Hypersurfaces of constant mean curvature with finite index and volume of polynomial growth

By

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1. Introduction.

(1.1) Let $x: M^n \to \bar{M}^{n+1}$ be an isometric immersion of a complete, noncompact, Riemannian n-manifold M^n into an oriented, complete, Riemannian $(n+1)$-manifold. Let $p \in M$ and denote by $B_r(p) \subset M$ the geodesic ball of center p and radius r. We say that the volume of M has polynomial growth if there exist positive numbers α and c such that $\mathrm{vol}(B_r(p)) \leqq cr^\alpha$. We want to prove the following result.

(1.2) Theorem. *Let M^n and $\bar{M}^{n+1}$ be as above and let $x: M^n \to \bar{M}^{n+1}$ have constant mean curvature H. Assume that the volume of M has polynomial growth and that $\mathrm{ind}\, M < \infty$. Then, there exists a constant $r_0 > 0$ such that*

$$H^2 \leqq - \inf_{M - B(r_0)} \overline{\mathrm{Ric}}(N).$$

Here N is a smooth unit normal field along M, $\overline{\mathrm{Ric}}(N)$ is the value of the Ricci curvature of $\bar{M}^{n+1}$ in the vector N, and $\mathrm{ind}\, M$ is defined as follows. Let L be the second-order differential operator on M given by

$$L = \Delta + \|B\|^2 + \overline{\mathrm{Ric}}(N),$$

where Δ is the Laplacian on M and $\|B\|^2$ is the second fundamental form of x. Associated to L is the quadratic form

$$I(f) = - \int_M f\, Lf\, dM,$$

defined on the vector space of functions f on M that have support on a compact domain $K \subset M$. For each such K, define the index $\mathrm{ind}_L K$ of L in K as the maximal dimension of a subspace where I is negative definite. The index $\mathrm{ind}\, M$ of L in M (or simply, the index of M) is then defined by

$$\mathrm{ind}\, M = \sup_{K \subset M} \mathrm{ind}_L K,$$

where the supremum is taken over all compact domains $K \subset M$.

One may also consider the index of the quadratic form I restricted to the subspace made up by those f's that satisfy the condition $\int_M f\,dM = 0$; this will be denoted by $\mathrm{Ind}_0 M$. However, it is easily checked that $\mathrm{Ind}\, M < \infty \Leftrightarrow \mathrm{Ind}_0 M < \infty$, so in the statement of Theorem (1.2) it is immaterial whether one takes $\mathrm{Ind}\, M$ or $\mathrm{Ind}_0 M$.

Theorem (1.2) has many interesting consequences. The Corollary below shows that if the Ricci curvature of $\bar{M}^{n+1}$ is nonnegative, any complete noncompact hypersurface of constant mean curvature with finite index and volume of polynomial growth is minimal.

(1.3) Corollary. *Let $x: M^n \to \bar{M}^{n+1}$ be as in Theorem* (1.2). *Assume, in addition, that $\overline{\mathrm{Ric}} \geqq 0$. Then $H \equiv 0$.*

Proof. Since $\inf\limits_{M - B(r_0)} \overline{\mathrm{Ric}}(N) = \beta \geqq 0$, we obtain from Theorem (1.2) that $H^2 \leqq -\beta$. Thus, $\beta = 0$ and $H \equiv 0$.

Corollary (1.3) should be compared with a similar recent result of Cheung [1]. His proof is entirely different from ours and he needs the following additional hypothesis to obtain the same result: a) x is proper; b) $\bar{M}^{n+1}$ has bounded geometry, in the sense that the sectional curvature is bounded from above and the injectivity radius is bounded from below; c) the growth condition for the volume takes the (slightly stronger) form $\sup\limits_r \frac{\mathrm{vol}(B_p(r))}{r^n} < \infty$.

(1.4) Remark. In the case $\bar{M}^{n+1} = R^{n+1}$, Corollary (1.3) generalizes a theorem of Chern [2] that complete graphs M in R^{n+1} with constant mean curvature are minimal. This follows from the facts that such graphs are strongly stable (i.e., $\mathrm{ind}\, M = 0$) and the volume of M grows polynomially. This also shows that the finiteness of the index and the polynomial growth of $\mathrm{vol}(M)$ are sufficient for the conclusion of Chern's theorem.

(1.5) Corollary. *Let $x: M^n \to \bar{M}^{n+1}$ be as in Theorem* (1.2). *Assume in addition that $\overline{\mathrm{Ric}} \leqq 0$ and that $\inf\limits_{\bar{M}} \overline{\mathrm{Ric}} = -\delta$, $\delta > 0$. Then $H^2 \leqq \delta$; in particular, if $\bar{M}^{n+1}$ is the hyperbolic space $H^{n+1}(-1)$ with constant sectional curvature -1, then $H^2 \leqq 1$.*

(1.6) Remark. The condition that $\mathrm{ind}\, M < \infty$ is certainly necessary for Theorem (1.2) as shown by the examples of the embedded Delaunay surfaces in R^3: they have infinite index and their volumes grow linearly.

2. Proof of Theorem (1.2).

(2.1) Fix a point $p \in M$ and denote by $B(r)$ the geodesic ball in M of center p and radius r. Let $r_0 > 0$ be a constant and denote by $A(r_0, r) = B(r) - B(r_0)$. We recall that the first eigenvalue $\lambda_1(A(r_0, r))$ is defined as the smallest λ that satisfies

(2.2) $$\Delta g + \lambda(A(r_0, r))\, g = 0,$$

for some nonzero function g on M with $g(\partial A) = 0$. The first eigenvalue of $M - B(r_0)$ will be defined by

$$\lambda_1(M - B(r_0)) = \inf_{r > r_0} \lambda_1(A(r_0, r)).$$

The following lemma is essentially due to Cheng and Yau ([3], Corollary 1, p. 345).

(2.3) Lemma. *Let u be a positive smooth function defined on a Riemannian manifold M, and let $r_0 > 0$ be a constant. Then*

$$\lambda_1(M - B(r_0)) \geqq \inf_{M - B(r_0)} \left(-\frac{\Delta u}{u}\right).$$

Proof. From [3], Theorem 4, we have that

$$\inf_{x \in A(r_0, r)} \left\{\frac{\Delta g}{g} - \frac{\Delta u}{u}\right\} < 0,$$

where $g \geqq 0$ is a smooth function on $A = A(r_0, r)$ with $(g(\partial A) = 0$ that satisfies (2.2) for $\lambda = \lambda_1$. Therefore,

$$\inf_{x \in A(r_0, r)} \left\{-\lambda_1(A(r_0, r)) - \frac{\Delta u}{u}\right\} < 0.$$

Thus

$$\lambda_1(A(r_0, r)) > \inf_{x \in A(r_0, r)} \left(-\frac{\Delta u}{u}\right)$$

and, by taking the infimum for $r > r_0$, the lemma follows.

(2.4) Lemma. *Assume that the volume of a Riemannian manifold M has polynomial growth, and let $r_0 > 0$ be a constant. Then*

$$\lambda_1(M - B(r_0)) = 0.$$

Proof. Let r be the distance function from the point $p \in M$. Fix a number $r_1 > r_0$, and define a radial function $f: B(r_1) \to \mathbb{R}$ by

$$f(r) = \begin{cases} = 0, & 0 \leqq r \leqq r_0 \\ = r - r_0, & r_0 \leqq r \leqq r_0 + \frac{1}{8} r_1 \\ = \frac{1}{8} r_1, & r_0 + \frac{1}{8} r_1 \leqq r \leqq \frac{7}{8} r_1 \\ = r_1 - r, & \frac{7}{8} r_1 \leqq r \leqq r_1 . \end{cases}$$

It is well-known that

$$\lambda_1(A(r_0, r_1)) \leqq \frac{\int_{A(r_0, r_1)} |\nabla f|^2 \, dM}{\int_{A(r_0, r_1)} f^2 \, dM}.$$

By using the special form of f, we easily see that

$$\frac{1}{64} r_1^2 \lambda_1(A(r_0, r_1)) \leqq \frac{V(A(r_0, r_1))}{V(A(r_0 + \frac{1}{8}r_1, \frac{7}{8}r_1))} = \frac{V(B(r_1)) - V(B(r_0))}{V(B(\frac{7}{8}r_1)) - V(B(r_0 + \frac{1}{8}r_1))}, \tag{2.5}$$

where by $V(\cdot)$ we mean the volume of the enclosed set.

Now, observe that if $r < s$ then $\lambda_1(A(r_0, r)) > \lambda_1(A(r_0, s))$. Thus, for all sequences $\{r_i\}$, $r_i < r_{i+1}$, $r_i \to \infty$,

$$\lim_{r_i} \lambda_1(A(r_0, r_i))$$

exists. So, by (2.5), if we prove that for some sequence $\{r_i\}$, $r_i \to \infty$, the expression

$$\frac{V(B(r_i)) - V(B(r_0))}{V(B(\frac{7}{8}r_i)) - V(B(r_0 + \frac{1}{8}r_i))}, \quad r_i \to \infty, \tag{2.6}$$

is bounded, then $\lim \lambda_1(A(r_0, r_i)) = 0$ for this sequence, hence for all others. Therefore $\lambda_1(M - B(r_0)) = 0$ and this will prove the Lemma.

To prove this, we use the fact the volume of M has polynomial growth, i.e., there exist positive numbers c and α such that $V(B(r)) \leqq cr^\alpha$. Therefore $V(B(r))/r^\alpha \leqq c$, and we can choose a sequence $\{r_i\}$, $r_i \to \infty$, such that

$$\lim_{r_i} \frac{V(B(r_i))}{r_i^\alpha} = c.$$

Consider this sequence $\{r_i\}$, and notice that

$$\lim_{r_i} \frac{V(B(kr_i))}{r_i^\alpha} = k^\alpha c, \quad k > 0.$$

Therefore,

$$\lim_{r_i \to \infty} \frac{V(B(r_i))}{r_i^\alpha} = c, \quad \lim_{r_i \to \infty} \frac{V(B(r_0))}{r_i^\alpha} = 0,$$

$$\lim_{r_i \to \infty} \frac{V(B(\frac{7}{8}r_i))}{r_i^\alpha} = \left(\frac{7}{8}\right)^\alpha c.$$

Observe now that if we choose r_i so that $r_0 < (\frac{3}{4} - \varepsilon) r_1$, for some $\varepsilon > 0$, we obtain that

$$r_0 + \tfrac{1}{8} r_i < (\tfrac{7}{8} - \varepsilon) r_i, \quad i = 1, \ldots,$$

which implies that

$$V(B(r_0 + \tfrac{1}{8}r_i)) < V(B((\tfrac{7}{8} - \varepsilon) r_i)),$$

hence

$$\frac{V(B(r_0 + \frac{1}{8}r_i))}{r_i^\alpha} \leqq \lim_{r_i \to \infty} \frac{V(B((\frac{7}{8} - \varepsilon) r_i))}{r_i^\alpha} = \left(\frac{7}{8} - \varepsilon\right)^\alpha c.$$

Therefore, there exists a subsequence of $\{r_i\}$, to be denoted again by $\{r_i\}$, such that

$$\lim_{r_i \to \infty} \frac{V(B(r_0 + \frac{1}{8} r_i))}{r_i^\alpha} \leqq \left(\frac{7}{8} - \varepsilon\right)^\alpha c.$$

It follows that, for this subsequence, the limit of (2.6) exists and is finite. This proves our claim and the Lemma.

(2.6) Proof of Theorem 1.2. In [4], Proposition 1, Fischer-Colbrie proved that if $\operatorname{ind} M < \infty$, there exist a compact set K and a positive function u on M such that on $M - K$, u satisfies

$$O = Lu = \Delta u + \|B\|^2 u + n\,\overline{\mathrm{Ric}}(N) u.$$

Let $p \in M$ and let $r_0 > 0$ be such that $K \subset B(r_0)$. By Lemma (2.4), $\lambda_1(M - B(r_0)) = 0$, and by Lemma (2.3),

$$0 = \lambda_1(M - B(r_0)) \geqq \inf_{M - B(r_0)} \left(-\frac{\Delta u}{u}\right) = \inf_{M - B(r_0)} \{\|B\|^2 + n\,\overline{\mathrm{Ric}}(N)\}$$

$$\geqq \inf_{M - B(r_0)} n(H^2 + \overline{\mathrm{Ric}}(N)),$$

since $\|B\|^2 \geqq nH^2$. Because $H = \text{const.}$, the theorem follows.

(2.7) Remark. As it can be seen from the proof, we have proved the following intrinsic result. Let M be a complete, noncompact Riemannian manifold and let $L = \Delta + q$ be an operator on M, where q is a smooth function on M. Assume that the index of L is finite and that the volume of M has polynomial growth. Then there exists $r_0 > 0$ such that $\inf_{M - B(r_0)} q \leqq 0$.

References

[1] L. F. Cheung, A non-existence theorem for stable constant mean curvature hypersurfaces. Manuscripta Math. **70**, 219–226 (1991).

[2] S. S. Chern, On the curvatures of a piece of hypersurface in Euclidean space. Abh. Math. Sem. Univ. Hamburg **29**, 77–91 (1965).

[3] S. Y. Cheng and S. T. Yau, Differential equations on Riemannian manifolds and their geometric applications. Comm. Pure Appl. Math. **28**, 333–354 (1975).

[4] D. Fischer-Colbrie, On complete minimal surfaces with finite Morse index in three-manifolds. Invent. Math. **82**, 121–132 (1985).

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