ASPECTS OF THE GEOMETRY AND TOPOLOGY OF EXPANDING HORIZONS

GREGORY J. GALLOWAY AND ABRAÃO MENDES

Abstract. The aim of this paper is to extend some basic results about marginally outer trapped surfaces to the context of surfaces having general null expansion. Motivated in part by recent work of Chai-Wan, we introduce the notion of $\mathfrak{g}$-stability for a general closed hypersurface Σ in an ambient initial data set and prove that, under natural energy conditions, Σ has positive Yamabe type, that is, Σ admits a metric of positive scalar curvature, provided Σ is $\mathfrak{g}$-stable. Similar results are obtained when Σ is embedded in a spacelike, or null, hypersurface of a spacetime satisfying the dominant energy condition. Conditions implying $\mathfrak{g}$-stability are also discussed. Finally, we obtain a spacetime positive mass theorem for initial data sets with compact boundary Σ of positive null expansion, assuming that the dominant energy condition is sufficiently strict near Σ. This extends recent results of Galloway-Lee and Lee-Lesourd-Unger.

1. Introduction

Marginally outer trapped surfaces are objects of considerable interest at the interface of spacetime geometry and the physics of black holes. The notion of a marginally outer trapped surface (or MOTS for short) was introduced early on in the development of the theory of black holes, in connection with gravitational collapse; see e.g [16]. MOTSs appeared in a more purely mathematical context in the work of R. Schoen and S.-T. Yau [22] concerning the existence of solutions of Jang's equation, in connection with their proof of the positivity of energy. MOTSs may be viewed as spacetime analogues of minimal surfaces in Riemannian geometry and, despite the absence of a variational characterization like that for minimal surfaces, satisfy a number of analogous properties.

MOTSs arise in various situations. For example, cross-sections of the event horizon in stationary black hole spacetimes (such as Schwarzschild and Kerr) are MOTSs. This can be roughly understood in terms of Hawking's area theorem. For dynamically evolving black hole spacetimes, the null geodesic generators of the event horizon have nonnegative expansion towards the future. However, in the steady state (stationary) limit, this expansion goes to zero. In dynamical black hole spacetimes (such as the Vaidya spacetime with null dust source), MOTSs typically occur inside the event horizon (as the boundary of the trapped region within a spacelike slice). In fact, there are general results showing that under appropriate conditions, MOTSs cannot occur outside the event horizon. (See Section 2 for a brief discussion.)

The aim of this note is to extend some basic results about MOTSs (which e.g. apply to stationary black hole spacetimes) to results about surfaces having general (nonzero) null expansion, which may be viewed as applicable to dynamical black hole spacetimes.

Surfaces of prescribed null expansion have recently been considered in [7], in which Riemannian band width estimates are extended to spacetime initial data sets.

Our first three results extend results on the topology of MOTSs [11, 12, 14] to surfaces of general nonnegative null expansion. Definitions of various terms will be given in the next section. The first result is a pure initial data result.

Theorem 1.1. *Let (M, g, K) be an initial data set and Σ be a closed hypersurface in M. If Σ is a $\mathfrak{g}$-stable surface in (M, g, K) with null expansion $\theta^+ = h \in C^\infty(\Sigma)$, $h \geq 0$ and either*

(i) $\mu - |J| \geq 0$ *and* $\tau \leq 0$ *along* Σ, $h \not\equiv 0$, *or*

(ii) $\mu - |J| \geq c_0$ *and* $h\tau \leq c_0$ *along* Σ *for some constant* $c_0 > 0$,

then Σ admits a metric of positive scalar curvature.

As is well-known there are many restrictions on the topology of manifolds that admit a metric of positive scalar curvature (see e.g. [8] for a recent reference).

We note that a result related to Theorem 1.1 was obtained in [7] using a slightly different version of stability, which is well-adapted to constructing surfaces of prescribed null expansion.

The next two results are spacetime results, and only require the (spacetime) dominant energy condition; no strictness is needed. They assume that Σ is a surface in a spacelike, or null, hypersurface, respectively. Physically, we may think of Σ as arising from the intersection of the event horizon (assumed to be sufficiently smooth) with these hypersurfaces.

Theorem 1.2. *Let $(\bar{M}, \bar{g})$ be a spacetime satisfying the dominant energy condition and M be a spacelike hypersurface in $(\bar{M}, \bar{g})$. If Σ is a $\mathfrak{g}$-locally weakly outermost surface in M with constant null expansion $\theta^+ = \theta_0 > 0$, then Σ admits a metric of positive scalar curvature.*

The next result assumes a natural condition on the inner null expansion of Σ.

Theorem 1.3. *Let $(\bar{M}, \bar{g})$ be a spacetime satisfying the dominant energy condition and $\mathcal{N}$ be a null hypersurface in $(\bar{M}, \bar{g})$. If Σ is a $\mathfrak{g}$-stable surface in $\mathcal{N}$ with null expansion scalars $\theta^+ = h \geq 0$, $h \not\equiv 0$, and $\theta^- < 0$, then Σ admits a metric of positive scalar curvature.*

Our final theorem pertains to the spacetime positive mass theorem [9]. Recent results have extended this theorem to initial data sets with boundary [13, 18], where the components are assumed to be weakly outer trapped ($\theta^+ \leq 0$), or weakly inner untrapped ($\theta^- \geq 0$). Here we show that the weakly outer trapped condition can be relaxed a bit, by requiring the dominant energy condition (for initial data sets) to be sufficiently strict near Σ.

Theorem 1.4. *Let (M, g, K) be an n-dimensional, $3 \leq n \leq 7$, complete asymptotically flat initial data set with compact boundary Σ. Assume that (M, g, K) satisfies the DEC* *such that $\mu - |J| \geq c$ on a normal neighborhood $U \cong [0, 2\epsilon] \times \Sigma$ of Σ for some constant $c > 0$. Suppose $\theta_0 := \sup_\Sigma \theta^+$ is positive, where θ^+ is the null expansion of Σ with respect to the normal pointing into M. Then $E \geq |P|$, provided $\theta_0 \leq c\epsilon/(2.18 + \epsilon \|\tau\|_{C^0(U)})$, where (E, P) is the ADM energy-momentum vector of (g, K).*

Proofs are presented in Section 3. In the next section, we introduce concepts pertinent to surfaces of general null expansion.

Acknowledgements. The work of GJG was partially supported by the Simons Foundation, under Award No. 850541. The work of AM was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Brazil (Grants 305710/2020-6, 405468/2021-0, and 309867/2023-1) and the Fundação de Amparo à Pesquisa do Estado de Alagoas - Fapeal, Brazil (Process No. E:60030.0000002254/2022).

2. Preliminaries

All manifolds in this paper are assumed to be smooth, orientable and connected, unless otherwise stated.

An *initial data set* (M, g, K) consists of an n-dimensional manifold M, $n \geq 3$, equipped with a Riemannian metric g and a symmetric $(0, 2)$-tensor K. The main physical example is when (M, g, K) is an initial data set in a spacetime (time-oriented Lorentzian manifold) $(\bar{M}, \bar{g})$, i.e. M is a spacelike hypersurface in $\bar{M}$, with induced metric g and second fundamental form K.

The *local energy density* μ and the *local current density* J of an initial data set (M, g, K) are given by

$$\mu = \frac{1}{2}(S - |K|^2 + (\operatorname{tr} K)^2) \quad \text{and} \quad J = \operatorname{div}(K - (\operatorname{tr} K)g),$$

where S is the scalar curvature of (M, g). When (M, g, K) is a spacetime initial data set, these quantities are given by $\mu = G(u, u)$ and $J = G(u, \cdot)$, where G is the Einstein tensor $G = \operatorname{Ric}_{\bar{M}} - \frac{1}{2} R_{\bar{M}} \bar{g}$. Moreover, the inequality,

$$\mu \geq |J| \tag{2.1}$$

on M is a consequence of the *spacetime dominant energy condition* (spacetime DEC),

$$G(X, Y) \geq 0$$

for all future-pointing causal vectors $X, Y \in T\bar{M}$. The inequality (2.1) plays the role of the dominant energy condition for initial data sets.

Though not strictly necessary, for the purpose of introducing certain concepts, we shall assume (M, g, K) is an initial data set in a spacetime $(\bar{M}, \bar{g})$. (In fact, this can always be arranged, cf. [2].) Then the second fundamental form K of M is given by,

$$K(X, Y) = \bar{g}(\bar{\nabla}_X u, Y)$$

for $X, Y \in TM$, where u is the future-pointing timelike unit normal to M and $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$. The mean curvature $\operatorname{tr} K$ of M in $(\bar{M}, \bar{g})$ is denoted by τ.

Let Σ be a closed embedded hypersurface in M with unit normal ν in M. By convention, we refer to ν as *outward pointing*. The *null expansion scalars* θ^+, θ^- of Σ in M with respect to ν are defined by,

$$\theta^{\pm} = \operatorname{div}_{\Sigma} \ell^{\pm},$$

where ℓ^+, ℓ^- are the future-pointing null normal fields $\ell^{\pm} = u \pm \nu$ along Σ. The *null second fundamental forms* χ^+, χ^- of Σ in $(\bar{M}, \bar{g})$ with respect to ν are defined by,

$$\chi^{\pm}(X, Y) = \bar{g}(\bar{\nabla}_X \ell^{\pm}, Y)$$

for $X, Y \in T\Sigma$. Note, in terms of initial data, $\chi^{\pm} = K|_{\Sigma} \pm A$, where A is the second fundamental form of Σ in M with respect to ν. Also, $\theta^{\pm} = \operatorname{tr} \chi^{\pm}$.

We now introduce a notion of stability for surfaces Σ of prescribed null expansion $\theta^+ = h$, which generalizes in a straightforward way the usual notion of stability of marginally outer trapped surfaces ($\theta^+ = 0$). (As noted in the introduction, a slightly different notion of stability for surfaces of prescribed null expansion was considered in [7]; see also [6].)

For a fixed $\epsilon_0 > 0$ sufficiently small, the map

$$\Psi : [0, \epsilon_0] \times \Sigma \to M, \quad \Psi(t, p) = \exp_p(t\nu(p)),$$

is well defined. Given $u \in C^{\infty}(\Sigma)$ positive with $\|u\|_{C^0} \leq \epsilon_0$, the map

$$\Psi^u : [0, 1] \times \Sigma \to M, \quad \Psi^u(t, p) = \exp_p(tu(p)\nu(p)), \tag{2.2}$$

is also well defined. We denote $\Psi^u(t, \Sigma)$ by Σ_t^u and the null expansion scalars of Σ_t^u by $\theta_u^{\pm}(t)$.

We say that Σ is a $\mathfrak{g}$-*stable surface* ($\mathfrak{g}$ for generalized) if

$$\frac{\partial \theta_u^+}{\partial t}\Big|_{t=0} \geq 0 \text{ for some } u > 0 \text{ with } \|u\|_{C^0} \leq \epsilon_0.$$

It is well known that (see e.g. [1, 3, 17]),

$$\frac{\partial \theta_u^+}{\partial t}\Big|_{t=0} = Lu,$$

where $L : C^{\infty}(\Sigma) \to C^{\infty}(\Sigma)$ is the elliptic operator

$$Lu = -\Delta u + 2\langle X, \nabla u \rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2} h^2 + h\tau \right) u, \tag{2.3}$$

where $h = \theta_u^+(0)$ is the null expansion of $\Sigma_0^u = \Sigma$,

$$Q = \frac{1}{2} R_{\gamma} - (\mu + J(\nu)) - \frac{1}{2} |\chi^+|^2,$$

$\gamma = \langle \, , \, \rangle$ is the induced metric on Σ, R_{γ} is the scalar curvature of γ, and X is the vector field tangent to Σ that is dual to the 1-form $K(\nu, \cdot)|_{\Sigma}$. Moreover, there is a real number λ, called the *principal eigenvalue* of L, satisfying $L\phi = \lambda\phi$ for some positive eigenfunction $\phi \in C^{\infty}(\Sigma)$, such that $\lambda \leq \operatorname{Re}(\mu)$ for any other eigenvalue μ of L. Also, the eigenspace of L associated with λ has dimension 1. By arguments essentially as in [1] (see also [17, Theorem A.10]), one has that Σ is $\mathfrak{g}$-stable if, and only if, $\lambda \geq 0$.

It is of interest to have conditions that imply $\mathfrak{g}$-stability. A basic criterion for $\mathfrak{g}$-stability is obtained by extending the notion of *(locally) weakly outermost* for MOTSs to surfaces of prescribed null expansion. We say that Σ is $\mathfrak{g}$-locally weakly outermost if for some $\epsilon_0 > 0$ sufficiently small, and for every $u \in C^\infty(\Sigma)$ positive with $\|u\|_{C^0} \leq \epsilon_0$, there is no $t \in [0,1]$ such that the inequality $\theta_u^+(t) < h$ holds pointwise with respect to Ψ^u, i.e. such that

$$\theta_u^+(t)(p_{t,u}) < h(p) \quad \text{for all} \quad p \in \Sigma, \tag{2.4}$$

where $p_{t,u} = \Psi^u(t,p)$. If Σ is $\mathfrak{g}$-locally weakly outermost then it is necessarily $\mathfrak{g}$-stable. In fact, if Σ is not $\mathfrak{g}$-stable, then $\lambda < 0$. Let $\phi > 0$ be a principal eigenfunction of L. Without loss of generality, we may assume that $\|\phi\|_{C^0} \leq \epsilon_0$. Therefore,

$$\frac{\partial \theta_\phi^+}{\partial t}\Big|_{t=0} = L\phi = \lambda\phi < 0.$$

Thus, since Σ is compact, $\theta_\phi^+(t) < \theta_\phi^+(0) = h$ for $t > 0$ sufficiently small.

We mention a simple condition that implies the $\mathfrak{g}$-locally weakly outermost condition.

Proposition 2.1. *Let Σ be a closed embedded hypersurface in an initial data set (M,g,K) with null expansion $\theta^+ = h$. Suppose there exists a variation $\Psi^{\hat{u}}$ of Σ, $\|\hat{u}\|_{C^0} \leq \hat{\epsilon}_0$, such that $\theta_{\hat{u}}^+(\hat{t}) \geq h$ pointwise for all $\hat{t} \in [0,1]$. Then Σ is $\mathfrak{g}$-locally weakly outermost.*

Proof. Let Ψ^u, $u > 0$, $\|u\|_{C^0} \leq \epsilon_0$, be any variation of Σ. Choose ϵ_0 sufficiently small so that $\Psi^u([0,1]\times\Sigma) \subset \Psi^{\hat{u}}([0,1]\times\Sigma)$. Suppose for some $t \in [0,1]$, $\theta_u^+(t) < h$ pointwise. By the compactness of Σ_t^u, there exists $\hat{t}$ such that Σ_t^u lies to the inside of $\Sigma_{\hat{t}}^{\hat{u}}$ and so that they meet tangentially at some point $q = \Psi^u(t,p)$. Restricting the size of Σ to a small neighborhood of p, we may assume there exists a constant a such that $\theta_u^+(t) < a < \theta_{\hat{u}}^+(\hat{t})$. But then the maximum principle for null expansion ([2, Prop. 2.4], [4, Prop. 3.1]) would require $\theta_u^+(t) = \hat{\theta}_{\hat{u}}^+(\hat{t})$, a contradiction. □

For example, the CMC spheres $r = r_0$ between the horizon and photon sphere ($2m < r_0 < 3m$) in the totally geodesic ($K = 0$) time slice $t = t_0$ of Schwarzschild spacetime satisfy the conditions of the proposition and hence are $\mathfrak{g}$-locally weakly outermost. At the end of this section, we include further comments about the $\mathfrak{g}$-locally weakly outermost condition.

We now extend the concept of $\mathfrak{g}$-stability for surfaces in initial data sets (e.g. in spacelike hypersurfaces) to surfaces in null hypersurfaces. Let $\mathcal{N}$ be a null hypersurface in a spacetime $(\bar{M},\bar{g})$ and Σ be a closed hypersurface in $\mathcal{N}$ that is spacelike in $(\bar{M},\bar{g})$. Fix a future-pointing null vector field ℓ^- on Σ that is orthogonal to Σ and tangent to $\mathcal{N}$.

As before, for $\epsilon_0 > 0$ sufficiently small, the map

$$F : [0,\epsilon_0]\times\Sigma \to \mathcal{N}, \quad F(t,p) = \exp_p(-t\ell^-(p))$$

is well defined. Now, extend ℓ^- to a neighborhood of Σ in $\mathcal{N}$ by

$$\ell^- = -\frac{\partial F}{\partial t}$$

and define ℓ^+ as the future-pointing null vector field that is normal to $\mathcal{N}$ and satisfies $\langle \ell^+, \ell^- \rangle = -2$. In this case, the *null expansion scalars* θ^+, θ^- of Σ in $\mathcal{N}$ are defined by

$$\theta^\pm = \operatorname{div}_\Sigma \ell^\pm.$$

Clearly, the map

$$F^u : [0,1] \times \Sigma \to \mathcal{N}, \quad F^u(t,p) = \exp_p(-tu(p)\ell^-(p))$$

is also well defined for $u \in C^\infty(\Sigma)$ positive with $\|u\|_{C^0} \leq \epsilon_0$. Denote $F^u(t,\Sigma)$ by Σ_t^u and the null expansion scalars of Σ_t^u in $\mathcal{N}$ by $\theta_u^\pm(t) = \operatorname{div}_{\Sigma_t^u} \ell^\pm$. In this case, from well-known formulas ([1, 3]), we have,

$$\frac{\partial \theta_u^+}{\partial t}\Big|_{t=0} = L_- u,$$

where $L_- : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the elliptic operator

$$L_- u = -\Delta u + 2\langle X, \nabla u\rangle + \left(\frac{1}{2}R_\gamma - G(\ell^+, \ell^-) - |X|^2 + \operatorname{div} X + h\theta^-\right) u, \tag{2.5}$$

where $h = \theta_u^+(0)$ and $\theta^- = \theta_u^-(0)$ are the null expansion scalars of Σ in $\mathcal{N}$.

We say that Σ is a $\mathfrak{g}$-*stable surface* in $\mathcal{N}$ if

$$\frac{\partial \theta_u^+}{\partial t}\Big|_{t=0} \geq 0 \text{ for some } u > 0 \text{ with } \|u\|_{C^0} \leq \epsilon_0.$$

Again, as in the initial data case, one has that Σ is $\mathfrak{g}$-stable if, and only if, $\lambda_1(L_-) \geq 0$, where $\lambda_1(L_-)$ is the principal eigenvalue of L_-. A completely analogous definition of $\mathfrak{g}$-locally weakly outermost holds in this null hypersurface case (just replace Ψ^u by F^u), and again $\mathfrak{g}$-locally weakly outermost implies $\mathfrak{g}$-stability.

In a similar manner, we have the following null hypersurface version of Proposition 2.1.

Proposition 2.2. *Let Σ be a closed embedded hypersurface in a null hypersurface $\mathcal{N}$. Suppose there exists a variation $F^{\hat{u}}$ of Σ, $\|\hat{u}\|_{C^0} \leq \hat{\epsilon}_0$, such that $\theta_{\hat{u}}^+(\hat{t}) \geq h$ pointwise for all $\hat{t} \in [0,1]$. Then Σ is $\mathfrak{g}$-locally weakly outermost.*

The proof is essentially the same as in Proposition 2.1, except a variation of the maximum principle referenced there is needed, one that applies to null, rather than spacelike, hypersurfaces. That the maximum principle for null expansion holds in this case follows from [20, Theorem 2], which is a consequence of the maximum principle for null hypersurfaces [10].

Vaidya spacetime, which may be viewed as a dynamical version of Schwarzschild spacetime, provides a nice illustration of Proposition 2.2. Using formulas in [5] for the Vaidya spacetime in ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ), together with the approximate location of the event horizon H (as analyzed in e.g. [21]), it can be shown for suitable mass functions that spherically symmetric cross-sections of H (which will have constant positive outward null expansion) satisfy the conditions of Proposition 2.2, and hence are $\mathfrak{g}$-locally weakly outermost.

The following will be important for some of our results. It was proved in [11] (based on the main argument in [14]).

Lemma 2.3. *Let (Σ, γ) be a closed Riemannian manifold and $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be a differential operator of the form*

$$Lu = -\Delta u + 2\langle X, \nabla u\rangle + (\mathcal{Q} - |X|^2 + \operatorname{div} X)u,$$

where X is a tangent vector field on Σ and $\mathcal{Q} = \frac{1}{2}R_\gamma - \mathcal{P}$ for some function $\mathcal{P} \geq 0$. If $\lambda_1(L) \geq 0$, then Σ admits a metric of positive scalar curvature, unless $\lambda_1(L) = 0$, $\mathcal{P} \equiv 0$, and (Σ, γ) is Ricci flat.

Further comments on the $\mathfrak{g}$-locally weakly outermost condition. It is a basic result in the theory of black holes that for suitably defined black hole spacetimes satisfying the null energy condition (NEC), $\operatorname{Ric}(X, X) \geq 0$ for all null vectors X, outer trapped surfaces cannot exist outside the event horizon. See, for example, [23, Prop. 12.2.2], which, under natural circumstances, extends to outer trapped (not just trapped) surfaces. This provides a certain rationale for assuming that a MOTS is weakly outermost, and hence stable. The proof is based on the following. Suppose Σ' is an outer trapped surface outside the event horizon. Then one can construct an outgoing future complete null normal geodesic η to Σ' without null focal points. But the assumption $\theta^+ < 0$, together with the NEC and the future completeness of η, imply that η must necessarily contain a focal point.

The result above can be modified in a way that takes into account the presence of matter. Suppose, instead of the NEC, the following holds: along each future complete outward null normal geodesic $s \to \eta(s)$ to Σ', with $\eta'(0) = \ell^+(p)$, $p = \eta(0) \in \Sigma'$,

$$\int_0^\infty \operatorname{Ric}(\eta', \eta')ds \geq \theta_0. \tag{2.6}$$

Then the null expansion θ^+ of Σ' cannot satisfy $\theta^+ < \theta_0$. This follows from e.g. [15, Prop. 2], which otherwise implies, under the condition (2.6), that there is a null focal point along any such η. In the case when matter (e.g. an accretion disk) is present in the vicinity of Σ (interpreted as a cross-section of the event horizon with small but positive outward null expansion), this provides some justification for the assumption that Σ is $\mathfrak{g}$-locally weakly outermost, and hence $\mathfrak{g}$-stable.

3. Proofs

3.1. **Proof of Theorem 1.1.**

It follows from (2.3) that

$$Lu = -\Delta u + 2\langle X, \nabla u\rangle + (\mathcal{Q} - |X|^2 + \operatorname{div} X)u,$$

where

$$\mathcal{Q} = Q - \frac{1}{2}h^2 + h\tau = \frac{1}{2}R_\gamma - \mathcal{P}$$

and

$$\mathcal{P} = \mu + J(\nu) + \frac{1}{2}|\chi^+|^2 + \frac{1}{2}h^2 - h\tau \geq \mu + J(\nu) - h\tau \geq 0,$$

where the last inequality follows from either of the assumptions (i) and (ii), since $\mu + J(\nu) \geq \mu - |J|$. If $P \equiv 0$, then the above inequalities imply that $h\tau = \mu + J(\nu)$, which in turn implies $h \equiv 0$. This contradicts both (i) and (ii), since in the latter case we have that $h\tau = c_0 > 0$. The result then follows from Lemma 2.3

3.2. **Proof of Theorem 1.2.** First, observe that it is not difficult to construct a spacelike hypersurface $\hat{M}$ in $(\bar{M}, \bar{g})$, with $\Sigma \subset \hat{M}$, satisfying the following conditions (see Figure 1):

- $\hat{M}$ meets M tangentially along Σ;
- $\hat{M}$ is in the causal past of M;
- $\hat{M}$ has mean curvature $\hat{\tau} \leq 0$.

If Σ is $\mathfrak{g}$-stable in $\hat{M}$, then we can apply Theorem 1.1 to conclude that Σ admits a metric of positive scalar curvature.

Suppose, by contradiction, that Σ is not $\mathfrak{g}$-stable in $\hat{M}$. In particular, for each outward neighborhood $\hat{U}_0$ of Σ in $\hat{M}$, there exists a hypersurface $\hat{\Sigma}$ whose null expansion $\hat{\theta}^+$ satisfies $\hat{\theta}^+ < \theta_0$ everywhere. Now, let $\mathcal{H}$ be the null hypersurface generated by the future directed outward null geodesics orthogonal to $\hat{\Sigma}$. Taking a smaller outward neighborhood $\hat{U}_0$ if necessary, $\mathcal{H}$ is a smooth null hypersurface that meets M in a closed hypersurface $\tilde{\Sigma}$ contained in U_0. Then, the Raychaudhuri equation for null geodesic congruences together with the null energy condition (which is a consequence of the dominant energy condition) give that the null expansion $\tilde{\theta}^+$ of $\tilde{\Sigma}$ is (pointwise) less than or equal to $\hat{\theta}^+$; hence, $\tilde{\theta}^+ < \theta_0$ everywhere, which contradicts the assumption that Σ is $\mathfrak{g}$-locally weakly outermost in M.

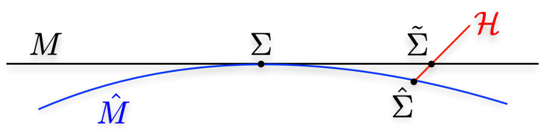

FIGURE 1.

3.3. **Proof of Theorem 1.3.** Similarly to the proof of Theorem 1.1, it follows from (2.5) that

$$L_{-}u = -\Delta u + 2\langle X, \nabla u\rangle + \left(\mathcal{Q} - |X|^2 + \operatorname{div} X\right) u,$$

where

$$\mathcal{Q} = \frac{1}{2} R_\gamma - \mathcal{P}$$

and

$$\mathcal{P} = G(\ell^+, \ell^-) - h\theta^- \geq -h\theta^- \geq 0.$$

Above, we have used the dominant energy condition. Since $h\theta^- \not\equiv 0$, the result follows from Lemma 2.3.

3.4. **Proof of Theorem 1.4.** On $U \cong [0, 2\epsilon] \times \Sigma$, g takes the form $g = dt^2 + \gamma_t$, where γ_t is the induced metric on $\Sigma_t \cong \{t\} \times \Sigma$.

Consider the modified initial data set $(M, g, \hat{K})$, where $\hat{K} = K - \frac{h}{n-1} g$. On U, the function h is defined by

$$h(t) = \begin{cases} \theta_0 \exp\left(1 - \frac{1}{1-(\frac{t}{\epsilon})^2}\right), & 0 \le t < \epsilon, \\ 0, & \epsilon \le t \le 2\epsilon, \end{cases}$$

and $h \equiv 0$ on $M \setminus U$. Clearly, $0 \le h \le h(0) = \theta_0$. Furthermore, simple computations show that $h'(t) \le 0$ and its minimum is attained at $t_0 = 3^{-\frac{1}{4}}\epsilon$ with $h'(t_0) \approx -2.17\,\theta_0/\epsilon$; in particular, $|Dh| \le 2.18\,\theta_0/\epsilon$.

Note that $\hat{\theta}^+ := \mathrm{tr}_\Sigma(\hat{K} + A) = \theta^+ - \theta_0 \le 0$, where A is the second fundamental form of Σ in (M, g) with respect to the normal pointing into M; that is, Σ is weakly outer trapped in $(M, g, \hat{K})$. Further, the DEC clearly holds on $M \setminus U$. On U, a computation shows that (cf. [19, Section 6])

$$\begin{aligned} \hat{\mu} - |\hat{J}| &= \mu + \frac{1}{2}\left(\frac{n}{n-1}h^2 - 2h\tau\right) - |J + Dh| \\ &\ge \mu - |J| + \frac{1}{2}\left(\frac{n}{n-1}h^2 - 2h\tau - 2|Dh|\right) \\ &\ge c - \theta_0\left(\|\tau\|_{C^0(U)} + 2.18\epsilon^{-1}\right) \ge 0. \end{aligned}$$

The theorem then follows from [18, Theorem 1.3].

References

1. Lars Andersson, Marc Mars, and Walter Simon, *Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes*, Adv. Theor. Math. Phys. **12** (2008), no. 4, 853–888. MR 2420905
2. Lars Andersson and Jan Metzger, *The area of horizons and the trapped region*, Comm. Math. Phys. **290** (2009), no. 3, 941–972. MR 2525646
3. ______, *Curvature estimates for stable marginally trapped surfaces*, J. Differential Geom. **84** (2010), no. 2, 231–265. MR 2652461
4. Abhay Ashtekar and Gregory J. Galloway, *Some uniqueness results for dynamical horizons*, Adv. Theor. Math. Phys. **9** (2005), no. 1, 1–30. MR 2193368
5. Abhay Ashtekar and Badri Krishnan, *Dynamical horizons and their properties*, Phys. Rev. D (3) **68** (2003), no. 10, 104030, 25. MR 2071054
6. Xiaoxiang Chai, *Hypersurfaces of prescribed null expansion*, 2021, arXiv:2107.12782.
7. Xiaoxiang Chai and Xueyuan Wan, *Band width estimates of CMC initial data sets*, 2022, arXiv:2206.02624.
8. Otis Chodosh and Chao Li, *Recent results concerning topological obstructions to positive scalar curvature*, Perspectives in scalar curvature. Vol. 2, World Sci. Publ., Hackensack, NJ, [2023] ©2023, pp. 215–230. MR 4577915
9. Michael Eichmair, Lan-Hsuan Huang, Dan A. Lee, and Richard Schoen, *The spacetime positive mass theorem in dimensions less than eight*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 1, 83–121. MR 3438380
10. Gregory J. Galloway, *Maximum principles for null hypersurfaces and null splitting theorems*, Ann. Henri Poincaré **1** (2000), no. 3, 543–567. MR 1777311

11. ______, *Rigidity of marginally trapped surfaces and the topology of black holes*, Comm. Anal. Geom. **16** (2008), no. 1, 217–229. MR 2411473
12. ______, *Rigidity of outermost MOTS: the initial data version*, Gen. Relativity Gravitation **50** (2018), no. 3, Art. 32, 7. MR 3768955
13. Gregory J. Galloway and Dan A. Lee, *A note on the positive mass theorem with boundary*, Lett. Math. Phys. **111** (2021), no. 4, Paper No. 111, 10. MR 4300252
14. Gregory J. Galloway and Richard Schoen, *A generalization of Hawking's black hole topology theorem to higher dimensions*, Comm. Math. Phys. **266** (2006), no. 2, 571–576. MR 2238889
15. Gregory J. Galloway and José M. M. Senovilla, *Singularity theorems based on trapped submanifolds of arbitrary co-dimension*, Classical Quantum Gravity **27** (2010), no. 15, 152002, 10. MR 2659235
16. Stephen W. Hawking and George F. R. Ellis, *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, vol. No. 1, Cambridge University Press, London-New York, 1973. MR 424186
17. Dan A. Lee, *Geometric relativity*, Graduate Studies in Mathematics, vol. 201, American Mathematical Society, Providence, RI, 2019. MR 3970261
18. Dan A. Lee, Martin Lesourd, and Ryan Unger, *Density and positive mass theorems for initial data sets with boundary*, Comm. Math. Phys. **395** (2022), no. 2, 643–677. MR 4487523
19. ______, *Density and positive mass theorems for incomplete manifolds*, Calc. Var. Partial Differential Equations **62** (2023), no. 7, Paper No. 194, 23. MR 4612768
20. Marc Mars, *Stability of marginally outer trapped surfaces and applications*, Recent trends in Lorentzian geometry, Springer Proc. Math. Stat., vol. 26, Springer, New York, 2013, pp. 111–138. MR 3064798
21. Alex B. Nielsen, *The spatial relation between the event horizon and trapping horizon*, Classical Quantum Gravity **27** (2010), no. 24, 245016, 14. MR 2739972
22. Richard Schoen and Shing-Tung Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** (1981), no. 2, 231–260. MR 612249
23. Robert M. Wald, *General relativity*, University of Chicago Press, Chicago, IL, 1984. MR 757180

Department of Mathematics, University of Miami, Coral Gables, FL, USA.
Email address: galloway@math.miami.edu

Instituto de Matemática, Universidade Federal de Alagoas, Maceió, AL, Brazil.
Email address: abraao.mendes@im.ufal.br