Hypersurfaces with null higher order mean curvature

Hilário Alencar* and Márcio Batista

Abstract. A hypersurface M^n immersed in a space form is r-minimal if its $(r+1)^{th}$-curvature (the $(r+1)^{th}$ elementary symmetric function of its principal curvatures) vanishes identically. Let W be the set of points which are omitted by the totally geodesic hypersurfaces tangent to M. We will prove that if an orientable hypersurface M^n is r-minimal and its r^{th}-curvature is nonzero everywhere, and the set W is nonempty and open, then M^n has relative nullity $n-r$. Also we will prove that if an orientable hypersurface M^n is r-minimal and its r^{th}-curvature is nonzero everywhere, and the ambient space is euclidean or hyperbolic and W is nonempty, then M^n is r-stable.

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1 Introduction

Let $\mathbb{Q}_c^{n+1}$ be a $(n+1)$-dimensional, simply-connected, complete Riemannian manifold with constant sectional curvature c. Let M^n be a n-dimensional connected manifold, and $x\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion. For every point $p \in M^n$, let $(\mathbb{Q}_c^n)_p$ be the totally geodesic hypersurface of $\mathbb{Q}_c^{n+1}$ tangent to $x(M^n)$ at $x(p)$.

We will denote by

$$W = \mathbb{Q}_c^{n+1} - \bigcup_{p\in M} (\mathbb{Q}_c^n)_p,$$

the set of points which are omitted by the totally geodesic hypersurfaces tangent to $x(M^n)$. We will work with an immersion whose set W is nonempty. In this direction, T. Hasanis and D. Koutroufiots, see [11], proved that

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Theorem A. *Let $x\colon M^2 \to \mathbb{Q}_c^3$ be a complete minimal immersion with $c \geq 0$. If W is nonempty, then x is totally geodesic.*

The proof is heavily based on the techniques of two dimensional manifolds. Now, let $x\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion. In [4], H. Alencar and K. Frensel extended the result above assuming an extra condition. In fact, they proved that

Theorem B. *Let M^n be a complete Riemannian manifold and $x\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric minimal immersion. If W is open and nonempty, then x is totally geodesic.*

On the other hand, H. Alencar, in [1], provides examples of non-totally geodesic minimal hypersurfaces in $\mathbb{R}^{2n}$, $n \geq 4$, with nonempty W, see the Example 3.2. These examples show that, in higher dimensions, it is necessary to add an extra hypothesis. In Theorem B, the extra condition is W open.

The aim of this work is to extend Theorem B for other immersions. In order to do that, we will introduce some definitions.

Let M^n be an oriented Riemannian manifold and $x\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion. Considering the symmetric functions S_r of the principal curvatures $k_1, \dots, k_n$:

$$S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r} \ (1 \leq r \leq n),$$

the r-mean curvature H_r of x is defined by $\binom{n}{r} H_r = S_r$.

It is convenient to introduce the Newton transformations defined inductively by

$$T_0 = I, \ T_r = S_r I - A T_{r-1}.$$

Here I is the identity matrix and A is the shape operator associated to x.

We say that $x\colon M^n \to \mathbb{Q}_c^{n+1}$ is r-minimal if $H_{r+1} = 0$, i.e., x is a critical point of the variational problem of minimizing the integral

$$\mathbb{A}_r(x) = \int_D \mathbb{F}_r\big(S_1, \dots S_r\big) dV,$$

where

$$\mathbb{F}_0 = 1, \ \mathbb{F}_1 = S_1, \dots, \ \mathbb{F}_r = S_r + \frac{c(n-r+1)}{r-1} \mathbb{F}_{r-2}, \quad 2 \leq r \leq n-1.$$

Associated to the second variation formula of $\mathbb{A}_r$ are the second order differential operators

$$L_r f = tr\big(T_r \nabla^2 f\big),$$

and

$$J_r f = L_r f - (r+2)S_{r+2}S_{r+2} f + c(n-r)S_r f,$$

where f is a differentiable function and $\nabla^2 f$ is the hessian of f.

When the ambient space is a space form, H. Rosenberg in [12] proved that $L_r f = \operatorname{div}(T_r \nabla f)$, where ∇f is the gradient in the induced metric, and thus L_r is self-adjoint operator.

Thus, a bilinear symmetric form I_r can be defined by

$$I_r(f, g) = -\int_M f J_r g \, dV,$$

where f and g are differentiable functions on M.

In [3], H. Alencar, M. do Carmo and M.F Elbert defined stability for r-minimal immersions:

Definition 1.1. Let D be a domain with compact closure and piecewise smooth boundary. D is r-stable if $I_r(f, f) > 0$ for all $f \in C_c^\infty(D)$ or if $I_r(f, f) < 0$ for all $f \in C_c^\infty(D)$. D is r- unstable if there exists a function $f \in C_c^\infty(D)$ such that $I_r(f, f) < 0$ and there exists a function $g \in C_c^\infty(D)$ such that $I_r(g, g) > 0$.

Now we recall the definition of relative nullity. Consider $\nu(p) = \dim \operatorname{Ker}(A)$, where A is the shape operator associated to the second fundamental form in p. The relative nullity is $\nu = \min_{p\in M^n} \nu(p)$.

Our first result reads as follows:

Theorem 1.1. *Let M^n be a complete and orientable Riemannian manifold and let $x\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric r-minimal immersion and $H_r \neq 0$ everywhere, $r \geq 1$. If W is open and nonempty, then $\nu = n - r$. In particular, $x(M^n)$ is foliated by complete totally geodesic submanifolds of dimension $n - r$.*

We observe that there exist examples of 1-minimal hypersurfaces with $H_1 \neq 0$ everywhere in $\mathbb{R}^{2n}$, $n \geq 5$, with nonempty W but $\nu \neq n - 1$, see the Example 3.1. These examples show that is necessary to add an extra hypothesis.

Corollary 1.1. *Let M^n be a complete Riemannian manifold and let $x\colon M^n \to \mathbb{S}^{n+1}$ be an isometric immersion with $H_2 = 0$ and mean curvature $H_1 \neq 0$ everywhere. There no exists an immersion x such that the set W is open and nonempty.*

Corollary 1.2. *Let M^n be a complete Riemannian manifold and let $x\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion with $H_3 = 0$ and scalar curvature $H_2 \neq 0$ everywhere. If W is open and nonempty, then $x(M^n) = \mathbb{S}^2 \times \mathbb{R}^{n-2}$.*

The second result is the following:

Theorem 1.2. *Let M^n be a complete and orientable Riemannian manifold, and let $x\colon M^n \to \mathbb{Q}_c^{n+1}$, $c \leq 0$, be an isometric r-minimal immersion and $H_r \neq 0$ everywhere, $r \geq 1$. If W is nonempty, then x is r-stable.*

2 Support function in space of constant curvature

Let us introduce the notions of position vector and support function in $\mathbb{Q}_c^{n+1}$. Consider an isometric immersion $x\colon M^n \to \mathbb{Q}_c^{n+1}$. Let s_c be a solution of the ordinary differential equation $y'' + cy = 0$, with initial conditions $y(0) = 0$ and $y'(0) = 1$. Then

$$s_c(r) = \begin{cases} r & \text{, if } \quad c = 0, \\ \dfrac{\sin(\sqrt{c}r)}{\sqrt{c}} & \text{, if } \quad c > 0, \\ \dfrac{\sinh(\sqrt{-c}r)}{\sqrt{-c}} & \text{, if } \quad c < 0. \end{cases}$$

For every point $p_0 \in \mathbb{Q}_c^{n+1}$, we will consider the function $r(.) = d(., p_0)$, where d is the distance function of $\mathbb{Q}_c^{n+1}$, and we will denote by grad r the gradient of the function r in $\mathbb{Q}_c^{n+1}$.

Using the analogy with the Euclidean Space, the vector field $X(p) = s_c(r)\,\mathrm{grad}\, r$ will be called *position vector* with origin p_0. When $c > 0$, the distance function is differentiable in $\mathbb{Q}_c^{n+1} - \{p_0, -p_0\}$. Therefore, in this case, the position vector with origin p_0 is differentiable in $\mathbb{Q}_c^{n+1} - \{p_0, -p_0\}$.

Let M^n be an oriented Riemannian manifold, $x\colon M^n \to \mathbb{Q}_c^{n+1}$ an isometric immersion, and N a unit normal vector field of x. The function $g\colon M \to \mathbb{R}$ defined by $g = \langle X, N\rangle$, where X is the position vector with origin p_0, will be called the *support function* of the immersion x. In the case $c > 0$, this function is differentiable if $x(M) \subseteq \mathbb{Q}_c^{n+1} - \{p_0, -p_0\}$.

For the case $c = 0$, $|g(p)|$ is the distance from p_0 to the tangent hyperplane to $x(M^n)$ at $x(p)$. In [4], the authors give a geometric interpretation of the support function in the case $c \neq 0$. We will describe the interpretation below.

In the case c > 0: We will suppose that $\mathbb{Q}_c^{n+1}$ is the sphere of radius $\frac{1}{\sqrt{c}}$ in $\mathbb{R}^{n+2}$. Then $|g(p)|$ is the euclidean distance from the point p_0 to the hyperplane which contains the totally geodesic hypersurface tangent to $x(M^n)$ at $x(p)$. In fact, since

$$p_0 = \cos(\sqrt{c}r(p))p - \frac{\sin(\sqrt{c}r(p))}{\sqrt{c}} \text{ grad } r(p),$$

we have

$$\langle p_0, N(p)\rangle = -\frac{\sin(\sqrt{c}r(p))}{\sqrt{c}}\langle \text{grad } r(p), N(p)\rangle = -g(p).$$

So, $|g(p)| = |\langle p_0, N(p)\rangle|$.

In the case c < 0: Let $\mathbb{L}^{n+2}$ be the euclidean space $\mathbb{R}^{n+2}$ endowed with the Lorenzian metric

$$\langle\langle u, v\rangle\rangle = u_1v_1 + \ldots + u_{n+1}v_{n+1} - u_{n+2}v_{n+2}.$$

Let $\mathbb{H}^{n+1}(c)$ be the hypersurface of $\mathbb{L}^{n+2}$ given by

$$\mathbb{H}^{n+1}(c) = \left\{ v \in \mathbb{L}^{n+2};\ v_{n+2} > 0, \langle\langle v, v\rangle\rangle = \frac{1}{c} \right\}.$$

It is well know that $\mathbb{H}^{n+1}(c)$ with the induced metric is a model of hyperbolic space $\mathbb{Q}_c^{n+1}$.

We can assume, without loss of generality, that $p_0 = \left(0, 0, \ldots, 0, \frac{1}{\sqrt{c}}\right)$. In this case, the euclidean distance from the point p_0 to the hyperplane that passes through the origin of $\mathbb{R}^{n+2}$ and contains the totally geodesic hypersurface, $(\mathbb{Q}_c^n)_p$, tangent to $x(M^n)$ at $x(p)$, is given by

$$\frac{|g(p)|}{\sqrt{1 + 2g(p)^2}}.$$

In fact, since

$$p_0 = \cosh(\sqrt{-c}r(p))p - \frac{\sinh(\sqrt{-c}r(p))}{\sqrt{-c}} \text{ grad } r(p),$$

we have that

$$\langle\langle p_0, N(p)\rangle\rangle = -g(p).$$

Let $N(p) = \big(N_1, \ldots, N_{n+1}, N_{n+2}\big)$. Then

$$\overline{N} = \frac{1}{\sqrt{1 + 2N_{n+2}^2}} \cdot N,$$

is a unit vector in $\mathbb{R}^{n+2}$ orthogonal to the hyperplane that passes through the origin of $\mathbb{R}^{n+2}$ and contains $(\mathbb{Q}_c^n)_p$. Therefore,

$$|\langle q, \overline{N}(p)\rangle| = \left|\frac{-N_{n+2}}{\sqrt{1 + 2N_{n+2}^2}}\right| = \frac{|g(p)|}{\sqrt{1 + 2g(p)^2}},$$

and this concludes the geometric interpretation.

Lemma 2.1. *Let $x\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion, and $0 \le r < n-1$, $p \in M^n$.*

(a) *If $S_{r+1}(p) = 0$, then T_r is semi-definite at p;*

(b) *If $S_{r+1}(p) = 0$ and $S_{r+2}(p) \neq 0$, then T_r is definite at p.*

Proof. See [5], Proposition 2.8, p. 192. □

An other important result is:

Lemma 2.2. *Let $x\colon M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion and $p \in M^n$.*

(a) *For $1 \le r < n$, one has $H_r^2 \ge H_{r-1}H_{r+1}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1} \neq 0$ in this case, then p is umbilical point;*

(b) *If, for some $1 \le r < n$, one has $H_r = H_{r+1} = 0$, then $H_j = 0$ for all $r \le j \le n$. In particular, at most $r-1$ of the principal curvatures are different from zero.*

Proof. See [6], Proposition 2.1, p. 176. □

The result below is standard, for completeness we will give a proof.

Lemma 2.3. *Let $x : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion. The operator L_r associated to the immersion x is elliptic if, and only if, T_r is positive definitive.*

Proof. Let $\left\{\frac{\partial}{\partial x_i}\right\}$ be a local frame of M^n at p. By direct computation, we have locally the expression of L_r:

$$L_r f(p) = \sum_{i,j,k} g^{kj} t_{ik} \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{i,j,k,l} g^{kj} t_{ik} \Gamma_{ij}^l \frac{\partial f}{\partial x_l},$$

where g_{ij} is the metric of M^n, Γ_{ij}^k are the Christoffel symbols and

$$t_{ij} = \left\langle T_r \left(\frac{\partial}{\partial x_i}\right), \frac{\partial}{\partial x_j}\right\rangle.$$

From the above local expression, it is easy to conclude that the linear operator L_r is elliptic if, and only if, T_r is positive definitive. □

Corollary 2.1. *Let $x : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion. If T_r is negative defined, then the operator $-L_r$ associated to the immersion x is elliptic.* □

3 r-minimal hypersurface with W nonempty

In this section we will prove the result on r-minimal hypersurface with W nonempty and open.

Theorem 3.1 (Thm. 1.1 in Introduction)**.** *Let M^n be a complete and orientable Riemannian manifold and let $x : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric r-minimal immersion and $H_r \neq 0$ everywhere, $r \geq 1$. If W is open and nonempty, then $\nu = n - r$. In particular, $x(M^n)$ is foliated by complete totally geodesic submanifolds of dimension $n - r$.*

Proof. Let $x : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric r-minimal immersion, i.e., $H_{r+1} = 0$ in M. Let $q \in W$ and X be the position vector with origin q. Fix an orientation, N, such that the support function $g(p) = \langle X(p), N(p)\rangle$ is positive.

In [2], Lemma 2, the authors proved that

$$L_r g = -\langle \nabla S_{r+1}, X^T\rangle - (r+1) S_{r+1}\theta_c - (S_1 S_{r+1} - (r+2) S_{r+2}) g,$$

where $\theta_c = s_c'$.

By using the equality with $H_{r+1} = 0$ we have

$$L_r g = (r + 2) S_{r+2} g. \tag{1}$$

Using Lemma 2.1(a) we have that T_r is semi-definite. Since H_r does not vanish, we have that H_r is positive or negative, because $c_r H_r = tr(T_r)$, where $c_r = (n - r)\binom{n}{r}$. Now we use Lemma 2.2 and obtain:

$$0 = H_{r+1}^2 \geq H_r H_{r+2}. \tag{2}$$

Using the information above, we claim that $H_{r+2} \equiv 0$.

In fact, first we assume that $H_{r+2} \geq 0$. Using (2) and hypothesis we conclude that $H_r < 0$ and thus T_r is negative defined. Applying Corollary 2.1 we have that $-L_r$ is elliptic. Whereas from (1) we have

$$(-L_r) g \leq 0.$$

Now, following exactly the proof given by H. Alencar and K. Frensel in [4], Theorem 3.1, we conclude that g attains its minimum in M^n. Then, from the Maximum Principle, g is constant. Since g is positive, we have that H_{r+2} vanishes.

Second, we assume $H_{r+2} \leq 0$. Using (2) and $H_r \neq 0$ we conclude that $H_r > 0$ and thus T_r is positive defined. Applying Lemma 2.3 we have that L_r is elliptic. Whereas from (1) we have

$$L_r g \leq 0.$$

Now, following exactly the proof given by H. Alencar and K. Frensel in [4], Theorem 3.1, we conclude that g attains its minimum in M^n. Then, from the Maximum Principle, g is constant. Since g is positive, we have that H_{r+2} vanishes.

Thus we conclude that $H_{r+2} \equiv 0$. Now, we use Lemma 2.2(b) to conclude that $H_j = 0$ for $j \geq r + 1$ and so that $\nu \geq n - r$. Since H_r does not change sign we have that $\nu = n - r$. In particular, we may apply Theorem 5.3 of [7] to deduce that $x(M^n)$ is foliated by complete totally geodesic submanifolds of dimension $n - r$. □

Example 3.1. We now describe an example of a 1-minimal hypersurface in $\mathbb{R}^{n+1}$ with W nonempty, but $\nu \neq n - 1$.

Let $G = O(p+1) \times O(p+1)$ be the standard action in $\mathbb{R}^{p+1} \times \mathbb{R}^{p+1}$, where p is an integer greater than 1. The orbit space of that action can be represented by

$$\pi\left(\mathbb{R}^{2p+2}\right) = \Omega = \left\{(x, y) \in \mathbb{R}^2; x \geq 0,\ y \geq 0\right\},$$

where $\pi(u, v) = (|u|, |v|)$. If $\gamma(t) = (x(t), y(t))$ is a curve in Ω, then an explicit parametrization of the hypersurface $M = \pi^{-1}(\gamma)$ is given by

$$f(t, a, b) = x(t)\Phi(a) \oplus y(t)\Psi(b),$$

where Ψ and Φ are parametrization of the unit sphere $\mathbb{S}^p$.

J. Sato, in [13], Theorem 1.2, classified the profile curve if $M = \pi^{-1}(\gamma)$ has null scalar curvature. These curves, called, type B are interesting for us. Such a curve γ, is regular, intersects orthogonally one of the half-axes $x \geq 0$ or $y \geq 0$ and asymptotes one half-straight line $\gamma_1(t) = (\cos(\alpha)t, \sin(\alpha)t)$ or $\gamma_2(t) = (\sin(\alpha)t, \cos(\alpha)t)$, where $t \geq 0$ and

$$\alpha = \frac{1}{4}\arccos\left(\frac{3-2p}{2p-1}\right),$$

when $t \to \pm\infty$.

Let $h(p) = \langle f(p), N(p)\rangle$ be the support function of the immersion f. The unit normal field is $N(t, a, b) = -y'(t)\Phi(a) \oplus x'(t)\Psi(b)$. Using the expression for f and N it is possible to verify that $h(p) = -u'(t)(x^2(t) + y^2(t))$, where $u = \arctan\left(\frac{y}{x}\right)$.

Moreover, when $p \geq 4$, J. Sato also proved in [13], Lemma 2.3, that $u' \neq 0$ for every orbit associated to a profile curve of the type B. Thus, h or $-h$ is positive in M, i.e., W is nonempty. In order to finish our example it is enough to use that

$$k_0 = -\frac{x''y' + x'y''}{\left[(x')^2 + (y')^2\right]^{3/2}},$$

$$k_i = \frac{y'}{x\sqrt{(x')^2 + (y')^2}},\ i = 1, \ldots, p,$$

$$k_j = -\frac{x'}{y\sqrt{(x')^2 + (y')^2}},\ j = p+1, \ldots, 2p,$$

and thus the rank of the second fundamental form of the immersion is greater than p, when $p \geq 2$. □

Example 3.2. H. Alencar described in [1] an example of a minimal hypersurface in $\mathbb{R}^{2m}$ with W nonempty, but the hypersurface is not totally geodesic. For completeness we will give a sketch of the example.

Let $G = SO(m) \times SO(m)$ be the standard action in $\mathbb{R}^m \times \mathbb{R}^m$, where m is an integer greater than 1. The orbit space of that action can be represented by

$$\pi\left(\mathbb{R}^{2m}\right) = \Omega = \left\{(x, y) \in \mathbb{R}^2; x \geq 0,\ y \geq 0\right\},$$

where $\pi(u, v) = (|u|, |v|)$. If $\gamma(t) = (x(t), y(t))$ is a curve in Ω, then an explicit parametrization of the hypersurface $M = \pi^{-1}(\gamma)$ is given by

$$f(t, a, b) = x(t)\Phi(a) \oplus y(t)\Psi(b),$$

where Ψ and Φ are parametrization of the unit sphere $\mathbb{S}^{m-1}$.

H. Alencar, in [1], Theorem 4.1 and 5.1, classified the profile curve if $M = \pi^{-1}(\gamma)$ has null mean curvature. These curves called as topological type A are interesting for us. Such a curve γ is regular, intersects orthogonally one of the half-axes $x \geq 0$ or $y \geq 0$ and it has asymptote the half-straight line $\gamma(t) = (t, t)$, where $t \geq 0$, when $t \to \pm\infty$.

Let $h(p) = \langle f(p), N(p)\rangle$ be the support function of the immersion f. The unit normal field is

$$N(t, a, b) = -y'(t)\Phi(a) \oplus x'(t)\Psi(b).$$

Using the expression for f and N it is possible to verify that

$$h(p) = -u'(t)(x^2(t) + y^2(t)),$$

where $u = \arctan\left(\frac{y}{x}\right)$.

Moreover, when $m \geq 4$, H. Alencar also proved in [1], Proposition 4.4, that $u' \neq 0$ for every orbit associated to a profile curve of the topological type A. Thus, h or $-h$ is positive in M, i.e., W is nonempty. □

Corollary 3.1 (Cor. 1.1 in Introduction)**.** *Let M^n be a complete Riemannian manifold and let $x\colon M^n \to \mathbb{S}^{n+1}$ be an isometric immersion with $H_2 = 0$ and mean curvature $H_1 \neq 0$ everywhere. There no exists an immersion x such that the set W is open and nonempty.*

Proof. Suppose that W is open and nonempty. Using Theorem 3.1 we have $\nu = n - 1$. On the other hand, as any principal curvature has a sign, because $H_1 \neq 0$ everywhere, we can apply the Theorem 2 in [9], p. 99, to conclude that there exists a principal curvature with an opposite sign. But this is impossible, because $\nu = n - 1$. □

Corollary 3.2 (Cor. 1.2 in Introduction)**.** *Let M^n be a complete Riemannian manifold and let $x\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion with $H_3 = 0$ and scalar curvature $H_2 \neq 0$ everywhere. If W is open and nonempty, then $x(M^n) = \mathbb{S}^2 \times \mathbb{R}^{n-2}$.*

Proof. Using Theorem 3.1 we have $\nu = n - 2$. Now, we can apply the Theorem 3.4 in [8], p. 11, to conclude that $x(M^n) = \mathbb{S}^2 \times \mathbb{R}^{n-2}$. □

4 r-stability

In this section we will prove the result on r-stable hypersurface with W nonempty. One has:

Theorem 4.1 (Thm. 1.2 in Introduction)**.** *Let M^n be a complete and orientable Riemannian manifold and let $x\colon M^n \to \mathbb{Q}_c^{n+1}$, $c \leq 0$, be an isometric r-minimal immersion and $H_r \neq 0$ everywhere, $r \geq 1$. If W is nonempty, then x is r-stable.*

Proof. Let $p_0 \in W$ and X be the position vector with origin p_0. Since $p_0 \in W$, we can choose an orientation N in M^n for which the support function g is positive. From Lemma 2 in [2] we have $L_r(g) - (r+2)S_{r+2}g = 0$, provided $H_{r+1} = 0$.

First, let us consider $H_r > 0$. In this case, the operator L_r is elliptic.

In [10], Proposition 3.13, M.F. Elbert proved that operator of type $L_r + q$, where q is a differentiable function on M^n, is positive if and only if there is a positive differentiable function f on M^n such that $L_r f + qf = 0$. Since

$$J_r g = L_r g - (r+2)S_{r+2}g = 0,$$

the operator J_r is positive definite, i.e.,

$$\int_M \left(\langle T_r \nabla f, \nabla f\rangle + (r+2)S_{r+2}f^2\right) dV > 0,$$

for every nonzero function f. Since $c \cdot H_r \leq 0$ we have:

$$\int_M \left(\langle T_r \nabla f, \nabla f\rangle + ((r+2)S_{r+2} - c(n-r)S_r)f^2\right) dV > 0,$$

for every nonzero function f. Then

$$J_r \quad L_r \quad (r \quad 2)S_{r+2} + c(n-r)S_r$$

is positive definite, i.e., x is r-stable.

Finally, consider $H_r < 0$. In this case, the operator $(-L_r)$ is elliptic.

In [10], Proposition 3.13, M.F. Elbert proved that operator of type $L_r + q$, where q is a differentiable function on M^n, is negative if and only if there is a positive differentiable function f on M^n such that $L_r f + qf = 0$. Since

$$J_r g = L_r g - (r+2)S_{r+2}g = 0,$$

the operator J_r is negative definite, i.e.,

$$\int_M \left(\langle T_r \nabla f, \nabla f\rangle + (r+2)S_{r+2}f^2\right) dV < 0,$$

for every nonzero function f. Since $c \cdot H_r \geq 0$ we have:

$$\int_M \left(\langle T_r \nabla f, \nabla f\rangle + ((r+2)S_{r+2} - c(n-r)S_r)f^2\right) dV < 0,$$

for every nonzero function f. Then

$$J_r = L_r - (r+2)S_{r+2} + c(n-r)S_r$$

is negative definite, i.e., x is r-stable. □

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Hilário Alencar
Universidade Federal de Alagoas
Instituto de Matemática
57072-970 Maceió, AL
BRAZIL

E-mail: hilario@pos.mat.ufal.br

Márcio Batista
Universidade Federal de Alagoas
Instituto de Matemática
57072-970 Maceió, AL
BRAZIL

E-mail: mhbs28@impa.br