Integral Formulas for the r-Mean Curvature Linearized Operator of a Hypersurface

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Abstract. For a normal variation of a hypersurface M^n in a space form Q_c^{n+1} by a normal vector field fN, R. Reilly proved:

$$\frac{d}{dt}\, S_{r+1}(t)\Big|_{t=0} = L_r f + (S_1 S_{r+1} - (r+2)S_{r+2})f + c(n-r)S_r f,$$

where L_r $(0 \leq r \leq n-1)$ is the linearized operator of the $(r+1)$-mean curvature S_{r+1} of M^n given by $L_r = \mathrm{div}(P_r\nabla)$; that is, $L_r =$ the divergence of the rth Newton transformation P_r of the second fundamental form applied to the gradient ∇, and $L_0 = \Delta$ the Laplacian of M^n.

From the Dirichlet integral formula for L_r,

$$\int_{M^n} (f\, L_r g + \langle P_r \nabla f, \nabla g\rangle) = 0,$$

new integral formulas are obtained by making different choices of f and g, generalizing known formulas for the Laplacian. The method gives a systematic process for proofs and a unified treatment for some Minkowski type formulas, via L_r.

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1. Introduction

Let $x\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a compact oriented Riemannian manifold M^n into the Euclidean space $\mathbb{R}^{n+1}$ with inner product $\langle\ ,\ \rangle$ and volume element dM. The Dirichlet integral formula for the Laplacian Δ of M^n,

$$\int_{M^n} (f\,\Delta g + \langle \nabla f, \nabla g\rangle)dM = 0,$$

gives rise to useful integral formulas for conveniently chosen functions f and g on M^n. For example, if $f = 1$ and $g = \langle x, x\rangle/2$ we obtain the Minkowski formula

$$\int_{M^n} (1 + H_1(\langle x, N\rangle)dM = 0,$$

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where N a unit normal vector field on M^n and H_1 is the normalized mean curvature of x given by $H_1 = (1/n)S_1$ with

$$S_1 = \sum_{i=1}^{n} \lambda_i$$

and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the second fundamental form B of x. Here $\lambda_i = \langle \nabla_{e_i} e_i, N \rangle$, $1 \le i \le n$, where $e_1, \ldots, e_n$ are the corresponding eigenvectors and ∇ is the covariant derivative of the ambient space (see Section 2).

If $x\colon M^n \to Q_c^{n+1}$ is an isometric immersion into a simply connected space form Q_c^{n+1}, that is, $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ with curvature $c = 0$, $c > 0$ and $c < 0$, respectively, let X_t be a normal variation of x and $S_1(t)$ the mean curvature of $X_t(M^n)$. It is known that

$$\frac{d}{dt}\, S_1(t)|_{t=0} = \Delta f + |B|^2\, f + cn\, f,$$

where $f = \langle \partial X_t/\partial t|_{t=0}, N\rangle$. This shows that the Laplacian is the linearized operator of S_1 arising from normal variations of x. For the r-mean curvature of x given by

$$S_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \ldots \lambda_{i_r}, \qquad 1 \le r \le n,$$

Reilly [20] proved that

$$\frac{d}{dt}\, S_{r+1}(t)|_{t=0} = L_r f + (S_1 S_{r+1} - (r+2) S_{r+2}) f + c(n-r) S_r f,$$

where L_r is the linearized operator of S_{r+1} arising from normal variations of x given by

$$L_r f = \mathrm{div}(P_r \nabla f)$$

and $S_0 = 1$. Here ∇f and div are, respectively, the gradient of f and the divergence operator on M^n and P_r is the rth Newton transformation, a polynomial in the second fundamental form B of x defined inductively by

$$P_0 \;=\; I,$$
$$P_r \;=\; S_r I - B P_{r-1}.$$

It follows that B and P_r have the same eigenvectors and each eigenvalue of P_r is the partial derivative of S_{r+1} with respect to the corresponding eigenvalue of B (see Section 3). The Dirichlet integral formula for L_r is then

$$\int_{M^n} (f\, L_r g + \langle P_r \nabla f, \nabla g\rangle) dM = 0, \tag{1.1}$$

where f and g are functions on M^n.

Denote by $\operatorname{grad} s$ the gradient of the distance function $s(\cdot) = d(\cdot, p_0)$ in Q_c^{n+1}, where $p_0 \in Q_c^{n+1}$ is a fixed point. Note that s is differentiable, except at p_0 and $-p_0$ for $c > 0$. Define the position vector X of M^n in Q_c^{n+1}, with respect to p_0, by

$$X = S_c(s) \operatorname{grad} s,$$

with $S_c(s) = s$, $\sin(s\sqrt{c})/\sqrt{c}$ or $\sinh(s\sqrt{-c})/\sqrt{-c}$, according to $c = 0$, $c > 0$ or $c < 0$ [2]. Note that for $c = 0$, we have $X = x$. Denote $\theta_c(s) = (d/ds)S_c(s)$ and $X^T =$ the component of X tangent to M^n.

We will prove the following

THEOREM 1. *Let $x\colon M^n \to Q_c^{n+1}$ be an isometric immersion of a compact oriented Riemannian manifold M^n and $0 \le p \le n$, $1 \le q \le n$ integers. Then, for any c,*

(a)
$$\int_{M^n} \Bigg(\langle X,N\rangle^p \Bigg\{ \left(\frac{\langle X,X\rangle}{2}\right)^{q-1} \Big[\theta_c((n-r)S_r\theta_c + (r+1)S_{r+1}\langle S,N\rangle) - \frac{c}{n}|X^T|^2(n-r)S_r\Big] + \frac{(q-1)}{n}\left\langle \frac{X,X}{2}\right\rangle^{q-2} \theta_c^2 |X^T|^2(n-r)S_r \Bigg\} - \frac{p}{n}\langle X,N\rangle^{p-1}\left(\frac{\langle X,X\rangle}{2}\right)^{q-1}\theta_c|X^T|^2(r+1)S_{r+1}\Bigg) dM = 0;$$

(b)
$$\int_{M^n} \Bigg(\langle X,N\rangle^p \Big\{\theta_c^{q-1}[(n-r)S_r\theta_c + (r+1)S_{r+1}\langle S,N\rangle] - \frac{c}{n}(q-1)\theta_c^{q-2}|X^T|^2(n-r)S_r\Big\} - \frac{p}{n}\theta_c^{q-1}\langle X,N\rangle^{p-1}|X^T|^2(r+1)S_{r+1}\Bigg) dM = 0;$$

(c)
$$\int_{M^n} \Bigg(\left(\frac{\langle X,X\rangle}{2}\right)^p \Big\{ \langle X,N\rangle^{q-1}[-(r+1)S_{r+1}\theta_c - (S_1S_{r+1} - (r+2)S_{r+2})\langle X,N\rangle - \langle(\nabla S_{r+1})^T, X\rangle] + \frac{(q-1)}{n}\langle X,N\rangle^{q-2}|X^T|^2(S_1S_{r+1} - (r+2)S_{r+2})\Big\} - \frac{p}{n}\left(\frac{\langle X,X\rangle}{2}\right)^{p-1}\langle X,N\rangle^{q-1}\theta_c|X^T|^2(r+1)S_{r+1}\Bigg) dM = 0.$$

These formulas are obtained choosing first $f = \langle X,N\rangle^p$ and $g = (\langle X,X\rangle/2)^q$ in (1.1), for (a); then, we choose $f = \langle X,N\rangle^p$ and $g = \theta_c^q$ in (1.1) to obtain (b) for

$c \neq 0$, and if $c = 0$, (b) comes from (a) with $q = 1$; finally, take $f = (\langle X, X\rangle/2)^p$ and $g = \langle X, N\rangle^q$ in (1.1) to prove (c).

The formulas in (a) and (b) generalize Minkowski formulas. In fact, if $c = 0$, $p = 0$ and $q = 1$ in (a) we obtain

$$\int_{M^n} (H_r + H_{r+1}\langle X, N\rangle) dM = 0, \tag{1.2}$$

proved by Hsiung in [11], where H_r is the normalized r-mean curvature given by $H_r = S_r / \binom{n}{r}$. For $p = 0$ and $q = 1$, (b) gives, for any c,

$$\int_{M^n} (H_r\theta_c + H_{r+1}\langle X, N\rangle) dM = 0, \tag{1.3}$$

which yields a Minkowski formula in S^{n+1} and H^{n+1} first proved by Bivens [5] (see also [7, 10, 14]). By taking $c = 0$ and $q = 1$ in (a) we obtain a formula proved by Shahin [21] and Gardner [9, eq. (2.7)], which has been proved for $n = 2$ by Chern in [6]. For $c = 0$, similar formulas to (a) were proved in [22].

Thus, Theorem 1 generalizes all these formulas offering a systematic process for the proofs. In fact, our method gives a unified treatment for some Minkowski type formulas via the $(r + 1)$-mean curvature linearized operator L_r of a hypersurface in a space form.

As an application of (b) with $p = 0$ and $q = 1$ we will prove the following

THEOREM 2. *Let M^n be a compact oriented Riemannian manifold and $x\colon M^n \to Q_c^{n+1}$ an isometric immersion with constant $(r + 1)$-mean curvature H_{r+1}, $0 \leq r \leq n - 1$. If $c > 0$ assume that $x(M^n)$ is contained in an open hemisphere of Q_c^{n+1}. Then, the set of points*

$$W = Q_c^{n+1} - \bigcup_{p \in M} (Q_c^n)_p$$

which are omitted by the totally geodesic hypersurfaces $(Q_c^n)_p$ tangent to $x(M^n)$ is non-empty if and only if $x(M^n)$ is a geodesic sphere.

For $r = 0$, this fact was proved by Alencar and Frensel in [2]. The condition that W is non-empty in Theorem 2 is equivalent to r-stability of compact hypersurfaces with H_{r+1} constant in Q_c^{n+1}; for the definitions of r-stability, see Section 5. There are several papers containing some generalization of Minkowski type formulas, for example [12, 13, 18, 22]. We would like to thank Udo Simon for bringing to our attention the work by Kohlman [14] and Simon [22].

2. Preliminaries

Let Q_c^{n+1} be a simply connected space form of constant curvature c. For $c = 0$, it is the Euclidean space $\mathbb{R}^{n+1}$. We assume that for $c > 0$, Q_c^{n+1} is the $(n+1)$-sphere with radius $1/\sqrt{c}$ in $\mathbb{R}^{n+2}$ and for $c < 0$, Q_c^{n+1} is the hyperbolic model $\mathbb{H}^{n+1}(c)$ in $\mathbb{R}^{n+2}$.

Let $x: M \to Q_c^{n+1}$ be an isometric immersion of an n-dimensional oriented Riemannian manifold M^n. Let X be the position vector of M^n with origin at $p_0 \in Q_c^{n+1}$, defined in the Introduction. By analogy with the Euclidean case, for a unit normal vector field N we call $\langle X, N\rangle$ the support function of the immersion from the point p_0.

To fix notation, we let ∇ be the covariant derivative in Q_c^{n+1} and B the second fundamental form of x whose matrix with respect to an orthonormal basis $e_1, \ldots, e_n$ is given by

$$h_{ij} = \langle \nabla_{e_i} e_j, N\rangle \,.$$

Fix a point $p_0 \in Q_c^{n+1}$ and consider the distance function $s(\cdot) = d(p_0, \cdot)$ in $Q_c^{n+1}(Q_c^{n+1} - \{p_0, -p_0\}$ for $c > 0)$. Let $e_1, \ldots, e_n$ be an orthonormal local basis on M^n. Then

$$\nabla_{e_i} \operatorname{grad} s = \frac{\theta_c}{S_c}\,(e_i - \langle \operatorname{grad} s, e_i\rangle \operatorname{grad} s). \tag{2.1}$$

In fact, if we decompose $e_i = \langle \operatorname{grad} s, e_i\rangle \operatorname{grad} s + v_i$, where v_i is in the plane spanned by e_i and $\operatorname{grad} s$, then

$$\nabla_{e_i} \operatorname{grad} s = \langle \operatorname{grad} s, e_i\rangle \nabla_{\operatorname{grad} s} \operatorname{grad} s + \nabla_{v_i} \operatorname{grad} s = \frac{\theta_c}{S_c}\, v_i.$$

In the last equality we used that v_i is tangent to a geodesic circle of radius s in Q_c^{n+1} whose geodesic curvature is θ_c/S_c.

From (2.1) we get

$$\nabla_{e_j} X = \theta_c[\operatorname{grad} s\,\langle \operatorname{grad} s, e_j\rangle + e_j - \langle \operatorname{grad} s, e_j\rangle \operatorname{grad} s] = \theta_c e_j. \tag{2.2}$$

Hence

$$\nabla_{e_i} \nabla_{e_j} X = \theta_c h_{ij} N + \sum_k \langle \nabla_{e_i} e_j, e_k\rangle\, e_k - c\,\langle X, e_i\rangle\, e_j,$$

where (h_{ij}) is the matrix of B with respect to e_i. For a geodesic frame $e_1, \ldots, e_n$ at a point of $\mathbb{R}^{n+1}$ this becomes

$$\nabla_{e_i} \nabla_{e_j} X = \theta_c h_{ij} N - c\,\langle X, e_i\rangle\, e_j. \tag{2.3}$$

For the unit normal vector field N and geodesic frame $e_1, \ldots, e_n$ we have

$$\begin{aligned} \nabla_{e_i} \nabla_{e_j} N &= \nabla_{e_i} \left(-\sum_k h_{jk} e_k \right) \\ &= -\sum_k (\nabla_{e_i} h_{jk}) e_k - h_{ij}^2\, N. \end{aligned}$$

Therefore, by (2.2) and the Codazzi equations we get

$$\nabla_{e_i}\nabla_{e_j}\langle X,N\rangle=-\theta_c h_{ij}-\sum_k(\nabla_{e_k}h_{ij})\langle X,e_k\rangle-h_{ij}^2\langle X,N\rangle. \tag{2.4}$$

3. The Operator L_r

Let $x\colon M\to Q_c^{n+1}$ be an isometric immersion of a Riemannian manifold M^n with second fundamental form B and eigenvalues $\lambda_1,\dots,\lambda_n$. The elementary symmetric functions S_r associated to B are defined by

$$S_r=\sum_{i_1<\cdots<i_r}\lambda_{i_1}\dots\lambda_{i_r},$$

and the r-mean curvature

$$H_r=\left(1/\binom{n}{r}\right)S_r. \tag{3.1}$$

Set $S_0=H_0=1$ and $S_r=H_r=0$ if $r\notin\{0,1,\dots,n\}$. The rth Newton transformation P_r is defined, inductively, by

$$P_0=I,$$

$$P_r=S_r I-B\,P_{r-1}.$$

Since P_r is a polynomial in B, we have that $BP_r=P_rB$ and B and P_r are simultaneously diagonalizable. If $\lambda_1,\dots,\lambda_n$ are eigenvalues of B, then the eigenvalues of P_r are the partial derivatives of $S_{r+1}=S_{r+1}(\lambda_1,\dots,\lambda_n)$ with respect to $\lambda_1,\dots,\lambda_n$, denoted by $S_r(B_1),\dots,S_r(B_n)$; that is,

$$S_r(B_j)=S_r(\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n),$$

the r-elementary symmetric function associated to the restriction B_j of B to the subspace orthogonal to the corresponding eigenvector e_j. Associated to P_r we have a second order differential operator L_r defined by

$$L_r f=\text{ trace }(P_r\text{ Hess }(f)), \tag{3.2}$$

where Hess (f) is the Hessian matrix of the function $f\colon M^n\to\mathbb{R}$. It follows that

$$L_r f=\text{div}(P_r\nabla f),$$

where ∇f is the gradient of f and div is the the divergence operator on M^n [17].

LEMMA 1. *Let $x\colon M^n\to Q_c^{n+1}$ be an isometric immersion of an n-dimensional oriented Riemannian manifold M^n into a space form Q_c^{n+1}. Then,*

(a) $L_r\theta_c=-c[(n-r)S_r\theta_c+(r+1)\langle X,N\rangle S_{r+1}]$, *if* $c\neq 0$;

(b) $\dfrac{1}{2}L_r|X|^2=\theta_c[(n-r)S_r\theta_c+(r+1)S_{r+1}\langle X,N\rangle]-\dfrac{c}{n}|X^T|^2(n-r)S_r$, *for any* c.

Proof. A direct computation with a geodesic frame $e_1, \ldots, e_n$ gives that

$$\nabla_{e_i} \nabla_{e_j} \theta_c = -c(\theta_c \delta_{ij} + h_{ij} \langle X, N \rangle).$$

Hence

$$\begin{aligned} L_r \theta_c &= -c\Big[\sum_{ij} (S_r \delta_{ij} - \cdots + (-1)^r h_{ij}^r)(\theta_c \delta_{ij} + h_{ij} \langle X, N \rangle)\Big] \\ &= -c[\theta_c \text{ trace } P_r + \langle X, N \rangle \text{ trace } (B\, P_r)] \\ &= -c[\theta_c (n-r) S_r + (r+1) \langle X, N \rangle S_{r+1}]. \end{aligned}$$

In the last equality we have used that

$$\text{trace } P_r = (n-r) S_r \tag{3.3}$$

and

$$(r+1) S_{r+1} = \text{trace } (B\, P_r), \tag{3.4}$$

which are proved in [3, lemma 2.1]. This proves (a).

To prove (b) we will use (2.2) and (2.3) to obtain

$$\begin{aligned} \nabla_{e_i} \nabla_{e_j} \langle X, X \rangle &= 2 \nabla_{e_i} \langle \nabla_{e_j} X, X \rangle \\ &= 2 \langle \nabla_{e_j} \nabla_{e_i} X, X \rangle + 2 \langle \nabla_{e_j} X, \nabla_{e_i} X \rangle \\ &= 2 \theta_c h_{ij} \langle X, N \rangle + 2 \theta_c^2 \delta_{ij} - 2c \langle X, e_i \rangle \langle X, e_j \rangle. \end{aligned}$$

Hence, by (3.2) we get

$$\begin{aligned} \frac{1}{2} L_r |X|^2 &= \frac{1}{2} \text{ trace } (P_r \nabla_{e_i} \nabla_{e_j} |X|^2) \\ &= \text{trace } (h_{ij} P_r \langle X, N \rangle) \theta_c + \text{ trace } (P_r) \theta_c^2 \\ &\quad - c \text{ trace } (\langle X, e_i \rangle \langle X, e_j \rangle P_r) \\ &= (n-r) S_r \theta_c^2 + (r+1) S_{r+1} \langle X, N \rangle \theta_c - \frac{c}{n} |X^T|^2 (n-r) S_r, \end{aligned}$$

by (3.3) and (3.4), if we choose $e_1, \ldots, e_n$ such that, at a point, $\langle X, e_i \rangle = \langle X, e_j \rangle$, $\forall\, i, j$. This finishes the proof of Lemma 1. □

LEMMA 2. *Let $x \colon M^n \to Q_c^{n+1}$ be an isometric immersion of an oriented Riemannian manifold M^n into a space form Q_c^{n+1}. Then*

$$\begin{aligned} L_r \langle X, N \rangle &= -(r+1) S_{r+1} \theta_c \\ &\quad - (S_1 S_{r+1} - (r+2) S_{r+2}) \langle X, N \rangle - \langle (\nabla S_{r+1})^T, X \rangle. \end{aligned}$$

Proof. By (3.2) and (2.4) we have

$$\begin{aligned} L_r \langle X, N\rangle &= \text{trace } (P_r \text{ Hess } \langle X, N\rangle) \\ &= -\theta_c \text{ trace } (P_r(h_{ij})) - \text{trace } \left(P_r\left(\sum_k \nabla_{e_k} h_{ij} \langle X, e_k\rangle\right)\right) \\ &\quad - \text{trace } (P_r(h_{ij}^2) \langle X, N\rangle). \end{aligned}$$

By using (3.3), (3.4) and the fact that

$$\text{trace } (P_r B^2) = S_1 S_{r+1} - (r+2) S_{r+2}$$

[3, lemma 2.1] one obtains

$$\begin{aligned} L_r \langle X, N\rangle &= -\theta_c (r+1) S_{r+1} - (S_1 S_{r+1} - (r+2) S_{r+2}) \langle X, N\rangle \\ &\quad - \sum_k \text{ trace } (\nabla_{e_k} h_{ij} P_r) \langle X, e_k\rangle \,. \end{aligned}$$

We claim that

$$\text{trace } \left(\sum_k (\nabla_{e_k} h_{ij} P_r) \langle X, e_k\rangle\right) = \langle (\nabla S_{r+1})^T, X\rangle.$$

In fact, by lemma A, (a) in [19] we have

$$\begin{aligned} r \nabla_{e_k} S_{r+1} &= \sum_{ij} h_{ij} (\nabla_{e_k} (P_r)_{ij}) \\ &= \sum_j \lambda_j (\nabla_{e_k} S_r(B_j)) \\ &= \sum_j \nabla_{e_k} (\lambda_j S_r(B_j)) - \sum_j \nabla e_k \lambda_j (S_r(B_j)), \end{aligned}$$

where λ_j and $S_r(B_j)$ are the eigenvalues of B and P_r, respectively.
On the other hand, by (3.4),

$$\nabla_{e_k} \text{ trace } (B P_r) = (r+1) \nabla_{e_k} S_{r+1}.$$

Hence,

$$\begin{aligned} r \nabla_{e_k} S_{r+1} &= \sum_j \nabla_{e_k} (\lambda_j S_r(B_j)) - \sum_j \nabla_{e_k} \lambda_j (S_r(B_j)) \\ &= \nabla_{e_k} \text{ trace } (B P_r) - \sum_j \nabla_{e_k} \lambda_j (S_r(B_j)) \\ &= (r+1) \nabla_{e_k} S_{r+1} - \text{ trace } \nabla_{e_k} h_{ij} ((P_r)_{ij}). \end{aligned}$$

This yields

$$\nabla_{e_k} S_{r+1} = \text{trace } \nabla_{e_k} h_{ij}((P_r)_{ij})$$

and so,

$$\left\langle (\nabla S_{r+1})^T, X \right\rangle = \text{ trace } \sum_k \nabla_{e_k} h_{ij}((P_r)_{ij}) \langle X, e_k \rangle ,$$

proving the claim. By substituting this in the last expression of $L_r \langle X, N \rangle$ above, we finish the proof of the lemma. □

For any differentiable functions f and g on M^n, the operator L_r satisfies

$$L_r\, fg = f\, L_r\, g + g\, L_r\, f + 2 \langle P_r \nabla f, \nabla g \rangle \tag{3.5}$$

and, if M^n is compact,

$$\int_{M^n} (f\, L_r\, g) dM = \int_{M^n} (g\, L_r\, f) dM, \tag{3.6}$$

(see [17]). Hence,

$$\int_{M^n} (f\, L_r\, g + \langle P_r \nabla f, \nabla g \rangle) dM = 0. \tag{3.7}$$

We will also need the formula

$$L_r\, f^p = p(f^{p-1}\, L_r\, f + \langle P_r\, \nabla f, \nabla f^{p-1} \rangle), \tag{3.8}$$

for any positive integer p.

The most striking property of L_r is that when M^n is compact (for $c > 0$ assume further that $x(M^n)$ is contained in an open hemisphere) and $S_{r+1} > 0$, the operator L_r is elliptic [14] (see also [3]).

4. Proofs of the Integral Formulas for L_r

Here we will prove Theorem 1. First, we need to compute $L_r\, (\langle X, X \rangle/2)^q$, $L_r \langle X, N \rangle^p$ and $L_r\, \theta_c^q$. Since

$$\nabla \left(\frac{\langle X, X \rangle}{2} \right) = \theta_c \sum_{i=1}^n \langle e_i, X \rangle\, e_i,$$

we get

$$\begin{aligned} \nabla \left(\frac{\langle X, X \rangle}{2} \right)^q &= q \left(\frac{\langle X, X \rangle}{2} \right)^{q-1} \theta_c \sum_{i=1}^n \langle e_i, X \rangle\, e_i \\ &= q \left(\frac{\langle X, X \rangle}{2} \right)^{q-1} \theta_c X^T. \end{aligned} \tag{4.1}$$

Hence, by (3.8) and Lemma 1,

$$\begin{aligned}
L_r\left(\frac{\langle X,X\rangle}{2}\right)^q &= q\Bigg[\left(\frac{\langle X,X\rangle}{2}\right)^{q-1} L_r\left(\frac{\langle X,X\rangle}{2}\right) \\
&\quad + \left\langle P_r\nabla\left(\frac{\langle X,X\rangle}{2}\right), \nabla\left(\frac{\langle X,X\rangle}{2}\right)^{q-1}\right\rangle\Bigg] \\
&= q\Bigg[\left(\frac{\langle X,X\rangle}{2}\right)^{q-1}\Big\{\theta_c[(n-r)S_r\theta_c + (r+1)S_{r+1}\langle X,N\rangle] \\
&\quad - c\frac{(n-r)}{n}S_r|X^T|^2\Big\} \\
&\quad + (q-1)\left(\frac{\langle X,X\rangle}{2}\right)^{q-2}\theta_c^2\left\langle P_rX^T, X^T\right\rangle\Bigg] \\
&= q\Bigg[\left(\frac{\langle X,X\rangle}{2}\right)^{q-1}\Big\{\theta_c[(n-r)S_r\theta_c + (r+1)S_{r+1}\langle X,N\rangle] \\
&\quad - c(n-r)S_r\frac{|X^T|^2}{n}\Big\} \\
&\quad + (q-1)\left(\frac{\langle X,X\rangle}{2}\right)^{q-2}\theta_c^2\frac{|X^T|^2}{n}(n-r)S_r\Bigg], \qquad (4.2)
\end{aligned}$$

since

$$\begin{aligned}
\left\langle P_rX^T, X^T\right\rangle &= \sum_k \langle e_k, X\rangle^2 \langle P_re_k, e_k\rangle \\
&= \frac{|X^T|^2}{n}(n-r)S_r, \qquad (4.3)
\end{aligned}$$

if we choose $e_1,\ldots,e_n$ such that, at a point,

$$\langle e_j, X\rangle = \langle e_k, X\rangle, \quad \forall\, j,k.$$

Now we compute $L_r(\langle X,N\rangle)^p$. We have

$$\begin{aligned}
\nabla(\langle X,N\rangle)^p &= p(\langle X,N\rangle)^{p-1}\nabla\langle X,N\rangle \\
&= -p(\langle X,N\rangle)^{p-1}\sum_{jk} h_{jk}\langle e_k, X\rangle e_j, \qquad (4.4)
\end{aligned}$$

since

$$\begin{aligned}
\nabla\langle X,N\rangle &= \sum_j(\langle\nabla_jX,N\rangle e_j + \langle X,\nabla_jN\rangle)e_j \\
&= -\sum_{jk} h_{jk}\langle e_k, X\rangle e_j.
\end{aligned}$$

Hence, by (3.8) and Lemma 2,

$$\begin{aligned} L_r \langle X, N\rangle^p \;&=\; p\left[\langle X, N\rangle^{p-1}\; L_r \langle X, N\rangle + \left\langle P_r \nabla \langle X, N\rangle\,, \nabla \langle X, N\rangle^{p-1}\right\rangle\right] \\ &=\; p\Big[\,\langle X, N\rangle^{p-1}\,(-(r+1)S_{r+1}\theta_c \\ &\quad - (S_1 S_{r+1} - (r+2)S_{r+2})\,\langle X, N\rangle - \langle (\nabla S_{r+1})^T, X\rangle) \\ &\quad + (p-1)\,\langle X, N\rangle^{p-2}\,\frac{|X^T|^2}{n}\,(S_1 S_{r+1} - (r+2)S_{r+2})\Big]. \end{aligned} \tag{4.5}$$

We used

$$\begin{aligned} &\langle P_r \nabla \langle X, N\rangle\,, \nabla \langle X, N\rangle^{p-1}\rangle \\ &\quad = (p-1)\,\langle X, N\rangle^{p-2} \sum_{ijk} h_{jk} h_{ik}\,\langle e_k, X\rangle\,\langle e_k, X\rangle\,\langle P_r e_j, e_i\rangle \\ &\quad = (p-1)\,\langle X, N\rangle^{p-2}\,\frac{|X^T|^2}{n} \sum_{ij} h_{ij}^2 (P_r)_{ij} \\ &\quad = (p-1)\,\langle X, N\rangle^{p-2}\,\frac{|X^T|^2}{n}\;\text{trace}\;(B^2 P_r) \\ &\quad = (p-1)\,\langle X, N\rangle^{p-2}\,\frac{|X^T|^2}{n}\,(S_1 S_{r+1} - (r+2)S_{r+2}), \end{aligned}$$

under the hypothesis that, at a point, $\langle e_k, X\rangle = \langle e_j, X\rangle$, $\forall\, j, k$.

To compute $L_r\theta_c$ we use that

$$\nabla\theta_c = -c\,X^T. \tag{4.6}$$

Hence (3.8) and Lemma 1 give that

$$\begin{aligned} L_r\theta_c^q \;&=\; q(\theta_c^{q-1}\,L_r\theta_c + \langle P_r\nabla\theta_c, \nabla\theta_c^{q-1}\rangle) \\ &=\; q(\theta_c^{q-1}[-c((n-r)S_r\theta_c + (r+1)S_{r+1}\,\langle X, N\rangle)] \\ &\quad + (q-1)\theta_c^{q-2}\,\langle P_r\nabla\theta_c, \nabla\theta_c\rangle) \\ &=\; q(\theta_c^{q-1}[-c((n-r)S_r\theta_c + (r+1)S_{r+1}\,\langle X, N\rangle)] \\ &\quad + (q-1)\theta_c^{q-2}\,c^2\langle P_r X^T, X^T\rangle) \\ &=\; q(\theta_c^{q-1}[-c((n-r)S_r\theta_c + (r+1)S_{r+1}\,\langle X, N\rangle)] \\ &\quad + c^2(q-1)\theta_c^{q-2}\frac{|X^T|^2}{n}\,(n-r)S_r). \end{aligned} \tag{4.7}$$

Now we prove Theorem 1.

Proof of (a). Choose $f = \langle X, N\rangle^p$ and $g = \left(\langle X, X\rangle/2\right)^q$ in (3.7) to obtain

$$\int_{M^n} \langle X, N\rangle^p \, L_r \left(\frac{\langle X, X\rangle}{2}\right)^q + \left\langle P_r \nabla \langle X, N\rangle^p , \nabla \left(\frac{\langle X, X\rangle}{2}\right)^q \right\rangle = 0. \qquad (4.8)$$

By (4.1) and (4.4)

$$\begin{aligned} &\left\langle P_r \nabla \langle X, N\rangle^p , \nabla \left(\frac{\langle X, X\rangle}{2}\right)^q \right\rangle \\ &\quad = \left\langle P_r \left[- p \langle X, N\rangle^{p-1} \sum_{jk} h_{jk} \langle e_k, X\rangle e_j \right], q \left(\frac{\langle X, X\rangle}{2}\right)^{q-1} \theta_c X^T \right\rangle \\ &\quad = -pq \langle X, N\rangle^{p-1} \left(\frac{\langle X, X\rangle}{2}\right)^{q-1} \theta_c \sum_{jk} h_{jk} \langle e_k, X\rangle \langle P_r e_j, X^T\rangle \\ &\quad = -pq \langle X, N\rangle^{p-1} \left(\frac{\langle X, X\rangle}{2}\right)^{q-1} \theta_c \frac{|X^T|^2}{n} \sum_{jk} h_{jk} (P_r)_{jk} \\ &\quad = -pq \langle X, N\rangle^{p-1} \left(\frac{\langle X, X\rangle}{2}\right)^{q-1} \theta_c \frac{|X^T|^2}{n} (r+1) S_{r+1}, \end{aligned} \qquad (4.9)$$

if we choose $e_1, \ldots, e_n$ such that, at a point, $\langle e_k, X\rangle = \langle e_j, X\rangle, \forall\, j, k$.

Now we use (4.2) and (4.9) in (4.8) to finish the proof of (a). □

Proof of (b). For $c \neq 0$ choose $f = \langle X, N\rangle^p$ and $g = \theta_c^q$ in (3.7). Then

$$\int_{M^n} \langle X, N\rangle^p \, L_r \theta_c^q + \langle P_r \nabla \langle X, N\rangle^p , \nabla \theta_c^q\rangle = 0. \qquad (4.10)$$

By (4.4) and (4.6) we have

$$\begin{aligned} \langle P_r \nabla \langle X, N\rangle^p , \nabla \theta_c^q\rangle &= cpq\, \theta_c^{q-1} \langle X, N\rangle^{p-1} \sum_{jk} h_{jk} \langle e_k, X\rangle \langle P_r e_j, X^T\rangle \\ &= cpq\, \theta_c^{q-1} \langle X, N\rangle^{p-1} \frac{|X^T|^2}{n} \sum_{jk} h_{jk} \langle P_r\rangle_{jk} \\ &= cpq\, \theta_c^{q-1} \langle X, N\rangle^{p-1} \frac{|X^T|^2}{n} (r+1) S_{r+1}. \end{aligned} \qquad (4.11)$$

Now use (4.7) and (4.11) in (4.10) to conclude the proof of (b) for $c \neq 0$. For $c = 0$, (b) comes from (a) with $q = 1$. □

Proof of (c). Now we choose $f = \left(\langle X, X\rangle/2\right)^p$ and $g = \langle X, N\rangle^q$ in (3.7) to obtain

$$\int_{M^n} \left(\frac{\langle X, X\rangle}{2}\right)^p L_r \langle X, N\rangle^q + \left\langle P_r \nabla \left(\frac{\langle X, X\rangle}{2}\right)^p , \nabla \langle X, N\rangle^q \right\rangle = 0. \qquad (4.12)$$

However, by (4.1) and (4.4) we have

$$\left\langle P_r \nabla \left(\frac{\langle X,X\rangle}{2}\right)^p, \nabla \langle X,N\rangle^q \right\rangle$$

$$= -pq\left(\frac{\langle X,X\rangle}{2}\right)^{p-1} \langle X,N\rangle^{q-1}\ \theta_c \frac{|X^T|^2}{n}\sum_{jk} h_{jk}(P_r)_{jk}$$

$$= -pq\left(\frac{\langle X,X\rangle}{2}\right)^{p-1} \langle X,N\rangle^{q-1}\ \theta_c \frac{|X^T|^2}{n}\,(r+1)S_{r+1}. \tag{4.13}$$

Therefore, (4.5) and (4.13) applied to (4.12) finish the proof of (c). We have thereby finished the proof of Theorem 1. □

COROLLARY. *Under the hypotheses of Theorem 1, if* $1 \le p \le n$*, then*

$$\int_{M^n} \Bigg(\langle X,N\rangle^p \Bigg\{ \left(\frac{\langle X,X\rangle}{2}\right)^{p-1}$$
$$\times \left[\theta_c((n-r)S_r\theta_c + (r+1)S_{r+1}\langle X,N\rangle) - \frac{c}{n}(n-r)S_r|X^T|^2\right]\Bigg\}$$
$$-\left(\frac{\langle X,X\rangle}{2}\right)^p + \Bigg\{ -\langle X,N\rangle^{p-1}\Big[(r+1)S_{r+1}\theta_c$$
$$-(S_1S_{r+1}-(r+2)S_{r+2})\langle X,N\rangle - \langle(\nabla S_{r+1})^T, X\rangle\Big]\Bigg\}$$
$$+\Bigg[\langle X,N\rangle^p\left(\frac{\langle X,X\rangle}{2}\right)^{p-2}\theta_c^2(n-r)S_r$$
$$-\langle X,N\rangle^{p-2}\left(\frac{\langle X,X\rangle}{2}\right)^p(S_1S_{r+1}-(r+2)S_{r+2})\Bigg]\Bigg)dM = 0.$$

Proof. Subtract (c) from (a) in Theorem 1 with $1 \le p = q \le n$. □

We observe that we could obtain the Corollary just using the self-adjointness of L_r given in (3.6) and the expressions of $L_r\langle X,N\rangle^p$ and $L_r\langle X,X/2\rangle^q$ given in (4.5) and (4.2), respectively.

5. Applications

Here we prove Theorem 2 and other facts as applications of integral formulas.

We will use the fact that if M^n is compact and $H_{r+1} > 0$ then

$$H_r \ge H_{r+1}^{r/r+1}, \qquad 1 \le r \le n-1 \tag{5.1}$$

with equality at umbilical points [16, lemma 1].

In the proof of Theorem 2 we will use that $W \neq \emptyset$ if and only if there exists a point $p_0 \in Q_c^{n+1}$ such that $\langle X, N\rangle$ never vanishes. Thus, if $H_{r+1} > 0$, also $H_r > 0$ by (5.1) and, with $p = 0$, $q = 1$ and $r = 0$ in Theorem 1(b) we must have

$$\langle X, N\rangle < 0 \tag{5.2}$$

for $c \leq 0$ since in this case $\theta_c \geq 1$. For $c > 0$, (5.2) also holds if we choose $-p_0$ instead of p_0, if necessary, because if $p_0 \in W$ also $-p_0 \in W$ and the corresponding support functions have opposite signs.

Proof of Theorem 2. We may assume that $H_{r+1} > 0$ by a proper choice of the normal vector N. By using (3.1) and the self-adjointness of L_r, Lemma 2 gives that

$$\int_M \left\{ -(r+2)\binom{n}{r+2} H_{r+2} + n\binom{n}{r+1} H_1 H_{r+1}\right\} \langle X, N\rangle \, dM$$
$$= -(r+1)\binom{n}{r+1} H_{r+1} \int_M \theta_c \, dM. \tag{5.3}$$

On the other hand, by using (5.1) and (5.2) in Theorem 1(b) with $p = 0$ and $q = 1$, we obtain

$$H_{r+1}^{r/r+1} \int_M \theta_c dM \leq \int_M H_r \theta_c dM = -H_{r+1} \int_M \langle X, N\rangle \, dM.$$

Substituting this in (5.3) we obtain

$$\int_M \left\{ -(r+2)\binom{n}{r+2} H_{r+2} + n\binom{n}{r+1} H_1 H_{r+1}\right\} \langle X, N\rangle \, dM$$
$$\geq (r+1)\binom{n}{r+1} H_{r+1}^{(r+2)/r+1)} \int_M \langle X, N\rangle \, dM. \tag{5.4}$$

Now we observe that if we denote

$$c(r) = (n-r)\binom{n}{r},$$

then

$$(r+2)\binom{n}{r+2} = c(r+1),$$

$$n\binom{n}{r+1} = \frac{n}{r+1} c(r)$$

and

$$(r+1)\binom{n}{r+1} = c(r).$$

Multiplying (5.4) by $(r+1)$ and using these equalities we get

$$\int_M \left\{ -(r+1)c(r+1)H_{r+2} + n\,c(r)H_1H_{r+1} - (r+1)c(r)H_{r+1}^{(r+2)/(r+1)} \right\} \langle X, N\rangle\, dM \geq 0.$$

Since $\langle X, N\rangle < 0$ and this integrand is greater or equal to zero, with equality at umbilic points [1, p. 392], the theorem is proved. □

THEOREM 3. Let M^n be a compact oriented Riemannian manifold and let $x\colon M^n \to Q_c^{n+1}$ be an isometric immersion with constant $(r+1)$-mean curvature. If $c > 0$, suppose further that $x(M^n)$ is contained in an open hemisphere. Then, W is non-empty if and only if x is r-stable.

Here r-stable means the following: M^n is a critical point of the functional

$$J_r = \int_{M^n} F_r(S_1, \ldots, S_r)dM + \lambda V$$

for all variations and the second derivative of J_r at M^n,

$$J_r''(f) = -(r+1)\int_{M^n} f\{L_rf + (S_1S_{r+1} - (r+2)S_{r+2})f + c(n-r)S_rf\}dM,$$

is non-negative for every normal variation $X\colon I \times M^n \to Q_c^{n+1}$ of M^n defined by fN satisfying $\int_{M^n} f\,dM = 0$. Here F_r is defined inductively:

$$F_0 = 1$$

$$F_1 = S_1$$

$$F_r = S_r + \frac{c(n-r+1)}{r-1} F_{r-2} \quad \text{for} \quad 2 \leq r \leq n-1,$$

λ is constant and

$$V(t) = \int_{[0,t]\times M^n} X^*\, dQ,$$

with $dQ =$ volume element of Q_c^{n+1}. For $r = 0$, this is the stability defined in [4].

Proof of Theorem 3. By the theorem in [3] if $c \neq 0$ or by theorem 2.1 in [1] if $c = 0$, x is r-stable if and only if $x(M^n)$ is a geodesic sphere. And by Theorem 2 above $x(M^n)$ is a geodesic sphere if and only if W is non-empty, proving the theorem. □

Trivially, a geodesic sphere of center p_0 in Q_c^{n+1} satisfies

$$H_r\theta_c + H_{r+1}\langle X, N\rangle \equiv 0, \quad 0 \leq r \leq n-1,$$

where X is the position vector relative to p_0, since $H_r = (\theta_c/S_c)^r$, if we choose $N = -X/S_c$.

The next theorem establishes the converse. A proof of the Euclidean version is given in [8].

THEOREM 4. *Let* $x\colon M^n \to Q_c^{n+1}$ *be an isometric immersion of a connected, compact and oriented Riemannian manifold* M^n *and* $p_0 \in Q_c^{n+1}$ *relative to which*

$$H_r\theta_c + H_{r+1}\langle X, N\rangle$$

does not change sign for some $0 \le r \le n-1$. *If* $c > 0$ *assume that* $x(M^n)$ *is contained in an open hemisphere of* Q_c^{n+1} *centered at* p_0. *Then* $x(M^n)$ *is a geodesic sphere.*

Proof. From the particular case of Theorem 1 given in (1.3), we obtain, for any c,

$$H_r\theta_c + H_{r+1}\langle X, N\rangle \equiv 0. \tag{5.5}$$

We first prove that $H_{r+1} > 0$. Clearly, for $c \le 0$ we always have that

$$\theta_c > 0. \tag{5.6}$$

If $c > 0$, (5.6) also holds by the hypothesis on p_0.

From the convexity of the ambient space and the compacity of M^n we may choose N to have an open set U where all eigenvalues of the second fundamental form of x are positive. Hence, $H_{r+1} > 0$ on U and we assume that it is the largest subset of M^n with such a property. We will show that $U = M^n$.

By (5.5) and (5.6),

$$\langle X, N\rangle < 0 \quad \text{on} \quad U,$$

since, by (5.1), also $H_r > 0$ on U.

On the other hand, by applying (5.1) to (5.5) we get, on U

$$\begin{aligned} 0 &= H_r\theta_c + H_{r+1}\langle X, N\rangle \\ &\ge H_{r+1}^{r/r+1}\theta_c + H_{r+1}\langle X, N\rangle \\ &= H_{r+1}^{r/r+1}\left(\theta_c + H_{r+1}^{1/r+1}\langle X, N\rangle\right). \end{aligned}$$

Hence

$$\theta_c + H_{r+1}^{1/r+1}\langle X, N\rangle \le 0 \quad \text{on} \quad U.$$

By continuity, also

$$\theta_c + H_{r+1}^{1/r+1}\langle X, N\rangle \le 0$$

on the closure $\overline{U}$ of U in M^n. Since by (5.6) θ_c is positive, also H_{r+1} must be positive on $\overline{U}$. This proves that $U = \overline{U}$ and since M^n is connected we then have $U = M^n$. Therefore, $H_{r+1} > 0$ on M^n.

Now, use (3.1) and (5.5) in Lemma 1 to obtain

$$L_r\theta_c \equiv 0 \text{ on } M^n, \text{ if } c \neq 0$$

and

$$L_r|X|^2 \equiv 0 \text{ on } M^n, \text{ if } c = 0.$$

Because $H_{r+1} > 0$ on M^n, we have that L_r is elliptic. Therefore,

$$\theta_c = \text{ const. on } M^n, \text{ if } c \neq 0$$

and

$$|X|^2 = \text{ const. on } M^n, \text{ if } c = 0.$$

It follows then that in any case $x(M^n)$ is a geodesic hypersphere in Q_c^{n+1}.

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