UPPER BOUNDS FOR THE FIRST EIGENVALUE OF THE OPERATOR $L_r$ AND SOME APPLICATIONS

HILÁRIO ALENCAR, MANFREDO DO CARMO, AND FERNANDO MARQUES

Abstract. We obtain upper bounds for the first eigenvalue of the linearized operator $L_r$ of the $r$-mean curvature of a compact manifold immersed in a space of constant curvature $\delta$. By the same method, we obtain an upper bound for the first eigenvalue of the stability operator associated to $L_r$ when $\delta < 0$.

1. Introduction

Let $M^n$ be a compact, connected, orientable Riemannian manifold, isometrically immersed in a simply connected space form $\mathcal{M}^{n+1}(\delta)$, with constant sectional curvature $\delta$. We obtain upper bounds for the first eigenvalue of some elliptic operators defined on $M$ (see below). In 1988, Heintze [H] proved that

$$\lambda_1^{\Delta} \leq n\delta + n\max H_1$$

for manifolds immersed in a hyperbolic space ($\delta < 0$) and that

$$\lambda_1^{\Delta} \leq n\delta + \frac{n}{\text{vol} M} \int_M H_1^2$$

for manifolds immersed in a sphere ($\delta > 0$), contained in a convex ball of radius $r \leq \frac{\pi}{\sqrt{\delta}}$ (if $\delta > 0$). Here $\lambda_1^{\Delta}$ denotes the first eigenvalue of the Laplacian on $M$ and $H_1$ denotes the mean curvature. (In fact, Heintze considered as ambient spaces Riemannian manifolds with curvature bounded above by $\delta$.) The latter inequality was obtained by Reilly [R] in 1977 for manifolds immersed in Euclidean space ($\delta = 0$), and generalized to arbitrary $\delta$ by El Soufi and Ilias [ESI] in 1992. In all of these estimates, equality holds precisely when $M$ is a geodesic sphere of $\mathcal{M}$. Both El Soufi and Ilias [ESI] and Heintze [H] applied these bounds to obtain the stability theorems of Barbosa and do Carmo [B-dC] for immersions in $\mathbb{R}^{n+1}$, and of Barbosa, do Carmo,
and Eschenburg [B-dC-E] for immersions in $S^{n+1}$ and $H^{n+1}$. (The restriction $r \leq \frac{n}{4\sqrt{\delta}}$ in the case of $S^{n+1}$ in [H] is stronger than that in [B-dC-E].)

We first introduce some notation. Consider the elementary symmetric functions $S_r$ ($r = 1, \ldots, n$) of the principal curvatures and the $r$-mean curvatures $H_r = \frac{S_r}{\binom{n}{r}}$.

Let $A$ be the second fundamental form associated to a globally defined normal unit vector field $N$. We define an operator $L_r$ by

$$L_r f = \text{div}(P_r \nabla_M f),$$

where $\nabla_M f$ stands for the gradient of $f$ in $M$ and $P_r$ denotes the classical Newton transformation defined inductively by

$$P_0 = I,$$

$$P_r = S_r I - A P_{r-1}. $$

Each $P_r$ is a self-adjoint operator whose trace is $c(r) H_r$, where $c(r) = (n - r) \binom{n}{r}$ (see [B-C, Lemma 2.1]). Note that $L_0 = \Delta$.

In general, the operator $L_r$ is not elliptic and some conditions are necessary to ensure the presence of ellipticity. However, in the theorems below, the hypotheses will guarantee that $L_r$ is elliptic; see, for instance, the remarks made at the beginning of the proofs of Theorems 1.1 and 1.2. Thus we can consider the first eigenvalue $\lambda_{L_r}^1$ of $L_r$. This is the object we study here.

Assume that $S_{r+1}$ is constant. Following [B-C] we say that the immersion is $r$-stable if $I_r(f) \geq 0$ for any $f: M \to \mathbb{R}$ satisfying $\int_M f dM = 0$, where

$$I_r(f) = -\int_M f \left\{ L_r(f) + \left[ \frac{n}{r+1} c(r) H_1 H_{r+1} - c(r+1) H_{r+2} + \delta c(r) H_{r+1} \right] f \right\}. $$

In 1993, Alencar, do Carmo, and Rosenberg [A-dC-R, Theorem 1.1] proved that if $H_{r+1}$ is positive (but not necessarily constant) on $M$, then

$$\lambda_{L_r}^1 \int_M H_r \leq c(r) \int_M H_{r+1}^2,$$

for manifolds immersed in Euclidean space, and equality holds if and only if $M$ is a sphere. They applied this result to obtain the theorem of Barbosa and do Carmo [B-dC] and a theorem of Alencar, do Carmo, and Colares [A-dC-C]. They also proved that an immersion of a hypersurface in $\mathbb{R}^{n+1}$ with $H_{r+1}$ constant is $r$-stable if and only if $M$ is a sphere. In 1995, Grosjean [G1] obtained sharp integral bounds for $\lambda_{L_r}^1$ of immersions in any space form $\overline{M} (\delta)$, under the additional hypothesis of convexity of the immersion.

In this paper, inspired by Heintze’s work [H], we obtain sharp upper bounds for $\lambda_{L_r}^1$ without the convexity hypothesis, in both the hyperbolic and spherical spaces. We prove the following theorems:
Theorem 1.1. Let $M^n$ be a compact manifold isometrically immersed in $\overline{M}^{n+1}(\delta)$, with $\delta < 0$. If $H_{r+1} > 0$ on $M$, then
\[
\lambda_1^{L_r} \leq \delta c(r) \min H_r + c(r) \frac{\max H_{r+1}^2}{\min H_r},
\]
and equality holds if and only if $M$ is a geodesic sphere.

Theorem 1.2. Let $M^n$ be a compact manifold isometrically immersed in $\overline{M}^{n+1}(\delta)$, with $\delta > 0$, and suppose $M$ is contained in a convex ball of radius $\frac{\pi}{4\sqrt{\delta}}$. If $H_{r+1} > 0$ on $M$, then
\[
\lambda_1^{L_r} \leq \left( \delta c(r) + c(r) \frac{\max H_{r+1}^2}{\min H_r^2} \right) \int_M H_r \text{vol } M,
\]
and equality holds if and only if $M$ is a geodesic sphere.

Furthermore, using the same techniques, we obtain an upper bound for the first eigenvalue of the operator $L_r - q$, where
\[
q = c(r + 1)H_{r+2} - \frac{nc(r)}{r+1} H_1 H_{r+1} - \delta c(r) H_r.
\]

From this the $r$-stability theorem for manifolds immersed in a hyperbolic space, proved in 1997 by Barbosa and Colares [B-C], will follow. (In fact, in [B-C] the $r$-stability theorem was proved for any manifold $\overline{M}^{n+1}(\delta)$.)

To state this theorem, we need some notation. We denote by $s_\delta$ the solution of the differential equation $y'' + \delta y = 0$, with the initial conditions $y(0) = 0$, $y'(0) = 1$. Set $c_\delta = s'_\delta$, then $c_\delta = -\delta s_\delta$ and $c_\delta^2 + \delta s_\delta^2 = 1$. We have $s_\delta(t) = \frac{1}{\sqrt{-\delta}} \sinh(\sqrt{-\delta}t)$ and $c_\delta(t) = \cosh(\sqrt{-\delta}t)$ in the case $\delta < 0$, and $s_\delta(t) = \sqrt{\delta} \sin(\sqrt{\delta}t)$, $c_\delta(t) = \cos(\sqrt{\delta}t)$ in the case $\delta > 0$. Note that if $\delta = 0$, $s_\delta(t) = t$.

We will prove the following result:

Theorem 1.3. If $\delta < 0$ and $H_{r+1} > 0$, there exists a point $p_0 \in \overline{M}$ such that, if $d = d(p_0, \cdot)$ denotes the distance function from $p_0$ in $\overline{M},$ then
\[
\lambda_1(L_r - q) \leq c(r) \int_M \left( \frac{\max H_{r+1}^2}{\min H_r} - H_1 H_{r+1} \right) s_\delta^2(d) \left( \int_M s_\delta^2(d) \right),
\]
and equality holds precisely when $M$ is a geodesic sphere.

As a consequence, we obtain:

Corollary 1.4 ([B-C]). The only $r$-stable compact immersed hypersurfaces in a hyperbolic space, with constant $H_{r+1} > 0$, are the geodesic spheres.
Remark 1.5. After writing this paper, we received a preprint of J.F. Grosjean (see [G2]) who obtained, independently, the results presented here.

We want to thank the referee for useful suggestions.

2. Preliminaries

Throughout this paper, we use the notations defined in the Introduction. Let $M^n$ and $\overline{M}^{n+1}(\delta)$ be as before, let $p_0 \in \overline{M}$, and let $d = d(p_0, \cdot)$ be the distance function from $p_0$ in $\overline{M}$. Let $x_i$ $(i = 1, \ldots, n+1)$ be the normal coordinates centered in $p_0$, with respect to some orthonormal basis in $T_{p_0} \overline{M}$. We denote by $\nabla$ and $\nabla_M$ the gradients taken in $\overline{M}$ and $M$, respectively.

Lemma 2.1. Suppose $x \in \overline{M}$. Assume $x \in B \left(p_0, \frac{\kappa_0}{2\sqrt{\delta}} \right)$ in the case $\delta > 0$. If $u, v \in T_x \overline{M}$ and $v$ is orthogonal to $\nabla d$, then

$$s_2^2(d) \sum_{i=1}^{n+1} \langle \langle \nabla x_i, u \rangle \langle \nabla x_i, v \rangle \rangle = \langle u, v \rangle.$$

Proof. The map

$$L_{\hat{x}} = (d \exp_{p_0})_{\hat{x}} : T_{p_0} \overline{M} \to T_x \overline{M},$$

where $\exp_{p_0}(\hat{x}) = x$, is a linear isomorphism. Using $\langle \nabla x_i, u \rangle = \langle L_{\hat{x}}^{-1} u(x_i) \rangle$, we obtain

$$s_2^2(d) \sum_{i=1}^{n+1} \langle \langle \nabla x_i, u \rangle \langle \nabla x_i, v \rangle \rangle = s_2^2(d) \langle L_{\hat{x}}^{-1}(u), L_{\hat{x}}^{-1}(v) \rangle.$$

Since $L_{\hat{x}}$ is a radial isometry, $L_{\hat{x}}^{-1}(v)$ is tangent to the sphere of radius $|\hat{x}|$ in $T_{p_0} \overline{M}$. Further, $L_{\hat{x}}^{-1}(u) = \hat{w} + \hat{r}$, where $\hat{w}$ is tangent and $\hat{r}$ is orthogonal to this sphere in $T_{p_0} \overline{M}$. Hence, $\langle L_{\hat{x}}^{-1}(u), L_{\hat{x}}^{-1}(v) \rangle = \langle \hat{w}, \hat{v} \rangle$, where $\hat{v} = L_{\hat{x}}^{-1}(v)$.

Let $\gamma : [0, d] \to \overline{M}$ be the normalized geodesic with

$$\gamma(0) = p, \quad \gamma(d) = x, \quad \gamma'(0) = \frac{\tilde{x}}{|\tilde{x}|} ,$$

where $|\tilde{x}| = d$. Let $J_v(t), J_w(t)$ be Jacobi fields along $\gamma$ such that

$$J_v(0) = J_w(0) = 0, \quad J'_v(0) = \frac{\hat{v}}{|\hat{v}|}, \quad J'_w(0) = \frac{\hat{w}}{|\hat{w}|}.$$

Since $\overline{M}$ has constant sectional curvature,

$$\langle J_v(d), J_w(d) \rangle = s_2^2(d) \frac{\hat{v} \cdot \hat{w}}{|\hat{v}| |\hat{w}|}.$$

Recall also that

$$J_v(t) = (d \exp_{p_0})_{t \hat{x}} \left( t \frac{\hat{v}}{|\hat{v}|} \right), \quad J_w(t) = (d \exp_{p_0})_{t \hat{x}} \left( t \frac{\hat{w}}{|\hat{w}|} \right).$$
Hence, \( \langle J_v(d), J_w(d) \rangle = \frac{d^2}{\|v\| \|w\|} \langle v, w \rangle \), where \( w = L_x(\tilde{w}) \), and using (1), we obtain \( \langle v, w \rangle = \frac{s^2}{d^2} \langle \tilde{v}, \tilde{w} \rangle \). Thus
\[
\frac{s^2}{d^2} \sum_{i=1}^{n+1} \langle \nabla x_i, u \rangle \langle \nabla x_i, v \rangle = \frac{s^2}{d^2} \langle \tilde{v}, \tilde{w} \rangle = \langle v, w \rangle = \langle v, u \rangle,
\]
which concludes the proof. \( \square \)

Define the position vector \( X \) of \( M^n \) in \( M^{n+1}(\delta) \) with respect to \( p_0 \) by \( X = s \delta (d) \nabla d \). Denote by \( X^T \) the component of \( X \) tangent to \( M \); i.e., \( X^T = s \delta (d) \nabla_M d \). Observe that
\[
\nabla_M c \delta = -\delta X^T.
\]

**Lemma 2.2.** With the above notation,
\[
\sum_{i=1}^{n+1} \langle P_r \nabla_M \left( \frac{s \delta}{d} x_i \right), \nabla_M \left( \frac{s \delta}{d} x_i \right) \rangle + \delta \langle P_r X^T, X^T \rangle = c(r) H_r.
\]

**Proof.** Using the fact that \( P_r \) is self-adjoint, we obtain
\[
\langle P_r \nabla_M \frac{s \delta}{d} x_i, \nabla_M \frac{s \delta}{d} x_i \rangle = \frac{x_i^2}{d^2} \left( c \delta - \frac{s \delta}{d} \right)^2 \langle P_r \nabla_M d, \nabla_M d \rangle
+ 2 \frac{x_i s \delta}{d^2} \left( c \delta - \frac{s \delta}{d} \right) \langle P_r \nabla_M d, \nabla_M x_i \rangle
+ \frac{s^2 \delta}{d^2} \langle P_r \nabla_M x_i, \nabla_M x_i \rangle.
\]
Since \( \sum_i x_i \nabla x_i = d \nabla d \) and
\[
\langle P_r \nabla_M d, \nabla_M x_i \rangle = \langle P_r \nabla_M d, \nabla x_i \rangle,
\]
because \( P_r \nabla_M d \) is tangent to \( M \), we have
\[
\sum_{i=1}^{n+1} \langle P_r \nabla_M \frac{s \delta}{d} x_i, \nabla_M \frac{s \delta}{d} x_i \rangle
= \left( c \delta \right) \langle P_r \nabla_M d, \nabla_M d \rangle
+ \left( \frac{s^2 \delta}{d^2} - \frac{s \delta}{d} \right) \sum_{i=1}^{n+1} \langle P_r \nabla_M x_i, \nabla_M x_i \rangle.
\]
Further
\[
\delta \langle P_r X^T, X^T \rangle = \delta \frac{s^2}{d^2} \langle P_r \nabla_M d, \nabla_M d \rangle.
\]
Thus
\[
\sum_{i=1}^{n+1} \left< P_r \nabla_M \left( \frac{s_d}{d} x_i \right), \nabla_M \left( \frac{s_d}{d} x_i \right) \right> + \delta \langle P_r X^T, X^T \rangle = \frac{s^2}{d^2} \sum_{i=1}^{n+1} \left< P_r \nabla_M x_i, \nabla_M x_i \right> + \left( 1 - \frac{s^2}{d^2} \right) \langle P_r \nabla_M d, \nabla_M d \rangle.
\]

Now let \( e_1, \ldots, e_n \in T_p M \) be an orthonormal basis such that \( e_n \) lies in the direction of \( \nabla_M d \) (if \( \nabla_M d \neq 0 \)). Then there are numbers \( \lambda \) and \( \mu \) satisfying \( e_n = \lambda \nabla d + \mu e^*_n \), where \( e^*_n \) is a unit vector orthogonal to \( \nabla d \). From this we easily obtain \( \nabla_M d = \lambda e_n \) and \( (e^*_n)^T = \mu e_n \).

A simple calculation gives
\[
\sum_{i=1}^{n+1} \left< P_r \nabla_M \left( \frac{s_d}{d} x_i \right), \nabla_M \left( \frac{s_d}{d} x_i \right) \right> + \delta \langle P_r X^T, X^T \rangle = \frac{n-1}{d^2} \sum_{j=1}^{n-1} \left< P_r e_j, e_j \right> + \frac{s^2}{d^2} \sum_{i=1}^{n+1} \left< P_r \nabla_M x_i, e_n \right> \langle \nabla_M x_i, e_n \rangle + \left( 1 - \frac{s^2}{d^2} \right) \lambda^2 \langle P_r e_n, e_n \rangle.
\]

Since \( e_j \) is orthogonal to \( \nabla d \) for all \( j = 1, \ldots, n-1 \), we can apply Lemma 2.1 and obtain
\[
(2) \quad \sum_{i=1}^{n+1} \left< P_r \nabla_M \left( \frac{s_d}{d} x_i \right), \nabla_M \left( \frac{s_d}{d} x_i \right) \right> + \delta \langle P_r X^T, X^T \rangle = \sum_{j=1}^{n-1} \left< P_r e_j, e_j \right> + \frac{s^2}{d^2} \sum_{i=1}^{n+1} \left< P_r \nabla_M x_i, e_n \right> \langle \nabla_M x_i, e_n \rangle + \left( 1 - \frac{s^2}{d^2} \right) \lambda^2 \langle P_r e_n, e_n \rangle.
\]

Observing that, for any \( x \in M \) and any \( u \in T_x M \),
\[
\sum_{i=1}^{n+1} \langle \nabla x_i, u \rangle \langle \nabla x_i, \nabla d \rangle = \langle u, \nabla d \rangle,
\]
we obtain, after some manipulation,
\[
\sum_{i=1}^{n+1} \left< P_r \nabla_M x_i, e_n \right> \langle \nabla_M x_i, e_n \rangle = \lambda \langle P_r e_n, \nabla d \rangle + \sum_{i=1}^{n+1} \langle \nabla x_i, P_r e_n \rangle \langle \nabla x_i, \mu e^*_n \rangle.
\]
Applying Lemma 2.1 again, because $e_n^*$ is orthogonal to $\nabla d$, the right-hand side of the last equation is equal to

$$\lambda \langle P_r e_n, \nabla d \rangle + \frac{d^2}{s^2(d)} \mu \langle P_r e_n, e_n^* \rangle.$$  

Substituting this in (2), we obtain

$$\sum_{i=1}^{n+1} \langle P_r \nabla M \left( s \delta \frac{d}{d} x_i \right), \nabla M \left( s \delta \frac{d}{d} x_i \right) \rangle + \delta \langle P_r X^T, X^T \rangle = \sum_{j=1}^{n} \langle P_r e_j, e_j \rangle = \text{trace}(P_r) = c(r) H_r. \quad \square$$

**Lemma 2.3.** If $H_{r+1} > 0$ and $c_\delta \geq 0$, then

$$\int_M c_\delta \leq \frac{\max H_{r+1}}{\min H_r} \int_M s_\delta.$$  

**Proof.** Recall that if $H_{r+1} > 0$, then $H_j > 0$, where $1 \leq j \leq r$ (see [B-C, Proposition 3.2]). Recall also Minkowski’s formula (see [A-C])

$$\int_M [H_r c_\delta + H_{r+1} \langle X, N \rangle] = 0.$$ 

Then, by the Cauchy-Schwarz inequality,

$$\int_M H_r c_\delta = -\int_M H_{r+1} \langle X, N \rangle \leq \int_M H_{r+1} |X| = \int_M H_{r+1} s_\delta \leq (\max H_{r+1}) \int_M s_\delta.$$ 

Since $c_\delta \geq 0$, we also have $(\min H_r) \int_M c_\delta \leq \int_M H_r c_\delta$. From these inequalities we obtain

$$(\min H_r) \int_M c_\delta \leq (\max H_{r+1}) \int_M s_\delta.$$ 

This concludes the proof. \quad \square

**3. Proofs of Theorems 1.1, 1.2, 1.3 and Corollary 1.4**

We are now in a position to prove the upper bounds for the first eigenvalue of $L_r$. Our proofs use the Rayleigh quotient, applied with suitable test functions.

In all proofs, $p_0 \in \overline{M}$ will be a point such that

$$\int_M s_\delta(d) \frac{d}{d} x_i = 0 \quad (i = 1, \ldots, n+1).$$
where \(d = d(p_0, \cdot)\). The existence of such a point, assuming that \(M\) lies in a convex ball of \(\overline{M}\), can be verified by a standard argument. Namely, if \(M\) lies in a convex ball \(B\), then

\[
Y_q = \int_M s_\delta d(q,p) \exp_q^{-1}(p) dp \in T_q\overline{M}
\]

defines a vector field in a neighborhood of \(B\) which, at the boundary, points towards the interior of \(B\). Thus, \(Y\) has a zero in \(B\), and if we take \(p_0\) as this zero, then \(p_0\) has the required property. Note that if \(B\) has radius less than \(\frac{\pi}{4\sqrt{\delta}}\), then \(M\) lies in a ball of radius \(<\frac{\pi}{2\sqrt{\delta}}\) around \(p_0\). As a consequence, \(c_\delta \geq 0\).

Proof of Theorem 1.1. Since \(H_{r+1} > 0\) and \(M\) is compact, \(L_r\) is elliptic (see [B-C, Proposition 3.2]).

Using the Rayleigh quotient with the test functions \(s_\delta d\), we obtain

\[
\lambda_1^{L_r} \int_M s_\delta^2 = \lambda_1^{L_r} \int_M \sum_{i=1}^{n+1} \left( \frac{s_\delta}{d} x_i \right)^2
\]

\[
\leq \int_M \sum_{i=1}^{n+1} \left( P_r \nabla_M \left( \frac{s_\delta}{d} x_i \right), \nabla_M \left( \frac{s_\delta}{d} x_i \right) \right)
\]

\[
= c(r) \int_M H_r - \delta \int_M \langle P_r X^T, X^T \rangle,
\]

where the last equality follows from Lemma 2.2.

From Stokes' theorem it follows that

\[
\int_M f L_r g + \langle P_r \nabla_M f, \nabla_M g \rangle = 0.
\]

Applying this with \(f = g = c_\delta\) and using the relation \(\nabla_M c_\delta = -\delta X^T\), we obtain

\[
\delta \int_M \langle P_r X^T, X^T \rangle = -\frac{1}{\delta} \int_M c_\delta L_r(c_\delta).
\]

Hence,

\[
\int_M \sum_{i=1}^{n+1} \left( P_r \nabla_M \left( \frac{s_\delta}{d} x_i \right), \nabla_M \left( \frac{s_\delta}{d} x_i \right) \right) = c(r) \int_M H_r + \frac{1}{\delta} \int_M c_\delta L_r(c_\delta).
\]

It is known that (see [A-C, Lemma 1])

\[
L_r(c_\delta) = -\delta [c(r) H_r c_\delta + c(r) \langle X, N \rangle H_{r+1}] .
\]

From this and the inequality

\[
-\langle X, N \rangle \leq |X| = s_\delta,
\]
we obtain
\[
\sum_{i=1}^{n+1} \left\langle P_i \nabla_M \left( \frac{s_\delta}{d} x_i \right), \nabla_M \left( \frac{s_\delta}{d} x_i \right) \right\rangle \leq \delta c(r) \int_M s_\delta^2 H_r - c(r) \int_M c_\delta \langle X, N \rangle H_{r+1} \\
\leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \int_M c_\delta s_\delta H_{r+1} \\
\leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \max H_{r+1} \int_M c_\delta s_\delta.
\]

If \( \delta \leq 0 \), it is also known that (see Lemma 2.8 in [H])
\[
\int_M s_\delta j \int_M s_\delta c_\delta \leq \left( \int_M s_\delta^2 \right) \int_M c_\delta.
\]
Using this inequality and Lemma 2.2, we have
\[
\sum_{i=1}^{n+1} \left\langle P_i \nabla_M \left( \frac{s_\delta}{d} x_i \right), \nabla_M \left( \frac{s_\delta}{d} x_i \right) \right\rangle \\
\leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \left( \frac{\max H_{r+1}}{\min H_r} \right)^2 \int_M s_\delta^2 \\
\leq \delta c(r) \left( \min H_r \right) \int_M s_\delta^2 + c(r) \left( \frac{\max H_{r+1}}{\min H_r} \right) \int_M s_\delta^2.
\]

By applying (3), we obtain
\[
\lambda_1^{L_r} \int_M s_\delta^2 \leq \delta c(r) \left( \min H_r \right) \int_M s_\delta^2 + c(r) \frac{\max H_{r+1}^2}{\min H_r} \int_M s_\delta^2.
\]
Dividing both sides by \( \int_M s_\delta^2 \) gives the desired estimate.

If equality holds, then we necessarily have
\[
-\langle X, N \rangle = |X||N|,
\]
and this implies that \( \nabla d \) is orthogonal to \( M \). Thus, \( d \) is constant on \( M \), and therefore \( M \) is a geodesic sphere around \( p_0 \). \( \square \)

**Proof of Theorem 1.2.** Since \( H_{r+1} > 0 \) and \( M \) is contained in a convex ball, \( L_r \) is again an elliptic operator (see [B-C, Proposition 3.2]). Put
\[
c = \frac{1}{\text{vol } M} \int_M c_\delta, \quad \text{so} \quad \int_M \frac{c_\delta - c}{\sqrt{\delta}} = 0.
\]
Recall that \( s_\delta(d) = \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}d) \) and \( c_\delta(d) = \cos(\sqrt{\delta}d) \), so \( |c| < 1 \).
Using the Rayleigh quotient with \( \frac{s_i(d)}{d} x_i \) and \( \frac{c_i - c}{\sqrt{\delta}} \) as test functions, we obtain

\[
\lambda_1^{L_r} \int_M \left[ s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right] 
\leq \int_M \sum_{i=1}^{n+1} \left( P_r \nabla_M \frac{s_\delta}{d} x_i, \nabla_M s_\delta x_i \right) + \int_M \left( P_r \nabla_M \left( \frac{c_\delta - c}{\sqrt{\delta}} \right), \nabla_M \left( \frac{c_\delta - c}{\sqrt{\delta}} \right) \right) 
= \int_M \left[ \sum_{i=1}^{n+1} \left( P_r \nabla_M \left( \frac{s_\delta}{d} x_i \right), \nabla_M \left( \frac{s_\delta}{d} x_i \right) \right) + \delta(P_r X^T, X^T) \right] = c(r) \int H_r,
\]

where the last equality follows from Lemma 2.2.

Further, a direct calculation gives

\[
\int_M \left[ s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right] = \frac{1}{\delta} (\text{vol } M)(1 - c^2).
\]

Thus,

\[
\lambda_1^{L_r} \leq \left( \frac{1}{1 - c^2} \right) \frac{c(r)}{\delta} \int H_r.
\]

We next prove that

\[
\frac{1}{1 - c^2} \leq 1 + \frac{1}{\delta} \max H_{r+1}^2.
\]

By Lemma 2.3 we have

\[
c^2 = \frac{1}{(\text{vol } M)^2} \left( \int_M \epsilon_\delta \right)^2 \leq \frac{1}{(\text{vol } M)^2} \left( \max H_{r+1} \right)^2 \left( \int_M s_\delta \right)^2,
\]

and the Cauchy-Schwarz inequality gives

\[
\left( \int_M s_\delta \right)^2 \leq \left( \int_M s_\delta^2 \right) \text{vol } M.
\]

Therefore

\[
(1 - c^2) \left( 1 + \frac{1}{\delta} \max H_{r+1}^2 \right) 
\geq 1 + \frac{1}{\delta} \max H_{r+1}^2 \frac{1}{\text{vol } M} \max H_{r+1} \int_M s_\delta^2 - c^2 \cdot \frac{1}{\delta} \max H_{r+1}^2
\geq 1 + \frac{1}{\delta} \max H_{r+1}^2 \frac{1}{\text{vol } M} \max H_{r+1} \int_M s_\delta^2 - \left( \frac{1}{\text{vol } M} \int_M c_\delta \right) \frac{1}{\delta} \max H_{r+1}^2
= 1 + \frac{\max H_{r+1}^2}{\min H_r^2} \left( \frac{1}{\delta} - \frac{1}{\delta \text{vol } M} \int_M (\delta s_\delta^2 + c_\delta^2) \right) = 1,
\]
where the last inequality follows from Cauchy-Schwarz inequality. Hence,
\[
\frac{1}{1-c^2} \leq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2}.
\]
From (5), we have
\[
\lambda_1^{L_r} \leq \left( \frac{\delta + \max H_{r+1}^2}{\min H_r^2} \right) c(r) \frac{\int H_r}{\text{vol } M}.
\]
If equality holds, we also have equality in Lemma 2.3, so \( -\langle X, N \rangle = |X||N| \) and therefore \( \nabla d \) is orthogonal do \( M \). Hence, \( d \) is constant on \( M \), and therefore \( M \) is a geodesic sphere around \( p_0 \). \( \square \)

**Proof of Theorem 1.3.** Using the Rayleigh quotient for the operator \( L_r - q \), with \( \frac{s}{d} x_i \) as test functions, we obtain
\[
(6) \quad \lambda_1(L_r - q) \int_M s_\delta^2 \leq \int_M \sum_{i=1}^{n+1} \left( \frac{s_\delta}{d} x_i \right) \left[ -L_r \left( \frac{s_\delta}{d} x_i \right) + q \left( \frac{s_\delta}{d} x_i \right) \right]
\]
\[
= \int_M \sum_{i=1}^{n+1} \left[ P_r \nabla_M \left( \frac{s_\delta}{d} x_i \right), \nabla_M \left( \frac{s_\delta}{d} x_i \right) \right] + \int_M q s_\delta^2.
\]
By (4) we have
\[
\int_M \sum_{i=1}^{n+1} \left[ P_r \nabla_M \left( \frac{s_\delta}{d} x_i \right), \nabla_M \left( \frac{s_\delta}{d} x_i \right) \right]
\]
\[
\leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \frac{\max H_{r+1}^2}{\min H_r} \frac{\int_M s_\delta^2}{\int_M s_\delta^2}.
\]
Applying this to (6), we obtain
\[
\lambda_1(L_r - q) \int_M s_\delta^2 \leq c(r) \left( \int_M s_\delta^2 \right) \frac{\max H_{r+1}^2}{\min H_r}
\]
\[
+ c(r+1) \int_M H_{r+2}s_\delta^2 - \frac{nc(r)}{r+1} \int_M H_1 H_{r+1}s_\delta^2.
\]
Since \( H_{r+2} \leq H_1 H_{r+1} \), with equality at unabilical points (see [A-dC-R, p. 392]), we obtain
\[
\lambda_1(L_r - q) \int_M s_\delta^2 \leq c(r) \left( \int_M s_\delta^2 \right) \frac{\max H_{r+1}^2}{\min H_r} - c(r) \int_M H_1 H_{r+1}s_\delta^2,
\]
because \( c(r+1) - \frac{nc(r)}{r+1} = -c(r) \).

Dividing both terms by \( \int_M s_\delta^2 \), the desired inequality follows. The case when equality holds is handled in the same way as in the previous cases. \( \square \)
Proof of Corollary 1.4. Let $M^n$ be a compact hypersurface, immersed in $\overline{M^{n+1}(\delta)}$, where $\delta < 0$, with $H_{r+1} > 0$ and constant. Suppose $M$ is $r$-stable, that is to say,

$$\int_M f(-L_r + q)(f) \geq 0 \quad \text{when} \quad \int_M f = 0.$$

Taking for $f$ an eigenfunction of $L_r - q$ belonging to $\lambda_1^{L_r - q}$, we obtain

$$\lambda_1^{L_r - q} \int_M f^2 \geq 0,$$

and thus

\begin{equation}
\lambda_1^{L_r - q} \geq 0. \tag{7}
\end{equation}

Also, if $H_{r+1} > 0$, then $H_j > 0$ for all $j = 1, \ldots, r$ (see [B-C, Proposition 3.1]) and $H_r \geq H_{r+1}^{r+1}$ (see [M-R, Lemma 1]). Hence,

\begin{equation}
\max H_{r+1}^2 \leq \frac{H_r^2}{H_{r+1}^{r+1}} = H_{r+1}^{\frac{r+1}{r+1}}. \tag{8}
\end{equation}

Since $H_1 \geq H_{r+1}^{\frac{1}{r+1}}$, it follows that

\begin{equation}
H_{r+1}^{\frac{r+2}{r+1}} - H_1 H_{r+1} \leq 0, \tag{9}
\end{equation}

with equality at umbilical points.

By (7), (8), (9) and Theorem 1.3, we conclude that $H_{r+1}^{\frac{r+2}{r+1}} - H_1 H_{r+1} = 0$ everywhere, and so $M$ is a geodesic sphere. \hfill \Box

References


Hilário Alencar, Departamento de Matemática, Universidade Federal de Alagoas, 57072-900, Maceió - AL, Brazil

E-mail address: hilario@mat.ufal.br

Manfredo do Carmo, Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, 22460-320, Rio de Janeiro - RJ, Brazil

E-mail address: manfredo@impa.br

Fernando Marques, Department of Mathematics, Cornell University, Ithaca, NY 14853-4089, USA

E-mail address: marques@polygon.math.cornell.edu