\pi}{2\sqrt{c}}}(Q)$.

Proof. Since $|\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(x(p))| \le 1$ and $s_c'(\rho(x(p))) \ge 0$, from (2.6) we have

$$L_r\big(\theta_c\big(\rho(x)\big)\big) \ge (n-r)\left[s_c'(\rho) S_r - \frac{r+1}{n-r}|S_{r+1}| s_c(\rho)\right].$$

Integrating this inequality, we get

$$\begin{aligned}(2.9)\qquad &\int_D L_r\big(\theta_c\big(\rho(x)\big)\big)\, dM \\ &\quad \ge (n-r)\int_D \left[s_c'\big(\rho(x)\big) S_r - \frac{r+1}{n-r}|S_{r+1}| s_c\big(\rho(x)\big)\right] dM.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\int_D L_r\big(\theta_c\big(\rho(x)\big)\big)\, dM &= \int_D \operatorname{div} P_r\Big(\operatorname{grad}_M\Big(\theta_c\Big(\rho\big(x(p)\big)\Big)\Big)\Big)\, dM \\ &= \int_D \operatorname{div}\big(s_c\rho\big(x(p)\big) P_r(\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho)^{\top}\big)\, dM \\ &= \int_{\partial D} s_c\big(\rho(x)\big)\big\langle P_r\big((\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho)^{\top}\big), \nu\big\rangle\, dA,\end{aligned}$$

where ν denotes the outward unit normal vector field on ∂D. Therefore, if $q = x(p)$,

$$\begin{aligned}&\int_{\partial D} s_c\big(\rho(q)\big)\big\langle P_r\big(\big(\operatorname{grad}_{\mathcal{Q}_c^{n+1}} \rho(q)\big)^{\top}\big), \nu\big\rangle\, dA \\ &\quad \ge (n-r)\int_D \left[s_c'\big(\rho(x)\big) S_r - \frac{r+1}{n-r}|S_{r+1}| s_c\big(\rho(x)\big)\right] dM,\end{aligned}$$

and the proposition is proved. □

We would like to point out that the above proposition is valid for a more general class of domains. For instance, it is valid in the setting of Gauss–Green Theorem (see [10], p. 478). In particular, if we take D to be the intersection of the extrinsic ball with M i.e. $D = \overline{B}_\mu \cap M$ in Proposition 2.7, we have the following

Theorem 2.8. *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature c and let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that there exists a nonnegative constant α such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r \tag{2.10}$$

for some $r \leq n-1$. If P_r is positive semidefinite, then for any $q \in M$ such that $S_r(q) \neq 0$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, q and M such that for every $\mu > \mu_0$,

$$\mathcal{A}_r\big(\overline{B}_\mu(q) \cap M\big) = \int_{\overline{B}_\mu(q)\cap M} S_r \, dM \geq \int_{\mu_0}^{\mu} C\mathrm{e}^{-\alpha\tau}\, d\tau,$$

where $\overline{B}_\mu(q)$ is the ball of radius μ and center q in $\mathcal{Q}_c^{n+1}$. For the case $c > 0$, we assume $\mu \leq \frac{\pi}{2\sqrt{c}}$.

Proof. We use the notation introduced in Proposition 2.7. Take $D_\tau = \overline{B}_\tau(q) \cap M$, $\mu \leq 2\pi/\sqrt{c}$, if $c > 0$. Since the immersion is proper, we have that $\partial D_\tau \neq \emptyset$, for all $0 < \tau < \mu$. Thus, by using (2.10) in equation (2.8), we obtain that

$$\begin{aligned}
&\int_{\partial D_\mu} s_c\big(\rho(x)\big)\big\langle P_r(\operatorname{grad}_M \rho), \nu\big\rangle \, dA \\
&\quad \geq (n-r)\int_{D_\mu} \big(s_c'\big(\rho(x)\big) - \alpha s_c\big(\rho(x)\big)\big) S_r \, dM \\
&\quad = (n-r)\int_0^{\mu}\int_{\partial D_\tau} \frac{s_c'(\rho(x)) - \alpha s_c(\rho(x))}{s_c(\rho(x))} \\
&\qquad \times s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho|^{-1} S_r \, dA\, d\tau,
\end{aligned} \tag{2.11}$$

where we have used the co-area formula (see [3], p. 80). Observe that the conormal vector ν to ∂D_τ is parallel to $\operatorname{grad}_M \rho$. This fact together with the fact that P_r is positive semidefinite, imply the following:

$$\big\langle P_r(\operatorname{grad}_M \rho), \nu\big\rangle \leq \operatorname{trace}(P_r)|\operatorname{grad}_M \rho| = (n-r)S_r|\operatorname{grad}_M \rho|.$$

Using the above equation and the fact that along ∂D_τ, $\rho(x) = \tau$, we get

$$\begin{aligned}
&\int_{\partial D_\mu} s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho| S_r \, dA \\
&\quad \geq \int_0^{\mu} \frac{s_c'(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \int_{\partial D_\tau} s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho|^{-1} S_r \, dA\, d\tau.
\end{aligned} \tag{2.12}$$

Now we define

$$\varphi(\tau) = \int_{\partial D_\tau} s_c\big(\rho(x)\big)|\operatorname{grad}_M \rho|^{-1} S_r \, dA.$$

Since $|\operatorname{grad}_M \rho| \le 1$, equation (2.12) implies

$$\varphi(\mu) \ge \int_0^\mu \frac{s_c'(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau)\, d\tau.$$

By writing

$$\phi(\mu) = \int_0^\mu \frac{s_c'(\tau) - \alpha s_c(\tau)}{s_c(\tau)} \varphi(\tau)\, d\tau,$$

one finds

$$\phi'(\mu) \ge \frac{s_c'(\mu) - \alpha s_c(\mu)}{s_c(\mu)} \phi(\mu).$$

Thus, by integrating from $\mu_0 > 0$ to μ, the above differential inequality arises

$$\ln \frac{\phi(\mu)}{\phi(\mu_0)} \ge \ln\left(\frac{s_c(\mu)}{s_c(\mu_0)}\right) - \alpha(\mu - \varepsilon) = \ln\left(\left(\frac{s_c(\mu)}{s_c(\mu_0)}\right) \mathrm{e}^{-\alpha(\mu-\mu_0)}\right).$$

Hence,

$$\phi(\mu) \ge \frac{\phi(\mu_0)}{s_c(\mu_0)} s_c(\mu) \mathrm{e}^{-\alpha\mu}.$$

Define

$$f(\mu) = \int_{D_\mu} S_r\, dM.$$

Again, by the co-area formula, it follows that

$$f(\mu) = \int_0^\mu \left(\int_{\partial D_\tau} |\operatorname{grad}_M \rho|^{-1} S_r\, dA\right) d\tau.$$

Since

$$f'(\mu) = \int_{\partial D_\mu} |\operatorname{grad}_M \rho|^{-1} S_r\, dA = \frac{1}{s_c(\mu)} \varphi(\mu) \ge \frac{\phi(\mu_0)}{s_c(\mu_0)} \mathrm{e}^{-\alpha\mu},$$

then for $\mu > \mu_0$,

$$f(\mu) \ge \int_{\mu_0}^\mu \frac{\phi(\mu_0)}{s_c(\mu_0)} \mathrm{e}^{-\alpha\tau}\, d\tau.$$

□

Corollary 2.9. *Let $\mathcal{Q}_c^{n+1}$ be a Riemannian manifold with constant sectional curvature $c \le 0$ and let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that $S_r \ge 0$, $S_r \not\equiv 0$ and $S_{r+1} \equiv 0$ for some $r \le n-1$. Then $\int_M S_r\, dM = \infty$.*

Proof. Since the immersion is proper, we have $\partial(M \cap \overline{B}_\mu(q))$ is nonempty for all $\mu > \mu_0$. By using Proposition 2.4, since $S_{r+1} = 0$, we have that P_r is semidefinite. Now, the condition $S_r \ge 0$ implies that P_r is positive semidefinite. Therefore, using Theorem 2.8, with $\alpha = 0$, for all $\mu > \mu_0$,

$$\int_{\overline{B}_\mu \cap M} S_r\, dM \ge \int_{\mu_0}^\mu C\mathrm{e}^{-\alpha\tau}\, d\tau = C(\mu - \mu_0).$$

Then

$$\int_M S_r\, dM = \infty. \qquad \square$$

REMARK 2.10. When r is odd, the condition $S_r \geq 0$ can be obtained by choosing the right orientation.

The condition of semi-positiveness of P_2 is satisfied when M is a hypersurface immersed in $\mathbb{R}^{n+1}$ with $S_3 = 0$ (which is called 2-minimal) and $S_2 > 0$. In fact, in this case P_2 is positive definite, since L_2 is elliptic (see Proposition 2.2). So we have

COROLLARY 2.11. *Let M^n be a complete 2-minimal noncompact properly immersed hypersurface of $\mathbb{R}^{n+1}$ with nonnegative scalar curvature. Then either the scalar curvature is zero or the total scalar curvature is infinite.*

REMARK 2.12. When $n = 3$ the corollary can be proved by using Theorem III in [13] without the assumption that the immersion is proper. In this case, M^n has to be a cylinder and the conclusion of the above corollary follows immediately.

REMARK 2.13. The condition of semi-positiveness of P_r is also satisfied when M is a hypersurface in $\mathbb{R}^{n+1}$ with nonnegative sectional or positive Ricci curvature, $\mathrm{Ric}_M > 0$. Indeed when $\mathrm{Ric}_M > 0$, for each point in M, the principal curvatures can be arranged as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i < 0 < \lambda_{i+1} \leq \cdots \leq \lambda_n$. The positivity of the Ricci curvature implies

$$\mathrm{Ric}_M(e_j) = \lambda_j \left(\sum_{k \neq j} \lambda_k \right) > 0 \quad \forall j = 1, \dots, n.$$

If $i \in \{1, \dots, n-1\}$, it follows from the above equation that

$$\sum_{k \neq j} \lambda_k < 0, \quad \text{when } j \leq i, \tag{2.13}$$

and

$$\sum_{k \neq j} \lambda_k > 0, \quad \text{when } j > i. \tag{2.14}$$

From (2.13), we have for $j_1 \leq i$,

$$\sum_{k \neq j_1} \lambda_k = \left(\sum_{k=1}^{i} \lambda_k - \lambda_{j_1} \right) + \sum_{k=i+1}^{n} \lambda_k < 0.$$

Thus

$$-\sum_{k=1}^{i} \lambda_k > \sum_{k=1}^{i} \lambda_k + \lambda_{j_1} > \sum_{k=i+1}^{n} \lambda_k.$$

On the other hand, using (2.14), for $j_2 > i$, we find

$$\sum_{k\neq j_2} \lambda_k = \left(\sum_{k=1}^{i} \lambda_k - \lambda_{j_2}\right) + \sum_{k=i+1}^{n} \lambda_k > 0,$$

hence

$$-\sum_{k=1}^{i} \lambda_k < \sum_{k=1}^{i} \lambda_k + \lambda_{j_1} < \sum_{k=i+1}^{n} \lambda_k,$$

which is a contradiction. Thus, all λ_i has the same sign (we are indebted to F. Fontenele for this observation). So we can choose an orientation such that P_r is positive definite and $S_r > 0$.

Thus we have the following consequence.

COROLLARY 2.14. *Let M^n be a complete noncompact properly immersed hypersurface of $\mathbb{R}^{n+1}$ with positive Ricci curvature. Assume that there exists a positive constant α such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r$$

for some $r \leq n-1$. Then, for any $q \in M$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, Q and M such that

$$\int_{\overline{B}_\mu(q)\cap M} S_r \, dM \geq \int_{\mu_0}^{\mu} C\mathrm{e}^{-\alpha\tau} \, d\tau,$$

where $\overline{B}_\mu(q)$ is the geodesic ball in $\mathbb{R}^{n+1}$ centered at q.

The following is a direct consequence of Theorem 2.8 and Proposition 2.3.

COROLLARY 2.15. *Let M^n be a complete noncompact properly immersed hypersurface of $\mathcal{Q}_c^{n+1}$. Assume that S_r is positive and there exists a positive constant α such that*

$$(r+1)|S_{r+1}| \leq (n-r)\alpha S_r$$

for some $r \leq n-1$. If there exists a point such that all principal curvatures are nonnegative, then, for any $q \in M$ and any $\mu_0 > 0$, there exists a positive constant C depending on μ_0, q and M such that

$$\int_{\overline{B}_\mu(q)\cap M} S_r \, dM \geq \int_{\mu_0}^{\mu} C\mathrm{e}^{-\alpha\tau} \, d\tau,$$

where $\overline{B}_\mu(q)$ is the geodesic ball in $\mathcal{Q}_c^{n+1}$ centered at q. For the case $c > 0$, we assume $\mu \leq \frac{\pi}{2\sqrt{c}}$.

3. Volume estimates in general manifolds

In this section we consider N^{n+p} a Riemannian manifold with sectional curvature bounded from above by a constant c. Let M^n be a submanifold isometrically immersed in $N = N^{n+p}$.

Let $F : N \to \mathbb{R}$ be a C^2 function and denote $f : M \to \mathbb{R}$ the function induced by F by restriction. Essentially, following the steps involved in the proof of Lemma 2.5, we obtain

$$\Delta f = \sum_{i=1}^{n} \operatorname{Hess}_N F(e_i, e_i) + n\langle \operatorname{grad}_N F, \mathbf{H}\rangle,$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame along M and $\mathbf{H}$ is the mean curvature vector. Similar to Proposition 2.7, we have

Proposition 3.1. *Let N be a Riemannian manifold with sectional curvature bounded from above by a constant c and M^n an immersed connected submanifold of N. We denote by $\overline{\rho}(x)$ the distance in N between x and $Q \in N^{n+p}$ and $\rho(x)$ the induced function of $\overline{\rho}$ by restriction. Then*

$$\int_{\partial D} s_c\big(\rho(x)\big)\langle \operatorname{grad}_M \rho, \nu\rangle \, dA \geq n \int_D \big(s_c'\big(\rho(x)\big) - |\mathbf{H}| s_c\big(\rho(x)\big)\big) \, dM, \tag{3.1}$$

where $q = x(p)$, $D \subset M$ is a bounded domain with nonempty boundary ∂D and $D \cap C_N(Q) = \emptyset$, where $C_N(Q)$ is the cut locus of the point Q in N, and ν is the conormal vector along ∂D.

Proof. Let $V = s_c(\overline{\rho}) \operatorname{grad}_N \overline{\rho}$ and $V^\top$ the orthogonal projection of V into the tangent space of M. Then we have $V^\top = s_c(\rho) \operatorname{grad}_M \rho$, where $\rho(x)$ is the induced function of $\overline{\rho}$ to M by restriction. Thus, Lemma 2.5 of [14], p. 713, implies, when $\overline{\rho} < \operatorname{inj}_N(Q)$,

$$\operatorname{Hess}_N F(X, X) \geq s_c'(\overline{\rho})\langle X, X\rangle. \tag{3.2}$$

Then

$$\langle \overline{\nabla}_{e_i} V, e_i\rangle \geq s_c'(\overline{\rho})$$

for all $\overline{\rho}$ when $c \leq 0$, and $\rho \leq \frac{\pi}{\sqrt{c}}$, when $c > 0$. We find that

$$\Delta\big(\theta_c\big(\rho(x)\big)\big) \geq n\big[s_c'(\rho) - s_c(\rho)|\mathbf{H}|\big].$$

Integrating this inequality and applying Stokes' formula, we get

$$\int_{\partial D} s_c\big\langle (\operatorname{grad}_N \overline{\rho})^\top, \nu\big\rangle \, dA \geq n \int_D \big[s_c'\big(\rho(x)\big) - s_c\big(\rho(x)\big)|\mathbf{H}|\big] \, dM,$$

and the proposition follows. ☐

Similar to Proposition 2.7, the above result is valid in a more general setting, such as extrinsic geodesic balls. Using this fact, we arrive at

Theorem 3.2. *Let M be a Riemannian manifold isometrically immersed in a geodesic ball $\overline{B}(O,\rho_0)\subset N^{n+p}$ with codimension p. Assume that the sectional curvature of N in $\overline{B}(O,\rho_0)$ is bounded from above by c and moreover that there exists a positive constant α such that*

$$|\mathbf{H}|\le\alpha.$$

Then

$$\operatorname{vol}\big(B_\mu(q)\big)\ge n\omega_n\int_0^\mu s_c(t)^{n-1}\mathrm{e}^{-n\alpha s}\,dt,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $B_\mu(q)$ is the intrinsic geodesic ball in M with center $q\in M$ and radius $\mu<\operatorname{inj}_N(q)$.

Proof. By taking $D=B_\tau(q)$ in Proposition 3.1, we obtain

$$\langle\operatorname{grad}_M\rho,\nu\rangle\le|\operatorname{grad}_M\rho|.$$

Thus,

$$\begin{aligned}(3.3)\qquad &\int_{\partial B_\tau(q)}\frac{s_c(\rho(x))}{n}|\operatorname{grad}_M\rho|\,dA\\ &\quad\ge\int_{B_\tau(q)}\big(s_c'\big(\rho(x)\big)-\alpha s_c\big(\rho(x)\big)\big)\,dM\\ &\quad=\int_0^\mu\int_{\partial B_\tau(q)}\frac{s_c'(\rho(x))-\alpha s_c(\rho(x))}{s_c(\rho(x))}s_c\big(\rho(x)\big)|\operatorname{grad}_M\rho|^{-1}\,dA\,d\tau,\end{aligned}$$

where we have used the co-area formula (see [3], p. 80). Since the intrinsic distance is not less than the extrinsic one and

$$\left(\frac{s_c'}{s_c}\right)'\le 0,$$

then

$$\begin{aligned}(3.4)\qquad &\frac{1}{n}\int_{\partial B_\mu(q)}s_c\big(\rho(x)\big)|\operatorname{grad}_M\rho|\,dA\\ &\quad\ge\int_0^\mu\frac{s_c'(\tau)-\alpha s_c(\tau)}{s_c(\tau)}\int_{\partial B_\tau(q)}s_c\big(\rho(x)\big)|\operatorname{grad}_M\rho|^{-1}\,dA\,d\tau.\end{aligned}$$

Now we define

$$\varphi(\tau)=\int_{\partial B_\tau(q)}s_c\big(\rho(x)\big)|\operatorname{grad}_M\rho|^{-1}\,dA.$$

Equation (3.4) implies that

$$\frac{1}{n}\varphi(\mu)\ge\int_0^\mu\frac{s_c'(\tau)-\alpha s_c(\tau)}{s_c(\tau)}\varphi(\tau)\,d\tau.$$

By writing

$$\phi(\mu)=\int_0^\mu\frac{s_c'(\tau)-\alpha s_c(\tau)}{s_c(\tau)}\varphi(\tau)\,d\tau,$$

we have

$$\phi'(\mu) \geq \frac{n(s_c'(\mu) - \alpha s_c(\mu))}{s_c(\mu)}\phi(\mu).$$

Thus, by integrating from $\varepsilon > 0$ to μ, with $\mu \leq \min\{\mathrm{inj}_N(q), \frac{\pi}{2\sqrt{c}}\}$ when $c > 0$, the above differential inequality arises

$$\frac{1}{n}\ln\frac{\phi(\mu)}{\phi(\varepsilon)} \geq \ln\left(\frac{s_c(\mu)}{\varepsilon}\right) - \alpha(\mu-\varepsilon) = \ln\left[\left(\frac{s_c(\mu)}{\varepsilon}\right)\mathrm{e}^{-\alpha(\mu-\varepsilon)}\right].$$

Hence,

$$\frac{\phi(\mu)}{\phi(\varepsilon)} \geq \left[\left(\frac{s_c(\mu)}{\varepsilon}\right)\mathrm{e}^{-\alpha(\mu-\varepsilon)}\right]^n. \tag{3.5}$$

Observe that by the mean value theorem,

$$\lim_{\varepsilon\to 0}\frac{\phi(\varepsilon)}{\varepsilon^n} = \omega_n.$$

Then

$$\phi(\mu) \geq \omega_n s_c(\mu)^n \mathrm{e}^{-n\alpha\mu}.$$

Now, define

$$f(\mu) = \int_{B_\mu(q)} dM = \mathrm{vol}\big(B_\mu(q)\big).$$

Again, by the co-area formula, we can write $f(\mu)$ as

$$f(\mu) = \int_0^\mu \left(\int_{\partial B_\tau(q)} |\operatorname{grad}_M \rho|^{-1}\, dA\right) d\tau.$$

Hence

$$f'(\mu) = \int_{\partial B_\mu(q)} |\operatorname{grad}_M \rho|^{-1}\, dA.$$

This equality together with $|\operatorname{grad}_M \rho| \leq 1$, and equation (3.3) imply that

$$\frac{s_c(\mu)}{n} f'(\mu) \geq \int_{\partial B_\mu(q)} \frac{s_c(\rho(x))}{n} |\operatorname{grad}_M \rho|\, dA \geq \int_0^\mu \big(s_c'(\tau) - \alpha s_c(\tau)\big) f'(\tau)\, d\tau.$$

Since

$$f'(\mu) \geq \frac{\varphi(\mu)}{s_c(\mu)},$$

then

$$f(\mu) \geq \int_0^\mu \omega_n n s_c(\tau)^{n-1}\mathrm{e}^{-n\alpha\tau}\, d\tau,$$

which concludes the proof. □

The following corollary follows immediately.

Corollary 3.3. (i) *Let M^n be an immersed minimal hypersurface of the Euclidean space $\mathbb{R}^{n+p}$. Then*

$$\operatorname{vol}\big(B_\mu(q)\big) \geq \omega_n \mu^n,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $B_\mu(q)$ is the intrinsic geodesic ball in M centered at $q \in M$.

(ii) *Let M^n be an immersed hypersurface of the hyperbolic space $\mathbb{H}^{n+p}(-1)$. Assume that there exists a positive constant α such that*

$$|H| \leq \alpha < \frac{n-1}{n}.$$

Then, there exists a constant $C > 0$ so that, for $\mu \geq 1$,

$$\operatorname{vol}\big(B_\mu(q)\big) \geq C e^{(n-1-n\alpha)\mu},$$

where $B_\mu(q)$ is the intrinsic geodesic ball in M with center $q \in M$.

4. Mean curvature integral

In this section, inspired by a recent work of Topping [18], we prove a type of mean curvature integral estimate for complete submanifold in a Euclidean space $\mathbb{R}^n$ and we apply it to surfaces in $\mathbb{R}^n$.

Theorem 4.1. *Let M^m be a m-dimensional complete noncompact Riemannian manifold isometrically immersed in $\mathbb{R}^n$. Then there exists a positive constant δ depending on m such that if*

$$\limsup_{r \to +\infty} \frac{V(x,r)}{r^m} < \delta, \tag{4.1}$$

where $V(x,r)$ denotes the volume of the geodesic ball $B_r(x)$, then

$$\limsup_{R \to +\infty} \frac{\int_{B_R(x)} |H|^{m-1}\, dM}{R} > 0. \tag{4.2}$$

In particular, $\int_M |H|^{m-1}\, dM = +\infty$.

We need the following lemma of Topping [18].

Lemma 4.2 ([18], Lemma 1.2). *Let M^m be a m-dimensional complete Riemannian manifold isometrically immersed in $\mathbb{R}^n$. Then a positive constant δ depending on m exists, such that for any $x \in M$ and $R > 0$, at least one of the following statements is true*:

(i) $\sup_{r \in (0,R]} r^{-\frac{1}{m-1}} [V(x,r)]^{-\frac{m-2}{m-1}} \int_{B(x,r)} |H|^{m-1}\, dM > \delta$,

(ii) $\inf_{r \in (0,R]} \frac{V(x,r)}{r^m} > \delta$.

Proof of Theorem 4.1. We can choose L large enough so that $V(z,L) \leq \delta L^m$ for all $z \in M$. Then, from Lemma 4.2, we have

$$\sup_{r\in(0,L]} r^{-\frac{1}{m-1}} \big[V(z,r)\big]^{-\frac{m-2}{m-1}} \int_{B_r(z)} |H|^{m-1}\, dM > \delta.$$

Since

$$\int_{B_r(z)} |H|\, dM \leq \bigg(\int_{B_r(z)} |H|^{m-1}\, dM\bigg)^{\frac{1}{m-1}} \cdot \big(V(z,r)\big)^{\frac{m-2}{m-1}}$$

for any $z \in M$, there exists a $r(z) \in (0,R]$ such that

$$\int_{B_r(z)} |H|^{m-1}\, dM > \delta^{m-1} r(z).$$

Fix a point $o \in M$, and let $\gamma : [0,+\infty) \to M$ be a ray parametrized by an arclength with $\gamma(0) = o$. For any fixed $R > 0$,

$$\gamma\big([0,R]\big) \subset \bigcup_{t\in[0,R]} B_{r(\gamma(t))}\big(\gamma(t)\big).$$

From a covering argument used in Theorem 1.1 of [18], we can find an at most countable sequence $t_1, t_2, \ldots, t_q, \ldots \in [0,R]$ such that $\sum_i r(\gamma(t_i)) \geq \frac{1}{4}R$. Thus, when $i \neq j$,

$$B_{r(\gamma(t_i))}\big(\gamma(t_i)\big) \cap B_{r(\gamma(t_j))}\big(\gamma(t_j)\big) = \emptyset.$$

Then

$$\begin{aligned}\int_{B_{2R}(o)} |H|^{m-1}\, dM &\geq \sum_i \int_{B_{r(\gamma(t_i))}(\gamma(t_i))} |H|^{m-1}\, dM \\ &\geq \delta^{m-1} \sum_i r\big(\gamma(t_i)\big) \\ &\geq \delta^{m-1} \frac{1}{4} R.\end{aligned}$$

And the result is proved. □

For complete surfaces in $\mathbb{R}^n$ that satisfy the Gauss–Bonnet relation, we obtain the following result.

Corollary 4.3. *Let δ be as in Theorem 4.1. If M is a complete noncompact surface in $\mathbb{R}^n$ satisfying*

$$2\pi\chi(M) - \int_M K\, dM < 2\delta, \tag{4.3}$$

where $\chi(M)$ is the Euler characteristic of M, then

$$\int_M |H|\, dM = +\infty.$$

Proof. From Theorem A of Shiohama [17], for any $q \in M$, we find that

$$\lim_{r\to\infty} \frac{2V(B_r(q))}{r^2} = 2\pi\chi(M) - \int_M K\,dM.$$

It should be noted here that there is a misprint in the denominator of this expression in Shiohama's paper. So,

$$\lim_{r\to\infty} \frac{V(B_r(q))}{\pi r^2} < \delta.$$

Thus, Theorem 4.1 implies the result. □

REMARK 4.4. The flat plane embedded in $\mathbb{R}^n$ shows that the condition (4.3) is necessary.

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Hilário Alencar, Instituto de Matemática, Universidade Federal de Alagoas, 57072-900 Maceió-AL, Brazil

E-mail address: `hilario@mat.ufal.br`

Walcy Santos, Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.Postal 68530, 21941-909, Rio de Janeiro-RJ, Brazil

E-mail address: `walcy@im.ufrj.br`

Detang Zhou, Instituto de Matemática, Universidade Federal Fluminense, Niterói, RJ 24020-140, Brazil

E-mail address: `zhou@impa.br`