SECOND EIGENVALUE ESTIMATES AND SPECTRAL RIGIDITY FOR SUBMANIFOLDS OF COMPACT RANK-ONE SYMMETRIC SPACES

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Abstract. In this paper, we establish upper bounds for the second eigenvalue of the Schrödinger operator $L = \Delta + |\sigma|^2 + k$ on k-dimensional closed submanifolds of the compact projective spaces $\mathbb{F}P^m$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, as well as the Cayley projective plane. The estimates are obtained by combining the standard spherical embeddings of these spaces with a conformal test-function argument. In the complex, quaternionic, and Cayley cases, the resulting bounds involve correction terms that record the position of the tangent spaces of the submanifold relative to the corresponding geometric structures. We also derive universal estimates depending only on the dimension of the submanifold and the underlying division algebra. We show that equality in the sharp estimates forces the submanifold to be totally umbilical. Finally, using known classifications of totally umbilical submanifolds, we compute the second eigenvalue for the standard real, complex, quaternionic, and Cayley models and identify the totally geodesic examples that attain the sharp upper bounds.

1. Introduction

The spectrum of geometric Schrödinger operators often reflects a subtle interaction between the intrinsic geometry of a submanifold and the extrinsic geometry of its immersion. A fundamental example is the Jacobi operator of a two-sided hypersurface,

$$J = \Delta + |\sigma|^2 + \mathrm{Ric}_{\overline{M}}(N, N),$$

which represents the linearization of the mean curvature and governs the second variation of area. Its negative spectrum determines the Morse index, while estimates for its first eigenvalue frequently lead to rigidity and classification results for minimal and constant-mean-curvature submanifolds.

A particularly effective approach to second eigenvalue estimates is based on conformal admissible functions. The starting point is the Hersch's center-of-mass argument [19], subsequently incorporated by Li and Yau into the theory of conformal volume [21]. In this method, an immersion into a sphere is composed with a suitable conformal transformation so that its coordinate functions become orthogonal to a first eigenfunction of the operator under consideration. These coordinate functions can then be used simultaneously in the Rayleigh characterization of the second eigenvalue.

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El Soufi and Ilias developed this method systematically for immersed submanifolds and related the resulting energy estimates to the mean curvature of the immersion; see [14, 15]. In particular, they obtained sharp upper bounds for the second eigenvalue of Schrödinger operators on compact submanifolds of simply connected space forms. For the Jacobi operator, their estimates show that geodesic spheres are the unique extremal models. This extends the Euclidean rigidity theorem of Harrell and Loss [18], according to which the second Jacobi eigenvalue of a closed Euclidean hypersurface is nonpositive and vanishes only for round spheres.

More recently, second Jacobi eigenvalues have been used to characterize special submanifolds in ambient spaces that are not necessarily space forms. The second named author [22] obtained spectral characterizations of the Clifford torus in $\mathbb{S}^3$ and of slices in certain warped products, including $\mathbb{R} \times \mathbb{S}^n$. Further developments relate low Jacobi eigenvalues to topology, Willmore-type quantities, and rigidity of surfaces in higher-dimensional spheres; see [1]. In a complementary direction, the Morse index of minimal hypersurfaces in real projective spaces has received renewed attention; see [2, 12]. These works indicate that the low spectrum contains considerably more geometric information than stability alone.

The purpose of the present paper is to extend this circle of ideas to submanifolds of

$$\mathbb{R}P^m, \qquad \mathbb{C}P^m, \qquad \mathbb{H}P^m,$$

and of the Cayley projective plane $\mathbb{O}P^2$. These are the nonspherical compact rank-one symmetric spaces. In contrast with real projective space, the complex, quaternionic, and Cayley projective spaces do not have constant sectional curvature. Consequently, the space-form estimates cannot be transferred directly without taking into account the complex, quaternionic, or octonionic geometry of the ambient space.

Our starting point is the standard equivariant embedding of a projective space into a unit sphere. Such embeddings belong to the classical theory of minimal immersions of compact symmetric spaces and submanifolds with parallel second fundamental form; see [16, 17, 25]. In the associative cases, they can be written uniformly in terms of rank-one Hermitian projections as

$$\Phi_{\mathbb{F}}([x]) = \sqrt{\frac{m+1}{m}} \left(xx^* - \frac{1}{m+1} I_{m+1} \right), \qquad \mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}.$$

Related descriptions of canonical projective and Grassmannian embeddings can be found in [20, 24]. The Cayley plane is treated through the exceptional Euclidean Jordan algebra $\mathcal{H}_3(\mathbb{O})$.

Our first result gives a common normalization and an explicit formula for the extrinsic geometry of these embeddings. If g_0 denotes the standard projective metric, then

$$\Phi_{\mathbb{F}} : \left(\mathbb{F}P^m, \frac{2(m+1)}{m} g_0 \right) \longrightarrow \mathbb{S}^{N_{\mathbb{F}}(m)}$$

is a minimal isometric embedding with parallel second fundamental form. Moreover, for every unit tangent vector V,

$$|B(V,V)|^2 = \frac{m-1}{m+1}. \tag{1}$$

Remarkably, the right-hand side of (1) depends only on the projective dimension and not on the division algebra.

The main new feature appears when B is traced over an arbitrary real tangent subspace. Let $E \subset T_{[x]}\mathbb{F}P^m$ be a real k-plane. In the complex and quaternionic cases, we introduce the invariant

$$\Theta_{\mathbb{F}}(E) = \begin{cases} 0, & \mathbb{F} = \mathbb{R}, \\ \sum_{i,j=1}^{k} \langle J u_i, u_j\rangle^2, & \mathbb{F} = \mathbb{C}, \\ \sum_{\alpha=1}^{3}\sum_{i,j=1}^{k} \langle J_\alpha u_i, u_j\rangle^2, & \mathbb{F} = \mathbb{H}, \end{cases}$$

where $\{u_1, \ldots, u_k\}$ is a g_0-orthonormal basis of E. This quantity measures the interaction of E with the Kähler or quaternionic Kähler structures. Lemma 2.2 gives the exact identity

$$|\mathrm{tr}_E\, B|^2 = \frac{k(m-k) + m\Theta_{\mathbb{F}}(E)}{m+1}. \tag{2}$$

Thus the correction term that appears below is not an artifact of the proof: it records the Kähler or quaternionic Kähler angles of the tangent spaces. Formula (2) reduces to the usual constant-curvature expression in the real case.

We now state our principal spectral result. We use the convention that $\Delta = \mathrm{div}\,\nabla$ and enumerate the eigenvalues of $-L$ increasingly.

Theorem A. *Let $\iota : \Sigma^k \longrightarrow (\mathbb{F}P^m, g_0)$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, be a closed immersion, and consider $L = \Delta + |\sigma|^2 + k$. Then*

$$\lambda_2(L) \le k + 2 + \frac{2}{k\,\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \Theta_{\mathbb{F}}(T_p\Sigma)\, d\Sigma. \tag{3}$$

Equality in (3) forces Σ to be totally umbilical in $\mathbb{F}P^m$.

Since $0 \le \Theta_{\mathbb{F}}(E) \le (\dim_{\mathbb{R}} \mathbb{F} - 1)k$, Theorem A immediately gives

$$\lambda_2(L) \le \begin{cases} k+2, & \mathbb{F} = \mathbb{R}, \\ k+4, & \mathbb{F} = \mathbb{C}, \\ k+8, & \mathbb{F} = \mathbb{H}. \end{cases}$$

The real-projective estimate is the direct analogue of the sharp space-form results of El Soufi and Ilias. In this case, the equality analysis can be completed by using the spectrum of geodesic spheres: $\lambda_2(L) = k+2$ forces the image of Σ to be totally geodesic. The complex and quaternionic estimates exhibit a phenomenon absent from space forms: the sharp tangent-plane correction depends on the position of $T_p\Sigma$ relative to the ambient special structures.

For the Cayley plane, the same strategy applies through the standard minimal embedding $\Phi_{\mathbb{O}} : \mathbb{O}P^2 \longrightarrow \mathbb{S}^{25}$. Because octonionic multiplication is not associative, it is preferable to retain the invariant $\mathcal{A}_{\mathbb{O}}(E) = |\operatorname{tr}_E B|^2$ rather than impose an artificial analogue of $\Theta_{\mathbb{F}}$. Our second main estimate is the following.

Theorem B. *Let Σ^k be a closed manifold immersed in $(\mathbb{O}P^2, g_0)$, and consider $L = \Delta + |\sigma|^2 + k$. Then*

$$\lambda_2(L) \leq 2k + \frac{3}{k\,\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \mathcal{A}_{\mathbb{O}}(T_p\Sigma)\, d\Sigma.$$

Equality forces Σ to be totally umbilical in $\mathbb{O}P^2$.

The operator considered here is a natural Schrödinger operator for submanifolds of arbitrary codimension. For a hypersurface of $\mathbb{R}P^m$, it agrees with the Jacobi operator, since

$$\mathrm{Ric}_{\mathbb{R}P^m}(N, N) = m - 1 = k.$$

For complex and quaternionic projective spaces, the ambient Ricci term has a different constant value. Corresponding estimates for the genuine Jacobi operator are therefore obtained from (3) by translating the spectrum by the appropriate constant. This distinction is important when comparing our results with the existing literature on Morse index and stability.

The paper is organized as follows. Section 2 develops the standard projective embeddings, computes their induced metrics and second fundamental forms, and establishes equation (2). The Cayley model is also treated there through the exceptional Jordan algebra. Section 3 reviews the conformal center-of-mass argument and the energy estimates used to construct admissible test-functions. In Section 4, we prove Theorems A and B (Theorems 4.1 and 4.5) and examine their equality cases. Finally, Section 5 analyzes the totally umbilical models and computes the second eigenvalue explicitly on the standard real, complex, quaternionic, and Cayley examples.

2. Notation and Background

2.1. The standard projective embeddings.

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and denote

$$d = \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4\}.$$

Also, fix $m \geq 1$ and set $n = m + 1$. We regard $\mathbb{F}^n$ as a right $\mathbb{F}$-vector space, endowed with its standard Hermitian product

$$\langle x, y\rangle_{\mathbb{F}} = x^* y = \sum_{\alpha=1}^{n} \overline{x}_\alpha y_\alpha.$$

Its underlying real inner product is

$$\langle x, y\rangle_{\mathbb{R}} = \mathrm{Re}\langle x, y\rangle_{\mathbb{F}}.$$

Denote by

$$\mathcal{H}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A^* = A\}$$

the real vector space of Hermitian $n \times n$ matrices over $\mathbb{F}$. Notice that $\mathcal{H}_n(\mathbb{F})$ is a real vector space, rather than an $\mathbb{F}$-vector space. Indeed, multiplication of a Hermitian matrix by a nonreal scalar does not, in general, preserve the Hermitian condition.

We equip $\mathcal{H}_n(\mathbb{F})$ with the Euclidean inner product

$$\langle A, B\rangle = \mathrm{Re}\,\mathrm{tr}(AB). \tag{4}$$

Since A and B are Hermitian, the cyclicity of the real part of the trace gives

$$\mathrm{Re}\,\mathrm{tr}(AB) = \mathrm{Re}\,\mathrm{tr}(BA).$$

Moreover,

$$|A|^2 = \mathrm{Re}\,\mathrm{tr}(A^2) = \sum_{\alpha=1}^{n} |a_{\alpha\alpha}|^2 + 2\sum_{\alpha<\beta} |a_{\alpha\beta}|^2,$$

and hence (4) is positive definite.

Let

$$\mathcal{H}_n^0(\mathbb{F}) = \{A \in \mathcal{H}_n(\mathbb{F}) : \mathrm{tr}\, A = 0\}$$

be the trace-free subspace. Because the diagonal entries of a Hermitian matrix are real, the trace defines a real linear map

$$\mathrm{tr} : \mathcal{H}_n(\mathbb{F}) \longrightarrow \mathbb{R}.$$

Consequently, $\mathcal{H}_n^0(\mathbb{F})$ has real codimension one in $\mathcal{H}_n(\mathbb{F})$.

The diagonal part of a Hermitian matrix contributes n real parameters. For every pair $\alpha < \beta$, the entry $a_{\alpha\beta} \in \mathbb{F}$ determines the entry $a_{\beta\alpha} = \overline{a}_{\alpha\beta}$ and contributes d real parameters. Therefore,

$$\dim_{\mathbb{R}} \mathcal{H}_n(\mathbb{F}) = n + d\frac{n(n-1)}{2}.$$

It follows that

$$\dim_{\mathbb{R}} \mathcal{H}_n^0(\mathbb{F}) = n + d\frac{n(n-1)}{2} - 1. \tag{5}$$

Now let $x \in \mathbb{F}^n$ be a unit vector and define $P_x = xx^*$. Explicitly, $(P_x)_{\alpha\beta} = x_\alpha \overline{x}_\beta$. For every $z \in \mathbb{F}^n$, we have $P_x z = x(x^*z) = x\langle x, z\rangle_{\mathbb{F}}$. Thus P_x is the orthogonal projection onto the $\mathbb{F}$-line $x\mathbb{F} = \{xq : q \in \mathbb{F}\}$. In particular,

$$P_x^* = P_x, \qquad P_x^2 = P_x, \qquad \mathrm{tr}\, P_x = |x|^2 = 1.$$

If $q \in \mathbb{F}$ satisfies $|q| = 1$, then

$$P_{xq} = (xq)(xq)^* = xq\overline{q}x^* = xx^* = P_x.$$

Hence P_x depends only on the projective class

$$[x] = x\mathbb{F} \in \mathbb{F}P^m.$$

Conversely, if $P_x = P_y$ for two unit vectors x, y, then their images coincide, so $x\mathbb{F} = y\mathbb{F}$. Therefore,

$$P_x = P_y \quad \Longleftrightarrow \quad [x] = [y].$$

Consequently, the map $[x] \longmapsto P_x$ realizes $\mathbb{F}P^m$ as the set of rank-one Hermitian projections of trace one.

Consider the centered and normalized projection

$$\Phi_{\mathbb{F}}([x]) = c_m \left(P_x - \frac{1}{n} I_n\right), \qquad c_m = \sqrt{\frac{n}{n-1}} = \sqrt{\frac{m+1}{m}}. \tag{6}$$

The matrix $P_x - \frac{1}{n} I_n$ is Hermitian and trace-free, since

$$\operatorname{tr}\left(P_x - \frac{1}{n} I_n\right) = 1 - \frac{1}{n} \operatorname{tr} I_n = 1 - \frac{n}{n} = 0.$$

Thus $\Phi_{\mathbb{F}}$ takes values in $\mathcal{H}_n^0(\mathbb{F})$.

We next compute its norm. Using

$$P_x^2 = P_x, \qquad \operatorname{tr} P_x = 1, \qquad |I_n|^2 = \operatorname{tr} I_n = n,$$

we obtain

$$\begin{aligned}
\left|P_x - \frac{1}{n} I_n\right|^2 &= \operatorname{Re}\operatorname{tr}\left(P_x - \frac{1}{n} I_n\right)^2 \\
&= \operatorname{tr} P_x^2 - \frac{2}{n} \operatorname{tr} P_x + \frac{1}{n^2} \operatorname{tr} I_n \\
&= 1 - \frac{2}{n} + \frac{1}{n} \\
&= \frac{n-1}{n}.
\end{aligned}$$

The choice of c_m in (6) therefore gives

$$|\Phi_{\mathbb{F}}([x])|^2 = \frac{n}{n-1} \frac{n-1}{n} = 1.$$

Hence

$$\Phi_{\mathbb{F}} : \mathbb{F}P^m \longrightarrow \mathbb{S}\big(\mathcal{H}_n^0(\mathbb{F})\big)$$

takes values in the unit sphere of $\mathcal{H}_n^0(\mathbb{F})$. Since the projection P_x uniquely determines the line $[x]$, $\Phi_{\mathbb{F}}$ is injective. The computation of its differential below shows that $\Phi_{\mathbb{F}}$ is an immersion. Because $\mathbb{F}P^m$ is compact, $\Phi_{\mathbb{F}}$ is therefore an embedding.

By (5), the dimension of the ambient unit sphere is

$$N_{\mathbb{F}}(m) = \dim_{\mathbb{R}} \mathcal{H}_n^0(\mathbb{F}) - 1 = n + d\frac{n(n-1)}{2} - 2.$$

Substituting $n = m + 1$, we obtain

projective space	$\dim_{\mathbb{R}} \mathbb{F}P^m$	ambient sphere
$\mathbb{R}P^m$	m	$\mathbb{S}^{\frac{(m+1)(m+2)}{2}-2}$,
$\mathbb{C}P^m$	$2m$	$\mathbb{S}^{(m+1)^2-2}$,
$\mathbb{H}P^m$	$4m$	$\mathbb{S}^{2(m+1)^2-(m+1)-2}$.

We now compute the metric induced by $\Phi_{\mathbb{F}}$. The projective space can be represented as the quotient $\mathbb{F}P^m = \mathbb{S}^{d(m+1)-1}/\mathbb{S}^{d-1}$, where the unit scalars act on the right by $x \longmapsto xq$. At a unit vector x, the vertical space of this quotient is $\mathcal{V}_x = \{x\eta : \eta \in \operatorname{Im}\mathbb{F}\}$. The horizontal space is $\mathcal{H}_x = \{u \in \mathbb{F}^n : x^*u = 0\} = x^\perp$. Consequently, the standard quotient metric g_0 is characterized by the isometric identification $T_{[x]}\mathbb{F}P^m \simeq x^\perp$, where $x^\perp$ is endowed with the underlying real inner product.

Let $u \in x^\perp$ and choose a horizontal curve $x(t)$ in the unit sphere such that $x(0) = x$, $x'(0) = u$. Differentiating $P_{x(t)} = x(t)x(t)^*$ at $t = 0$, we obtain

$$\frac{d}{dt}\bigg|_{t=0} P_{x(t)} = ux^* + xu^*.$$

Therefore,

$$d\Phi_{\mathbb{F}}(u) = c_m(xu^* + ux^*).$$

To compute its norm, set $A = xu^*$, $A^* = ux^*$. Since $x^*u = 0$, we have

$$A^2 = x(u^*x)u^* = 0, \qquad (A^*)^2 = u(x^*u)x^* = 0.$$

Moreover,

$$AA^* = xu^*ux^* = |u|^2 P_x \qquad \text{and} \qquad A^*A = ux^*xu^* = uu^*.$$

Therefore, it follows that

$$\begin{aligned} |xu^* + ux^*|^2 &= \operatorname{Re}\operatorname{tr}\left((A + A^*)^2\right) \\ &= \operatorname{Re}\operatorname{tr}(AA^* + A^*A) \\ &= |u|^2 \operatorname{tr} P_x + \operatorname{tr}(uu^*) \\ &= 2|u|^2. \end{aligned}$$

Consequently,

$$|d\Phi_{\mathbb{F}}(u)|^2 = 2c_m^2|u|^2 = \frac{2(m+1)}{m}|u|^2.$$

In particular, $d\Phi_{\mathbb{F}}$ is injective, which confirms that $\Phi_{\mathbb{F}}$ is an immersion.

Thus the metric induced from the unit sphere is

$$g_\Phi = s_m g_0, \qquad s_m = \frac{2(m+1)}{m}.$$

Equivalently,

$$\Phi_{\mathbb{F}} : \left(\mathbb{F}P^m, \frac{2(m+1)}{m} g_0\right) \longrightarrow \mathbb{S}^{N_{\mathbb{F}}(m)}$$

is an isometric embedding.

With the normalization adopted here, g_0 has sectional curvature identically equal to 1 when $\mathbb{F} = \mathbb{R}$. In the complex case,

$$K_{g_0}(X, Y) = 1 + 3\langle JX, Y\rangle^2,$$

for every orthonormal pair X, Y. Hence

$$1 \leq K_{g_0} \leq 4.$$

In the quaternionic case, if (J_1, J_2, J_3) is a local admissible quaternionic frame, then

$$K_{g_0}(X, Y) = 1 + 3\sum_{\alpha=1}^{3} \langle J_\alpha X, Y\rangle^2,$$

and again

$$1 \leq K_{g_0} \leq 4.$$

After the homothety $g_\Phi = s_m g_0$, all sectional curvatures are divided by s_m.

2.2. **The second fundamental form.** For $u, v \in x^\perp$, we use the notation

$$u \odot v = uv^* + vu^*$$

and write $\langle u, v\rangle_{\mathbb{R}} = \mathrm{Re}(u^* v)$ for the underlying real inner product on $\mathbb{F}^n$.

Recall that

$$d\Phi_{\mathbb{F}}(v) = c_m(xv^* + vx^*).$$

To compute the Euclidean second fundamental form of $\Phi_{\mathbb{F}}$, extend u and v locally as horizontal vector fields and choose the extension of v so that its covariant derivative on $\mathbb{F}P^m$ vanishes at the point under consideration. Differentiating in the u direction, we obtain

$$D_u\Big(d\Phi_{\mathbb{F}}(v)\Big) = c_m\Big(uv^* + vu^* + x(D_u v)^* + (D_u v)x^*\Big).$$

At the chosen point, the normal component of $D_u v$ in the direction of x is

$$(D_u v)^\perp = -\langle u, v\rangle_{\mathbb{R}} x.$$

The remaining component represents the intrinsic covariant derivative and vanishes at that point by our choice of extension. It follows that the Euclidean second fundamental form is

$$\overline{B}(u, v) = c_m \left(u \odot v - 2\langle u, v\rangle_{\mathbb{R}} P_x\right).$$

We next determine its radial component. Since

$$\Phi_{\mathbb{F}}([x]) = c_m \left(P_x - \frac{1}{n} I_n\right),$$

we first observe that $\langle u \odot v, P_x \rangle = 0$. Indeed, using $x^* u = x^* v = 0$, we have

$$\operatorname{Re}\operatorname{tr}(uv^* P_x) = \operatorname{Re}\operatorname{tr}(uv^* x x^*) = 0,$$

and similarly for $vu^* P_x$. Moreover, $\operatorname{tr}(u \odot v) = 2\langle u, v \rangle_{\mathbb{R}}$ and

$$\left\langle P_x, P_x - \frac{1}{n} I_n \right\rangle = 1 - \frac{1}{n} = \frac{n-1}{n}.$$

Therefore,

$$\begin{aligned}\left\langle \overline{B}(u,v), \Phi_{\mathbb{F}}([x]) \right\rangle &= c_m^2 \left\langle u \odot v - 2\langle u, v \rangle_{\mathbb{R}} P_x, P_x - \frac{1}{n} I_n \right\rangle \\ &= c_m^2 \left(-\frac{2}{n} \langle u, v \rangle_{\mathbb{R}} - 2\frac{n-1}{n} \langle u, v \rangle_{\mathbb{R}} \right) \\ &= -2c_m^2 \langle u, v \rangle_{\mathbb{R}}.\end{aligned}$$

The outward unit normal of the ambient unit sphere is $\Phi_{\mathbb{F}}([x])$. Hence the Euclidean and spherical second fundamental forms of $\Phi_{\mathbb{F}}$ are related by

$$\overline{B}(u,v) = B(u,v) - g_\Phi(u,v)\Phi_{\mathbb{F}}([x]).$$

Since $g_\Phi(u,v) = 2c_m^2 \langle u, v \rangle_{\mathbb{R}}$, we obtain

$$\begin{aligned} B(u,v) &= \overline{B}(u,v) + 2c_m^2 \langle u, v \rangle_{\mathbb{R}} \Phi_{\mathbb{F}}([x]) \\ &= c_m \left(u \odot v - 2\langle u, v \rangle_{\mathbb{R}} P_x \right) + 2c_m^3 \langle u, v \rangle_{\mathbb{R}} \left(P_x - \frac{1}{n} I_n \right).\end{aligned}$$

Using $c_m^2 = \frac{n}{n-1} = \frac{m+1}{m}$, this simplifies to

$$B(u,v) = c_m \left(u \odot v + \frac{2}{m} \langle u, v \rangle_{\mathbb{R}} P_x - \frac{2}{m} \langle u, v \rangle_{\mathbb{R}} I_n \right). \tag{7}$$

The vectors u and v in (7) are measured with respect to the quotient metric g_0. Since

$$g_\Phi = s_m g_0, \qquad s_m = \frac{2(m+1)}{m},$$

if u is g_0-unit, then

$$\bar{u} = \frac{1}{\sqrt{s_m}} u = \sqrt{\frac{m}{2(m+1)}}\, u$$

is unit with respect to g_Φ.

Proposition 2.1. *The standard embedding $\Phi_{\mathbb{F}} : (\mathbb{F}P^m, s_m g_0) \longrightarrow \mathbb{S}^N$ is minimal. Moreover, its second fundamental form is parallel. If $V \in T_{[x]}\mathbb{F}P^m$ is g_Φ-unit and $u \in x^\perp$ is the corresponding g_0-unit vector, then*

$$B(V,V) = \sqrt{\frac{m}{m+1}} \left(P_u - \frac{1}{m}(I_n - P_x) \right). \tag{8}$$

In particular,

$$|B(V,V)|^2 = \frac{m-1}{m+1}.$$

Proof. Let $d = \dim_{\mathbb{R}} \mathbb{F}$ and choose a real orthonormal basis $\{e_1, \dots, e_{dm}\}$ of $x^\perp$ with respect to g_0. The corresponding g_Φ-orthonormal basis is $\bar{e}_a = \frac{1}{\sqrt{s_m}} e_a$, $a = 1, \dots, dm$. The real orthonormal basis may be chosen from an $\mathbb{F}$-orthonormal basis of $x^\perp$ by adjoining its complex or quaternionic multiples. Consequently,

$$\sum_{a=1}^{dm} P_{e_a} = d(I_n - P_x).$$

Taking $u = v = e_a$ in (7), we have $B(e_a, e_a) = 2c_m\Big(P_{e_a} + \frac{1}{m} P_x - \frac{1}{m} I_n\Big)$. Hence

$$\begin{aligned} \sum_{a=1}^{dm} B(\bar{e}_a, \bar{e}_a) &= \frac{2c_m}{s_m} \left[\sum_{a=1}^{dm} P_{e_a} + \frac{dm}{m} P_x - \frac{dm}{m} I_n \right] \\ &= \frac{2c_m}{s_m} \left[d(I_n - P_x) + dP_x - dI_n \right] \\ &= 0. \end{aligned}$$

Thus the mean curvature vector vanishes, and hence $\Phi_{\mathbb{F}}$ is minimal. Moreover, $\Phi_{\mathbb{F}}$ is the standard embedding of the symmetric R-space $\mathbb{F}P^m$; therefore, its second fundamental form is parallel [16].

We now compute $B(V, V)$. Since $V = \bar{u} = \frac{1}{\sqrt{s_m}} u$, bilinearity and (7) give

$$B(V, V) = \frac{1}{s_m} B(u, u) = \frac{2c_m}{s_m} \left(P_u + \frac{1}{m} P_x - \frac{1}{m} I_n \right).$$

Using $\frac{2c_m}{s_m} = \sqrt{\frac{m}{m+1}}$, we obtain

$$B(V, V) = \sqrt{\frac{m}{m+1}} \left(P_u - \frac{1}{m}(I_n - P_x) \right),$$

which proves (8).

Finally, decompose

$$\mathbb{F}^n = x\mathbb{F} \oplus u\mathbb{F} \oplus \{x, u\}_{\mathbb{F}}^\perp.$$

The endomorphism in (8) has eigenvalues

$$0$$

on $x\mathbb{F}$,

$$\sqrt{\frac{m}{m+1}} \left(1 - \frac{1}{m} \right)$$

on $u\mathbb{F}$, and

$$-\sqrt{\frac{m}{m+1}} \frac{1}{m}$$

on $\{x, u\}_{\mathbb{F}}^\perp$. The last eigenvalue has $\mathbb{F}$-multiplicity $m - 1$. Therefore,

$$|B(V, V)|^2 = \frac{m}{m+1} \left[\left(1 - \frac{1}{m} \right)^2 + \frac{m-1}{m^2} \right] = \frac{m-1}{m+1}.$$

□

In particular,

$$|B(V,V)|^2 = \frac{1}{3} \quad \text{on} \quad \mathbb{F}P^2, \qquad |B(V,V)|^2 = \frac{1}{2} \quad \text{on} \quad \mathbb{F}P^3.$$

Thus the Veronese embedding

$$\mathbb{R}P^3 \longrightarrow \mathbb{S}^8$$

satisfies

$$|B(V,V)|^2 = \frac{1}{2},$$

while the standard embeddings

$$\mathbb{R}P^2 \longrightarrow \mathbb{S}^4, \qquad \mathbb{C}P^2 \longrightarrow \mathbb{S}^7, \qquad \mathbb{H}P^2 \longrightarrow \mathbb{S}^{13}$$

all satisfy

$$|B(V,V)|^2 = \frac{1}{3}.$$

2.3. **The trace of B over a tangent subspace.** The formulas involving several orthonormal tangent vectors depend on the geometry of the tangent subspace. In the real-projective case, no additional structure is present. In the complex and quaternionic cases, however, the result depends on the interaction between the tangent subspace and the complex or quaternionic structures.

Let $E \subset T_{[x]}\mathbb{F}P^m$ be a real k-dimensional subspace and let $\{u_1, \dots, u_k\}$ be a g_0-orthonormal basis of E. Recall that, under the horizontal identification, $T_{[x]}\mathbb{F}P^m \simeq x^\perp$, the vectors u_i may be regarded as elements of $x^\perp$.

We define

$$\Theta_{\mathbb{F}}(E) = \begin{cases} 0, & \mathbb{F} = \mathbb{R}, \\ \displaystyle\sum_{i,j=1}^{k} \langle Ju_i, u_j \rangle_{\mathbb{R}}^2, & \mathbb{F} = \mathbb{C}, \\ \displaystyle\sum_{\alpha=1}^{3} \sum_{i,j=1}^{k} \langle J_\alpha u_i, u_j \rangle_{\mathbb{R}}^2, & \mathbb{F} = \mathbb{H}, \end{cases}$$

where J is the complex structure of $\mathbb{C}P^m$ and (J_1, J_2, J_3) is a local admissible quaternionic frame on $\mathbb{H}P^m$.

If π_E denotes the orthogonal projection onto E, then in the complex case

$$\Theta_{\mathbb{C}}(E) = \sum_{i=1}^{k} |\pi_E(Ju_i)|^2,$$

while in the quaternionic case

$$\Theta_{\mathbb{H}}(E) = \sum_{\alpha=1}^{3} \sum_{i=1}^{k} |\pi_E(J_\alpha u_i)|^2.$$

These expressions show that $\Theta_{\mathbb{F}}(E)$ is independent of the chosen orthonormal basis of E. In the quaternionic case, it is also independent of the admissible frame: any two admissible frames are related by an orthogonal transformation in SO(3), and the sum over α is invariant under such transformations.

The quantity $\Theta_{\mathbb{F}}(E)$ measures the interaction between the complex or quaternionic structure and the subspace E. Its values range from 0, when the corresponding structure is orthogonal to E, to the maximal value, attained precisely when E is invariant under the complex or quaternionic structure. More precisely, in the complex case, $\Theta_{\mathbb{C}}(E) = 0$ if $J(E) \perp E$ and $\Theta_{\mathbb{C}}(E) = k$ if E is J-invariant. Likewise, in the quaternionic case, $\Theta_{\mathbb{H}}(E) = 3k$ if E is invariant under J_1, J_2, and J_3.

Observe that, since orthogonal projections do not increase length, we have

$$|\pi_E(J_\alpha u_i)|^2 \leq |J_\alpha u_i|^2 = 1.$$

Thus,

$$0 \leq \Theta_{\mathbb{F}}(E) \leq (d-1)k, \qquad d = \dim_{\mathbb{R}} \mathbb{F}. \tag{9}$$

Lemma 2.2. *Let $\{\bar{u}_1, \ldots, \bar{u}_k\}$ be the g_Φ-orthonormal basis of E corresponding to $\{u_1, \ldots, u_k\}$; namely, $\bar{u}_i = \frac{1}{\sqrt{s_m}} u_i$. Then*

$$\left| \sum_{i=1}^{k} B(\bar{u}_i, \bar{u}_i) \right|^2 = \frac{k(m-k) + m\Theta_{\mathbb{F}}(E)}{m+1}.$$

In particular, when $\mathbb{F} = \mathbb{R}$,

$$\left| \sum_{i=1}^{k} B(\bar{u}_i, \bar{u}_i) \right|^2 = \frac{k(m-k)}{m+1}.$$

Proof. For each i, formula (8) gives

$$B(\bar{u}_i, \bar{u}_i) = \sqrt{\frac{m}{m+1}} \left(P_{u_i} - \frac{1}{m}(I_n - P_x) \right).$$

Hence

$$\sum_{i=1}^{k} B(\bar{u}_i, \bar{u}_i) = \sqrt{\frac{m}{m+1}} \left(Q_E - \frac{k}{m}(I_n - P_x) \right), \tag{10}$$

where

$$Q_E = \sum_{i=1}^{k} P_{u_i}.$$

Although Q_E is defined using an orthonormal basis of the real subspace E, it is not generally the orthogonal projection onto an $\mathbb{F}$-subspace. This is precisely why the complex and quaternionic correction terms appear. Since every u_i is unit, $\operatorname{tr} P_{u_i} = 1$, and therefore

$$\operatorname{tr} Q_E = k.$$

Moreover, because $u_i \in x^\perp$,

$$P_{u_i} P_x = 0.$$

Therefore, it follows that $Q_E(I_n - P_x) = Q_E$. We also have $(I_n - P_x)^2 = I_n - P_x$ and $\operatorname{tr}(I_n - P_x) = n - 1 = m$. Taking the squared norm in (10), we find

$$\begin{aligned}\left|\sum_{i=1}^{k} B(\bar{u}_i, \bar{u}_i)\right|^2 &= \frac{m}{m+1}\left|Q_E - \frac{k}{m}(I_n - P_x)\right|^2 \\ &= \frac{m}{m+1}\left[\operatorname{tr}(Q_E^2) - \frac{2k}{m}\operatorname{tr} Q_E + \frac{k^2}{m^2}\operatorname{tr}(I_n - P_x)\right] \\ &= \frac{m}{m+1}\left(\operatorname{tr}(Q_E^2) - \frac{k^2}{m}\right).\end{aligned}$$

Thus it remains to compute $\operatorname{tr}(Q_E^2)$.

By the definition of Q_E,

$$\operatorname{Re}\operatorname{tr}(Q_E^2) = \sum_{i,j=1}^{k} \operatorname{Re}\operatorname{tr}(P_{u_i} P_{u_j}).$$

For rank-one projections,

$$\begin{aligned}\operatorname{Re}\operatorname{tr}(P_{u_i} P_{u_j}) &= \operatorname{Re}\operatorname{tr}(u_i u_i^* u_j u_j^*) \\ &= |u_i^* u_j|^2.\end{aligned}$$

Therefore,

$$\operatorname{tr}(Q_E^2) = \sum_{i,j=1}^{k} |u_i^* u_j|^2.$$

In the real case, $u_i^* u_j = \langle u_i, u_j \rangle_{\mathbb{R}} = \delta_{ij}$, and hence

$$\operatorname{tr}(Q_E^2) = k.$$

In the complex case, the Hermitian product decomposes as

$$|u_i^* u_j|^2 = \langle u_i, u_j \rangle_{\mathbb{R}}^2 + \langle J u_i, u_j \rangle_{\mathbb{R}}^2.$$

Since the basis is real orthonormal, $\sum_{i,j=1}^{k} \langle u_i, u_j \rangle_{\mathbb{R}}^2 = k$. Consequently,

$$\operatorname{tr}(Q_E^2) = k + \Theta_{\mathbb{C}}(E).$$

Similarly, in the quaternionic case,

$$|u_i^* u_j|^2 = \langle u_i, u_j \rangle_{\mathbb{R}}^2 + \sum_{\alpha=1}^{3} \langle J_\alpha u_i, u_j \rangle_{\mathbb{R}}^2,$$

and therefore

$$\operatorname{tr}(Q_E^2) = k + \Theta_{\mathbb{H}}(E).$$

Thus, in all three cases,

$$\operatorname{tr}(Q_E^2) = k + \Theta_{\mathbb{F}}(E).$$

Substituting this identity into the previous norm calculation gives

$$\begin{aligned}\left|\sum_{i=1}^{k} B(\bar{u}_i, \bar{u}_i)\right|^2 &= \frac{m}{m+1}\left(k + \Theta_{\mathbb{F}}(E) - \frac{k^2}{m}\right) \\ &= \frac{k(m-k) + m\Theta_{\mathbb{F}}(E)}{m+1},\end{aligned}$$

which proves the result. □

2.4. **The Cayley projective plane.** The octonionic case requires a separate treatment because the algebra of octonions is not associative. In particular, expressions involving products of three or more octonionic matrices cannot be manipulated as in the real, complex, or quaternionic cases.

Let

$$\mathcal{H}_3(\mathbb{O}) = \{A \in M_3(\mathbb{O}) : A^* = A\}$$

be the real vector space of 3×3 Hermitian octonionic matrices. An element of $\mathcal{H}_3(\mathbb{O})$ has the form

$$A = \begin{pmatrix} a_1 & z_1 & z_2 \\ \overline{z}_1 & a_2 & z_3 \\ \overline{z}_2 & \overline{z}_3 & a_3 \end{pmatrix},$$

where $a_1, a_2, a_3 \in \mathbb{R}$, $z_1, z_2, z_3 \in \mathbb{O}$. Since $\dim_{\mathbb{R}} \mathbb{O} = 8$, we have

$$\dim_{\mathbb{R}} \mathcal{H}_3(\mathbb{O}) = 3 + 3 \cdot 8 = 27.$$

The space $\mathcal{H}_3(\mathbb{O})$ is endowed with the Jordan product

$$A \circ B = \frac{1}{2}(AB + BA).$$

Although ordinary octonionic matrix multiplication is not associative, this symmetrized product gives $\mathcal{H}_3(\mathbb{O})$ the structure of the exceptional simple Euclidean Jordan algebra. Its natural Euclidean inner product is $\langle A, B\rangle = \operatorname{tr}(A \circ B)$. In particular, $|A|^2 = \operatorname{tr}(A \circ A)$.

Let

$$\mathcal{H}_3^0(\mathbb{O}) = \{A \in \mathcal{H}_3(\mathbb{O}) : \operatorname{tr} A = 0\}.$$

This is a 26-dimensional Euclidean vector space. Therefore, its unit sphere is isometric to $\mathbb{S}^{25}$.

The Cayley projective plane can be realized as the set of primitive idempotents of trace one:

$$\mathbb{O}P^2 = \{P \in \mathcal{H}_3(\mathbb{O}) : P \circ P = P, \quad \operatorname{tr} P = 1\}.$$

Here, primitive means that P cannot be decomposed as the sum of two nonzero orthogonal idempotents. These elements play the role of the rank-one projections xx^* in the associative cases.

For $P \in \mathbb{O}P^2$, consider

$$\Phi_{\mathbb{O}}(P) = \sqrt{\frac{3}{2}}\left(P - \frac{1}{3}I_3\right). \tag{11}$$

Since $\operatorname{tr} P = 1$, we have

$$\operatorname{tr}\left(P - \frac{1}{3}I_3\right) = 1 - \frac{1}{3}\operatorname{tr} I_3 = 0.$$

Thus $\Phi_{\mathbb{O}}(P)$ belongs to $\mathcal{H}_3^0(\mathbb{O})$. Moreover, using $P \circ P = P$, we obtain

$$\begin{aligned}\left|P - \frac{1}{3}I_3\right|^2 &= \operatorname{tr}\left[\left(P - \frac{1}{3}I_3\right) \circ \left(P - \frac{1}{3}I_3\right)\right] \\ &= \operatorname{tr} P - \frac{2}{3}\operatorname{tr} P + \frac{1}{9}\operatorname{tr} I_3 = \frac{2}{3}.\end{aligned}$$

Consequently, $|\Phi_{\mathbb{O}}(P)|^2 = 1$. Hence (11) defines the standard embedding

$$\Phi_{\mathbb{O}} : \mathbb{O}P^2 \longrightarrow \mathbb{S}^{25}.$$

To describe its differential, recall the Peirce decomposition associated with an idempotent P:

$$\mathcal{H}_3(\mathbb{O}) = \mathcal{V}_1(P) \oplus \mathcal{V}_{\frac{1}{2}}(P) \oplus \mathcal{V}_0(P),$$

where

$$\mathcal{V}_\lambda(P) = \{X \in \mathcal{H}_3(\mathbb{O}) : P \circ X = \lambda X\}.$$

The tangent space of the Cayley plane at P is

$$T_P\mathbb{O}P^2 = \mathcal{V}_{\frac{1}{2}}(P).$$

Indeed, differentiating the identity $P \circ P = P$ along a curve $P(t) \subset \mathbb{O}P^2$ with $P(0) = P$ and $P'(0) = X$, we find $2P \circ X = X$, and therefore $P \circ X = \frac{1}{2}X$.

The differential of (11) is

$$d\Phi_{\mathbb{O}}(X) = \sqrt{\frac{3}{2}}\,X.$$

We normalize the standard Cayley metric g_0 by $g_0(X,Y) = \frac{1}{2}\langle X, Y\rangle$ on $\mathcal{V}_{\frac{1}{2}}(P)$. With this convention,

$$\langle d\Phi_{\mathbb{O}}(X), d\Phi_{\mathbb{O}}(Y)\rangle = \frac{3}{2}\langle X, Y\rangle = 3g_0(X,Y).$$

Thus $\Phi_{\mathbb{O}} : (\mathbb{O}P^2, 3g_0) \longrightarrow \mathbb{S}^{25}$ is an isometric embedding.

The compact exceptional Lie group F_4 acts transitively on the Cayley projective plane $\mathbb{O}P^2$, and the isotropy subgroup at each point is isomorphic to $\operatorname{Spin}(9)$. Consequently,

$$\mathbb{O}P^2 \cong F_4/\operatorname{Spin}(9).$$

The standard embedding $\Phi_{\mathbb{O}} \colon \mathbb{O}P^2 \longrightarrow \mathbb{S}^{25}$ is F_4-equivariant. It is minimal and has parallel second fundamental form; consequently, its image is an extrinsically symmetric submanifold of $\mathbb{S}^{25}$; see [4, 16].

The isotropy representation of Spin(9) is transitive on the unit sphere of $T_P\mathbb{O}P^2$. Consequently, the function

$$V \longmapsto |B(V,V)|^2$$

is constant on the unit tangent bundle. A computation using the Peirce decomposition gives

$$|B(V,V)|^2 = \frac{1}{3}, \qquad |V|^2_{3g_0} = 1. \tag{12}$$

Thus the Cayley projective plane has the same value of $|B(V,V)|^2$ as the standard embeddings of the real, complex, and quaternionic projective planes.

For a real k-plane $E \subset T_P\mathbb{O}P^2$, let $\{e_1,\dots,e_k\}$ be any $3g_0$-orthonormal basis of E. We define

$$\mathcal{A}_{\mathbb{O}}(E) = \left|\sum_{i=1}^{k} B(e_i,e_i)\right|^2.$$

The vector $\operatorname{tr}_E B = \sum_{i=1}^{k} B(e_i,e_i)$ is the trace of the restriction of B to $E\times E$. Therefore, it does not depend on the chosen orthonormal basis of E, and $\mathcal{A}_{\mathbb{O}}(E)$ is well defined.

For $k = 1$, formula (12) gives $\mathcal{A}_{\mathbb{O}}(E) = \frac{1}{3}$. For the full tangent space, minimality gives $\mathcal{A}_{\mathbb{O}}(T_P\mathbb{O}P^2) = 0$. More generally, the triangle inequality and (12) imply the rough estimate

$$0 \le \mathcal{A}_{\mathbb{O}}(E) \le \frac{k^2}{3}.$$

Unlike the complex and quaternionic cases, there is no globally defined family of almost complex structures that immediately produces an analogue of $\Theta_{\mathbb{F}}(E)$. It is therefore more natural to retain the invariant $\mathcal{A}_{\mathbb{O}}(E)$ in the eigenvalue estimate. This also avoids manipulating octonionic matrix products as if they were associative.

3. Conformal geometry and admissible functions

The use of conformal transformations in eigenvalue estimates goes back to Hersch's work on the first eigenvalue of the two-sphere and was developed systematically by Li and Yau through the notion of conformal volume [19, 21]. Its higher-dimensional formulation, together with its relation to minimal immersions and the first eigenvalue, was subsequently studied by El Soufi and Ilias [14]. We recall below the two ingredients needed in our argument.

Our sign convention is $\Delta = \operatorname{div}\nabla$, so that the eigenvalues μ of Δ, $\Delta u + \mu u = 0$, are nonnegative. All integrals in this section are computed with respect to the induced metric on Σ.

The first result is a weighted version of the conformal center-of-mass argument. The original unweighted construction is due to Hersch, while the formulation for conformal immersions and positive weights is part of the Li-Yau method.

Lemma 3.1 (Li-Yau center-of-mass argument)**.** *Let $\phi : \Sigma^k \longrightarrow \mathbb{S}^N$ be an immersion of a closed manifold, and let $u > 0$ be a smooth function on Σ. Then there exists a* *conformal diffeomorphism* $F : \mathbb{S}^N \longrightarrow \mathbb{S}^N$ *such that, writing*

$$\Psi = F \circ \phi = (\Psi_1, \dots, \Psi_{N+1}),$$

one has

$$\int_\Sigma u\Psi_\alpha \, d\Sigma = 0, \qquad \alpha = 1, \dots, N+1.$$

Equivalently,

$$\int_\Sigma u\Psi \, d\Sigma = 0$$

as a vector in $\mathbb{R}^{N+1}$*.*

Proof. The conformal transformations of $\mathbb{S}^N$ can be parametrized, up to orthogonal transformations, by the open unit ball $\mathbb{B}^{N+1}$. Denote the corresponding family by F_a, where $a \in \mathbb{B}^{N+1}$, and define

$$\mathcal{C}(a) = \frac{1}{\int_\Sigma u \, d\Sigma} \int_\Sigma u \, F_a \circ \phi \, d\Sigma.$$

The map

$$\mathcal{C} : \mathbb{B}^{N+1} \longrightarrow \mathbb{B}^{N+1}$$

is continuous. Furthermore, as a approaches the boundary, the conformal transformation F_a concentrates the image toward the boundary point determined by a. The standard degree argument of Hersch and Li-Yau then shows that $\mathcal{C}$ must vanish at some $a_0 \in \mathbb{B}^{N+1}$. Taking $F = F_{a_0}$ gives

$$\int_\Sigma u \, F \circ \phi \, d\Sigma = 0,$$

which is the desired conclusion. □

The importance of Lemma 3.1 in our setting is that the weight u can be chosen to be a positive first eigenfunction of a Schrödinger operator. The coordinate functions of Ψ are then orthogonal to the first eigenspace and are therefore admissible in the variational characterization of the second eigenvalue. Moreover, since Ψ takes values in the unit sphere,

$$\sum_{\alpha=1}^{N+1} \Psi_\alpha^2 = 1, \qquad \sum_{\alpha=1}^{N+1} |\nabla \Psi_\alpha|^2 = |\nabla \Psi|^2. \tag{13}$$

These identities allow the individual Rayleigh inequalities to be summed into a single geometric estimate.

The second ingredient controls the energy of the conformally modified immersion. It is a consequence of the conformal volume estimates of El Soufi and Ilias [14]; see also [13].

Lemma 3.2 (El Soufi-Ilias)**.** *Let* $\phi : \Sigma^k \longrightarrow \mathbb{S}^N$ *be an isometric immersion of a closed manifold, and let* $\vec{H}$ *be its mean curvature vector in* $\mathbb{S}^N$*, with the convention* $\vec{H} = \frac{1}{k} \operatorname{tr} \tau$*, where* τ *is the second fundamental form of* ϕ*. Then, for every conformal diffeomorphism*

$F : \mathbb{S}^N \longrightarrow \mathbb{S}^N$,

$$\int_\Sigma |\nabla(F \circ \phi)|^2 \, d\Sigma \le k \int_\Sigma \left(1 + |\vec{H}|^2\right) d\Sigma. \tag{14}$$

If $k \ge 3$ and equality holds, then the energy density $|\nabla(F \circ \phi)|^2$ is constant on Σ.

Remark 3.3. For $F = \mathrm{Id}_{\mathbb{S}^N}$, the immersion ϕ is isometric and hence $|\nabla\phi|^2 = k$. Thus (14) is immediate in this case. The content of the El Soufi-Ilias estimate is that the energy remains controlled after an arbitrary conformal transformation of the ambient sphere, even though $F \circ \phi$ is generally no longer isometric. The right-hand side is expressed solely in terms of the geometry of the original immersion.

We shall apply these lemmas as follows. Given a positive first eigenfunction u of the relevant Schrödinger operator, we choose F by Lemma 3.1. Each coordinate function Ψ_α is then an admissible test-function for the second eigenvalue. After summing the associated Rayleigh inequalities, identities (13) and estimate (14) convert the resulting analytic expression into an estimate involving the mean curvature of the spherical immersion. This is the conformal mechanism underlying the proofs in Section 4.

4. Second eigenvalue estimates

Let $\iota : \Sigma^k \longrightarrow (\mathbb{F}P^m, g_0)$ be a closed immersion, and let σ and $\vec{H}_\sigma$ be its second fundamental form and mean curvature vector, respectively. We consider

$$L = \Delta + |\sigma|^2 + k$$

and enumerate the eigenvalues of $-L$ increasingly. Thus,

$$\lambda_2(L) = \inf_{\substack{f \ne 0 \\ \int_\Sigma uf = 0}} \frac{\int_\Sigma (|\nabla f|^2 - (|\sigma|^2 + k) f^2)}{\int_\Sigma f^2},$$

where $u > 0$ is a first eigenfunction.

We are now in a position to establish the main eigenvalue estimate for closed submanifolds of real, complex, and quaternionic projective spaces.

Theorem 4.1. *Let $\iota : \Sigma^k \longrightarrow (\mathbb{F}P^m, g_0)$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, be a closed immersed submanifold, and consider the Schrödinger operator $L = \Delta + |\sigma|^2 + k$, where σ is the second fundamental form of the immersion with respect to g_0. Then*

$$\lambda_2(L) \le k + 2 + \frac{2}{k \,\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \Theta_{\mathbb{F}}(T_p\Sigma) \, d\Sigma. \tag{15}$$

In particular, denoting $d = \dim_{\mathbb{R}} \mathbb{F}$, then

$$\lambda_2(L) \le k + 2d. \tag{16}$$

Equality in (15) implies that Σ is totally umbilical in $\mathbb{F}P^m$. Moreover, in the real-projective case, equality

$$\lambda_2(L) = k + 2$$

forces the image of Σ *to be totally geodesic.*

Proof. We first pass to the normalization induced by the standard projective embedding. Recall that $\Phi_{\mathbb{F}} : (\mathbb{F}P^m, g_\Phi) \longrightarrow \mathbb{S}^N$ is an isometric immersion, where $g_\Phi = s_m g_0$, $s_m = \frac{2(m+1)}{m}$. Denote by $\widetilde{\Delta}$, $\widetilde{\sigma}$, and $d\widetilde{\Sigma}$ the Laplacian, the second fundamental form, and the volume element associated with the induced metric g_Φ on Σ. Since $g_\Phi = s_m g_0$ with s_m constant, we have

$$\widetilde{\Delta} = \frac{1}{s_m}\Delta, \qquad |\widetilde{\sigma}|^2 = \frac{1}{s_m}|\sigma|^2.$$

Consequently,

$$\widetilde{L} = \widetilde{\Delta} + |\widetilde{\sigma}|^2 + \frac{k}{s_m} = \frac{1}{s_m}L,$$

and therefore

$$\lambda_2(L) = s_m \lambda_2(\widetilde{L}). \tag{17}$$

Consider now the composed immersion $\phi = \Phi_{\mathbb{F}} \circ \iota : (\Sigma, g_\Phi) \longrightarrow \mathbb{S}^N$. Let τ and $\vec{H}_\tau$ denote its second fundamental form and mean curvature vector, respectively. We use the convention $\vec{H}_\tau = \frac{1}{k}\sum_{i=1}^k \tau(e_i, e_i)$, where $\{e_1, \ldots, e_k\}$ is a local g_Φ-orthonormal frame on Σ.

Let B denote the second fundamental form of $\Phi_{\mathbb{F}} : (\mathbb{F}P^m, g_\Phi) \longrightarrow \mathbb{S}^N$. The composition formula for second fundamental forms gives

$$\tau(X, Y) = B(X, Y) + \widetilde{\sigma}(X, Y). \tag{18}$$

Here $B(X, Y)$ is normal to $\mathbb{F}P^m$ in $\mathbb{S}^N$, whereas $\widetilde{\sigma}(X, Y)$ is tangent to $\mathbb{F}P^m$ and normal to Σ. Thus these two terms belong to mutually orthogonal normal subbundles. Taking the trace of (18), we obtain

$$k\vec{H}_\tau = \sum_{i=1}^k B(e_i, e_i) + k\vec{H}_{\widetilde{\sigma}},$$

and orthogonality yields

$$k|\vec{H}_\tau|^2 = \frac{1}{k}\left|\sum_{i=1}^k B(e_i, e_i)\right|^2 + k|\vec{H}_{\widetilde{\sigma}}|^2. \tag{19}$$

Let $u > 0$ be a first eigenfunction of $\widetilde{L}$. By Lemma 3.1, there exists a conformal diffeomorphism Γ of $\mathbb{S}^N$ such that the coordinate functions of $\Gamma \circ \phi$ are orthogonal to u. Hence they are admissible test-functions in the variational characterization of $\lambda_2(\widetilde{L})$. Summing the corresponding Rayleigh inequalities and applying Lemma 3.2, we find

$$\lambda_2(\widetilde{L}) \operatorname{Vol}(\Sigma, g_\Phi) \le k \int_\Sigma \left(1 + |\vec{H}_\tau|^2\right) d\widetilde{\Sigma} - \int_\Sigma \left(|\widetilde{\sigma}|^2 + \frac{k}{s_m}\right) d\widetilde{\Sigma}. \tag{20}$$

Using (19), the integrand on the right-hand side in (20) can be written as

$$k + \frac{1}{k}\left|\sum_{i=1}^k B(e_i, e_i)\right|^2 + k|\vec{H}_{\widetilde{\sigma}}|^2 - |\widetilde{\sigma}|^2 - \frac{k}{s_m}.$$

The standard trace inequality for the second fundamental form gives $|\widetilde{\sigma}|^2 \geq k|\vec{H}_{\widetilde{\sigma}}|^2$. Moreover, equality holds at a point if and only if $\widetilde{\sigma}(X,Y) = g_\Phi(X,Y)\vec{H}_{\widetilde{\sigma}}$, that is, if and only if the immersion is umbilical at that point. Therefore, (20) implies

$$\lambda_2(\widetilde{L})\,\mathrm{Vol}(\Sigma, g_\Phi) \leq \int_\Sigma \left[k + \frac{1}{k}\left|\sum_{i=1}^{k} B(e_i,e_i)\right|^2 - \frac{k}{s_m} \right] d\widetilde{\Sigma}. \tag{21}$$

By Lemma 2.2, the trace of B over the tangent plane $T_p\Sigma$ satisfies

$$\frac{1}{k}\left|\sum_{i=1}^{k} B(e_i,e_i)\right|^2 = \frac{k(m-k) + m\Theta_{\mathbb{F}}(T_p\Sigma)}{k(m+1)}.$$

Since $s_m = 2(m+1)/m$, a direct simplification gives

$$k + \frac{k(m-k) + m\Theta_{\mathbb{F}}(T_p\Sigma)}{k(m+1)} - \frac{k}{s_m} = \frac{k+2}{s_m} + \frac{m}{k(m+1)}\Theta_{\mathbb{F}}(T_p\Sigma). \tag{22}$$

Substituting (22) into (21), dividing by $\mathrm{Vol}(\Sigma, g_\Phi)$, and using $\frac{m}{m+1} = \frac{2}{s_m}$, we obtain

$$\lambda_2(\widetilde{L}) \leq \frac{k+2}{s_m} + \frac{2}{s_m k\,\mathrm{Vol}(\Sigma, g_\Phi)} \int_\Sigma \Theta_{\mathbb{F}}(T_p\Sigma)\, d\widetilde{\Sigma}.$$

Because $g_\Phi = s_m g_0$, both the volume and the integral are multiplied by the same constant factor $s_m^{k/2}$. Hence

$$\frac{1}{\mathrm{Vol}(\Sigma, g_\Phi)} \int_\Sigma \Theta_{\mathbb{F}}(T_p\Sigma)\, d\widetilde{\Sigma} = \frac{1}{\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \Theta_{\mathbb{F}}(T_p\Sigma)\, d\Sigma.$$

Multiplying the preceding inequality by s_m and invoking (17), we arrive at

$$\lambda_2(L) \leq k + 2 + \frac{2}{k\,\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \Theta_{\mathbb{F}}(T_p\Sigma)\, d\Sigma,$$

which proves (15). By (9), $\Theta_{\mathbb{F}}(T_p\Sigma) \leq (d-1)k$ at every point. Therefore

$$\frac{2}{k\,\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \Theta_{\mathbb{F}}(T_p\Sigma)\, d\Sigma \leq 2(d-1),$$

and (16) follows:

$$\lambda_2(L) \leq k + 2d.$$

Finally, suppose that equality holds in (15). Since the only geometric inequality used after the conformal test-function argument is

$$|\widetilde{\sigma}|^2 - k|\vec{H}_{\widetilde{\sigma}}|^2 \geq 0,$$

equality in the final estimate forces

$$|\widetilde{\sigma}|^2 = k|\vec{H}_{\widetilde{\sigma}}|^2$$

everywhere on Σ. Thus the trace-free second fundamental form vanishes identically, and Σ is totally umbilical in $\mathbb{F}P^m$.

In the real-projective case, $\Theta_{\mathbb{R}} \equiv 0$, so (15) becomes

$$\lambda_2(L) \leq k+2.$$

If equality holds, the preceding argument shows that Σ is totally umbilical. The classification of closed totally umbilical submanifolds of real projective space reduces the problem to a geodesic k-sphere. Lemma 4.2 below shows that a non-totally-geodesic sphere has $\lambda_2(L) = 0$. Hence equality $\lambda_2(L) = k+2$ can occur only in the totally geodesic case. □

Lemma 4.2. *Let $S^k(r)$ be a geodesic k-sphere of radius $r \in (0, \pi/2]$ in $(\mathbb{R}P^m, g_0)$. For*

$$L = \Delta + |\sigma|^2 + k,$$

one has

$$\lambda_2(L) = \begin{cases} 0, & 0 < r < \pi/2, \\ k+2, & r = \pi/2. \end{cases}$$

For $r = \pi/2$, $S^k(r)$ is a totally geodesic copy of $\mathbb{R}P^k$ in $(\mathbb{R}P^m, g_0)$.

Proof. Suppose first that $0 < r < \pi/2$. The metric induced on $S^k(r)$ is the round metric of radius $\sin r$. Its first nonzero Laplace eigenvalue is therefore

$$\mu_1(\Delta) = \frac{k}{\sin^2 r}.$$

The sphere is totally umbilical, with each principal curvature equal to $\cot r$, and hence $|\sigma|^2 = k \cot^2 r$. Since the potential $|\sigma|^2 + k$ is constant, the eigenvalues of L in the convention

$$Lu + \lambda u = 0$$

are obtained by subtracting this potential from the eigenvalues of Δ. The first eigenfunction of L is constant, while the second eigenvalue is

$$\lambda_2(L) = \frac{k}{\sin^2 r} - k\cot^2 r - k = k\left(\frac{1}{\sin^2 r} - \frac{\cos^2 r}{\sin^2 r} - 1\right) = 0.$$

When $r = \pi/2$, the limiting geodesic sphere is a totally geodesic copy of $\mathbb{R}P^k$, and thus $\sigma = 0$. The spectrum of the unit $\mathbb{R}P^k$ is obtained from the spectrum of $\mathbb{S}^k$ by retaining only the antipodally even spherical harmonics. The degree-one harmonics do not descend to $\mathbb{R}P^k$, while the first nonconstant descending harmonics have degree two. Their eigenvalue is

$$2(2+k-1) = 2(k+1).$$

Therefore

$$\lambda_2(L) = 2(k+1) - k = k+2,$$

as claimed. □

Remark 4.3. With our normalization, the projective spaces are Einstein, with

projective space	Ric_{g_0}
$\mathbb{C}P^m$	$2(m+1)g_0$
$\mathbb{H}P^m$	$4(m+2)g_0$
$\mathbb{O}P^2$	$36g_0$

Consequently, the Jacobi operator $J = \Delta + |\sigma|^2 + \mathrm{Ric}_{g_0}(N, N)$ differs from the operator considered here only by a constant. Therefore, the results of this paper also yield corresponding estimates for the second eigenvalue of J. We do not pursue these consequences here.

The real projective case takes a particularly elegant and remarkably clean form, for which we also provide a direct proof.

Corollary 4.4. *Let $\iota : \Sigma^k \longrightarrow (\mathbb{R}P^m, g_0)$ be a closed immersion, where g_0 has constant sectional curvature one, and consider $L = \Delta + |\sigma|^2 + k$. Then*

$$\lambda_2(L) \leq k + 2.$$

Moreover, equality holds if and only if ι is an embedding onto a totally geodesic copy of $\mathbb{R}P^k$ in $\mathbb{R}P^m$. Hence,

$$(\Sigma, \iota^* g_0) \cong (\mathbb{R}P^k, g_0).$$

Proof. For $\mathbb{F} = \mathbb{R}$, the tangent-plane correction term vanishes identically, that is, $\Theta_{\mathbb{R}}(T_p\Sigma) = 0$. The estimate therefore follows immediately from Theorem 4.1. If equality holds, the same theorem implies that Σ is totally umbilical. The classification of totally umbilical submanifolds of real projective space, together with Lemma 4.2, excludes the non-totally-geodesic extrinsic spheres. Hence the image of Σ is totally geodesic.

It remains to exclude a nontrivial covering of the totally geodesic copy of $\mathbb{R}P^k$. Since Σ is closed and connected, the induced map $\iota \colon \Sigma \longrightarrow \mathbb{R}P^k$ is a Riemannian covering. If $k \geq 2$, the only nontrivial connected cover is the antipodal covering $S^k \longrightarrow \mathbb{R}P^k$. In that case, since $L = \Delta + k$ and $\lambda_1(\Delta_{S^k}) = k$, one has $\lambda_2(L) = 0$, contrary to the equality value $k + 2$. If $k = 1$, a covering of degree q gives $\lambda_2(L) = \frac{4}{q^2} - 1$, so equality forces $q = 1$. Hence the covering is trivial and ι is an embedding onto a totally geodesic copy of $\mathbb{R}P^k$.

The converse for embedded submanifolds follows from $\lambda_2(\Delta_{\mathbb{R}P^k}) = 2(k+1)$ and $\sigma = 0$. □

The same method applies to the Cayley projective plane, although the geometry of its standard spherical embedding produces a different tangent-plane correction term.

Theorem 4.5. *Let $\iota : \Sigma^k \longrightarrow (\mathbb{O}P^2, g_0)$ be a closed immersed submanifold, and let $L = \Delta + |\sigma|^2 + k$. Then*

$$\lambda_2(L) \leq 2k + \frac{3}{k\,\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \mathcal{A}_{\mathbb{O}}(T_p\Sigma)\, d\Sigma.$$

Equality implies that Σ is totally umbilical in $\mathbb{O}P^2$.

Proof. Consider the standard isometric immersion $\Phi_{\mathbb{O}} : (\mathbb{O}P^2, 3g_0) \longrightarrow \mathbb{S}^{25}$ and the composition $\phi = \Phi_{\mathbb{O}} \circ \iota : \Sigma \longrightarrow \mathbb{S}^{25}$. With respect to the homothetic metric $\widetilde{g} = 3g_0$, define

$$\widetilde{L} = \widetilde{\Delta} + |\widetilde{\sigma}|^2 + \frac{k}{3}.$$

The constant scaling gives

$$\widetilde{L} = \frac{1}{3}L, \qquad \lambda_2(L) = 3\lambda_2(\widetilde{L}).$$

Repeating the conformal test-function argument used in the proof of Theorem 4.1, we obtain

$$\lambda_2(\widetilde{L})\,\mathrm{Vol}(\Sigma, \widetilde{g}) \leq \int_\Sigma \left[k + \frac{1}{k}\left|\sum_{i=1}^k B(e_i, e_i)\right|^2 - \frac{k}{3}\right] d\widetilde{\Sigma}.$$

The trace formula for the Cayley projective embedding transforms the integrand into

$$\frac{2k}{3} + \frac{1}{k}\mathcal{A}_{\mathbb{O}}(T_p\Sigma).$$

It follows that

$$\lambda_2(\widetilde{L}) \leq \frac{2k}{3} + \frac{1}{k\,\mathrm{Vol}(\Sigma, \widetilde{g})} \int_\Sigma \mathcal{A}_{\mathbb{O}}(T_p\Sigma)\, d\widetilde{\Sigma}.$$

Since the quotient between the integral and the volume is unchanged under the constant rescaling $g_0 \longmapsto 3g_0$, multiplication by 3 gives

$$\lambda_2(L) \leq 2k + \frac{3}{k\,\mathrm{Vol}(\Sigma, g_0)} \int_\Sigma \mathcal{A}_{\mathbb{O}}(T_p\Sigma)\, d\Sigma.$$

As in the proof of Theorem 4.1, equality forces $|\widetilde{\sigma}|^2 = k|\vec{H}_{\widetilde{\sigma}}|^2$. Thus the trace-free part of $\widetilde{\sigma}$, and equivalently that of σ, vanishes identically. Hence Σ is totally umbilical in $\mathbb{O}P^2$. □

Remark 4.6. The correction term $\Theta_{\mathbb{F}}$ in (15) is an intrinsic feature of the complex and quaternionic projective settings. Indeed, for an arbitrary real k-plane $E \subset T_p\mathbb{F}P^m$, the quantities

$$\left|\sum_{i=1}^k B(e_i, e_i)\right|^2 \qquad \text{and} \qquad \sum_{i,j=1}^k |B(e_i, e_j)|^2$$

are not determined solely by k. In the complex case, they also depend on the position of E relative to the Kähler structure, or equivalently on its Kähler angles. In the quaternionic case, the corresponding dependence is governed by the quaternionic Kähler angles. The function $\Theta_{\mathbb{F}}(E)$ records precisely this additional tangent-plane information.

Therefore, the real-projective computation cannot be extended to $\mathbb{C}P^m$ or $\mathbb{H}P^m$ by merely replacing the underlying division algebra.

5. The totally umbilical equality models

We now examine the totally umbilical submanifolds appearing in the equality analysis of Theorem 4.1.

5.1. **Totally umbilical submanifolds of complex projective space.** Throughout this section, $\mathbb{C}P^m$ is endowed with the Fubini-Study metric g_0 normalized to have constant holomorphic sectional curvature 4. We use the spectral convention

$$Lu + \lambda u = 0, \qquad L = \Delta + |\sigma|^2 + k.$$

We recall the classification of totally umbilical submanifolds in complex space forms due to Chen and Ogiue [10]; see also [8, 9, 11]. Let Σ^k, $k \geq 2$, be a connected totally umbilical submanifold of $\mathbb{C}P^m$. Then, locally, Σ is one of the following:

(1) a totally geodesic complex projective subspace

$$\mathbb{C}P^\ell \subset \mathbb{C}P^m, \qquad k = 2\ell;$$

(2) a totally geodesic totally real subspace

$$\mathbb{R}P^k \subset \mathbb{C}P^m;$$

(3) an extrinsic sphere contained in a totally geodesic totally real projective subspace,

$$S^k(r) \subset \mathbb{R}P^{k+1} \subset \mathbb{C}P^m, \qquad 0 < r < \frac{\pi}{2}.$$

We compute the second eigenvalue of L for each of these models.

Proposition 5.1. *Let Σ^k be a closed, connected, totally umbilical submanifold of $(\mathbb{C}P^m, g_0)$, with $k \geq 2$. Then the following statements hold.*

(1) *If $\Sigma = \mathbb{C}P^\ell \subset \mathbb{C}P^m$, $k = 2\ell$, is a totally geodesic complex projective subspace, then $\lambda_2(L) = k + 4$.*
(2) *If $\Sigma = \mathbb{R}P^k \subset \mathbb{C}P^m$ is the standard totally geodesic totally real projective subspace, then $\lambda_2(L) = k + 2$.*
(3) *If the immersion is the double covering $S^k \longrightarrow \mathbb{R}P^k \subset \mathbb{C}P^m$, then $\lambda_2(L) = 0$.*
(4) *If $\Sigma = S^k(r) \subset \mathbb{R}P^{k+1} \subset \mathbb{C}P^m$, $0 < r < \frac{\pi}{2}$, is a non-totally-geodesic extrinsic sphere, then $\lambda_2(L) = 0$.*

Proof. We consider the four cases separately.

Case 1: the complex projective subspace. Let $\Sigma = \mathbb{C}P^\ell \subset \mathbb{C}P^m$, $k = 2\ell$. The inclusion is totally geodesic, and hence $\sigma = 0$, $L = \Delta + k$. For the Fubini-Study metric of holomorphic sectional curvature 4, the Laplace spectrum of $\mathbb{C}P^\ell$ is

$$\mu_j = 4j(j + \ell), \qquad j = 0, 1, 2, \ldots.$$

In particular, $\mu_0 = 0$, $\mu_1 = 4(\ell+1)$. Since the potential of L is the constant $k = 2\ell$, the first eigenvalue of L is $\lambda_1(L) = -k$, and the second eigenvalue is

$$\lambda_2(L) = \mu_1 - k = 4(\ell+1) - 2\ell = 2\ell + 4 = k + 4.$$

The tangent bundle of $\mathbb{C}P^\ell$ is invariant under the complex structure. Hence

$$\Theta_{\mathbb{C}}(T_p\Sigma) = k$$

at every point. Therefore the right-hand side of (15) is

$$k + 2 + \frac{2}{k\,\mathrm{Vol}(\Sigma)} \int_\Sigma k\, d\Sigma = k + 4.$$

Thus the totally geodesic complex projective subspace realizes equality in (15) and also in the universal estimate $\lambda_2(L) \le k + 4$.

Case 2: the real projective subspace. Let $\Sigma = \mathbb{R}P^k \subset \mathbb{C}P^m$, $k \le m$, be the standard totally geodesic and totally real inclusion. The induced metric has constant sectional curvature 1, and $\sigma = 0$. The Laplace eigenvalues of the unit real projective space are

$$\mu_j = 2j(2j + k - 1), \qquad j = 0, 1, 2, \ldots.$$

Indeed, only the even spherical harmonics on S^k descend through the antipodal quotient. Thus $\mu_0 = 0$, $\mu_1 = 2(k+1)$. It follows that

$$\lambda_2(L) = \mu_1 - k = 2(k+1) - k = k + 2.$$

Since $\mathbb{R}P^k$ is totally real and isotropic in $\mathbb{C}P^m$, meaning that $J(T_{[x]}\mathbb{R}P^k) \perp T_{[x]}\mathbb{R}P^k$ for each $[x] \in \mathbb{R}P^k$, we have $\Theta_{\mathbb{C}}(T_p\Sigma) = 0$, $p \in \Sigma$. Consequently, the right-hand side of (15) is precisely $k+2$. Hence the totally geodesic real projective subspace also realizes equality in the sharp estimate, although it does not realize equality in the coarser estimate $\lambda_2(L) \le k + 4$.

Case 3: the spherical covering of the real projective subspace. Consider the totally geodesic immersion $\iota : S^k \longrightarrow \mathbb{R}P^k \subset \mathbb{C}P^m$ given by the antipodal covering. The induced metric on S^k is the unit round metric, and again $\sigma = 0$. However, the first nonzero Laplace eigenvalue of S^k is

$$\mu_1(S^k) = k,$$

because the degree-one spherical harmonics are now admissible. Therefore

$$\lambda_2(L) = \mu_1(S^k) - k = 0.$$

This shows that the value of $\lambda_2(L)$ depends not only on the image of the immersion but also on its parametrization. In particular, having totally geodesic image does not by itself guarantee equality in (15).

Case 4: the extrinsic spheres. Let $S^k(r) \subset \mathbb{R}P^{k+1} \subset \mathbb{C}P^m$, $0 < r < \frac{\pi}{2}$, be a geodesic sphere of radius r. Since $\mathbb{R}P^{k+1} \subset \mathbb{C}P^m$ is totally geodesic, the second fundamental form of $S^k(r)$ in $\mathbb{C}P^m$ agrees with its second fundamental form in $\mathbb{R}P^{k+1}$.

The induced metric is the round metric of radius $\sin r$. Therefore the first nonzero Laplace eigenvalue is

$$\mu_1 = \frac{k}{\sin^2 r}.$$

The sphere is totally umbilical, with principal curvatures equal to $\cot r$, and consequently $|\sigma|^2 = k \cot^2 r$. It follows that

$$\lambda_2(L) = \frac{k}{\sin^2 r} - k\cot^2 r - k = \frac{k}{\sin^2 r} - \frac{k\cos^2 r}{\sin^2 r} - k = 0.$$

Since, the sphere is contained in the totally real isotropic subspace $\mathbb{R}P^{k+1}$, we have $\Theta_{\mathbb{C}}(T_pS^k(r)) = 0$. The right-hand side of (15) is therefore $k+2$, whereas

$$\lambda_2(L) = 0.$$

Thus a non-totally-geodesic totally umbilical submanifold does not realize equality in the sharp estimate. □

Corollary 5.2. *Let $\iota : \Sigma^k \longrightarrow \mathbb{C}P^m$, $k \geq 2$, be a closed connected immersion. If equality holds in (15), then the image of Σ is totally geodesic. More precisely, up to an ambient isometry, the equality models are*

$$\mathbb{C}P^\ell \subset \mathbb{C}P^m, \qquad k = 2\ell,$$

and

$$\mathbb{R}P^k \subset \mathbb{C}P^m.$$

In the real-projective case, equality requires the immersion to have the spectrum of $\mathbb{R}P^k$; in particular, the double covering $S^k \longrightarrow \mathbb{R}P^k$ does not realize equality.

Proof. Equality in (15) implies, by Theorem 4.1, that Σ is totally umbilical. The classification recalled above leaves three possible types. By Proposition 5.1, every non-totally-geodesic extrinsic sphere satisfies $\lambda_2(L) = 0$ and hence cannot realize equality. Therefore the image must be either a totally geodesic complex projective subspace or a totally geodesic real projective subspace. The computation for the double covering shows that the parametrization must also be taken into account. □

Remark 5.3. The two totally geodesic equality models correspond to the two extreme types of tangent planes in complex projective space. For a complex k-plane, $\Theta_{\mathbb{C}} = k$, and equality gives $\lambda_2(L) = k+4$. For a totally real isotropic k-plane, $\Theta_{\mathbb{C}} = 0$, and equality gives $\lambda_2(L) = k+2$. The correction term $\Theta_{\mathbb{C}}$ therefore distinguishes spectrally between the complex and totally real isotropic equality models.

5.2. **Totally umbilical submanifolds of quaternionic projective space.** We next consider quaternionic projective space. Throughout this subsection, $\mathbb{H}P^m$ is endowed with its standard metric g_0, normalized to have quaternionic sectional curvature 4. As before, we use the convention

$$Lu + \lambda u = 0, \qquad L = \Delta + |\sigma|^2 + k.$$

According to the classification of Chen [7], every totally umbilical submanifold of real dimension $k > 4$ in $\mathbb{H}P^m$ is locally one of the following:

(1) a totally geodesic quaternionic projective subspace

$$\mathbb{H}P^\ell \subset \mathbb{H}P^m, \qquad k = 4\ell;$$

(2) a totally geodesic complex projective subspace

$$\mathbb{C}P^\ell \subset \mathbb{H}P^m, \qquad k = 2\ell;$$

(3) a totally geodesic totally real projective subspace

$$\mathbb{R}P^k \subset \mathbb{H}P^m;$$

(4) an extrinsic sphere contained in a totally geodesic totally real projective subspace,

$$S^k(r) \subset \mathbb{R}P^{k+1} \subset \mathbb{H}P^m, \qquad 0 < r < \frac{\pi}{2}.$$

The next result computes the second eigenvalue of the operator $L = \Delta + |\sigma|^2 + k$ on these models.

Proposition 5.4. *Let Σ^k be one of the standard totally umbilical submanifolds of $(\mathbb{H}P^m, g_0)$ listed above. We have:*

(1) *if $\Sigma = \mathbb{H}P^\ell \subset \mathbb{H}P^m$, $k = 4\ell$, then $\lambda_2(L) = k + 8$;*
(2) *if $\Sigma = \mathbb{C}P^\ell \subset \mathbb{H}P^m$, $k = 2\ell$, then $\lambda_2(L) = k + 4$;*
(3) *if $\Sigma = \mathbb{R}P^k \subset \mathbb{H}P^m$, then $\lambda_2(L) = k + 2$;*
(4) *if the immersion is the double covering $S^k \longrightarrow \mathbb{R}P^k \subset \mathbb{H}P^m$, then $\lambda_2(L) = 0$;*
(5) *if $\Sigma = S^k(r) \subset \mathbb{R}P^{k+1} \subset \mathbb{H}P^m$, $0 < r < \frac{\pi}{2}$, then $\lambda_2(L) = 0$.*

Proof. We analyze each model separately.

Case 1: quaternionic projective subspaces. Let $\Sigma = \mathbb{H}P^\ell \subset \mathbb{H}P^m$, $k = 4\ell$. The inclusion is totally geodesic, so that $\sigma = 0$, $L = \Delta + k$. For the standard metric of quaternionic sectional curvature 4, the eigenvalues of the Laplacian on $\mathbb{H}P^\ell$ are

$$\mu_j = 4j(j + 2\ell + 1), \qquad j = 0, 1, 2, \ldots;$$

see, for instance, [3, 5]. Hence $\mu_0 = 0$, $\mu_1 = 8(\ell + 1)$. It follows that

$$\lambda_2(L) = \mu_1 - k = 8(\ell + 1) - 4\ell = 4\ell + 8 = k + 8.$$

For a quaternionic tangent plane, all three local quaternionic structures preserve the tangent space. Consequently, $\Theta_{\mathbb{H}}(T_p\Sigma) = 3k$. The right-hand side of (15) is therefore

$$k + 2 + \frac{2}{k\,\mathrm{Vol}(\Sigma)} \int_\Sigma 3k\, d\Sigma = k + 8.$$

Thus $\mathbb{H}P^\ell$ realizes equality both in the sharp estimate and in the universal estimate $\lambda_2(L) \le k + 8$.

Case 2: complex projective subspaces. Let $\Sigma = \mathbb{C}P^\ell \subset \mathbb{H}P^m$, $k = 2\ell$, be a standard totally geodesic complex projective subspace. Its induced metric is the Fubini-Study metric of holomorphic sectional curvature 4, and $\sigma = 0$. Therefore $\mu_1(\Delta) = 4(\ell + 1)$, and hence

$$\lambda_2(L) = 4(\ell + 1) - 2\ell = 2\ell + 4 = k + 4.$$

Along $\mathbb{C}P^\ell$, one of the three local quaternionic structures preserves the tangent space, whereas the other two map it into the normal space. Therefore, it follows that $\Theta_{\mathbb{H}}(T_p\Sigma) = k$. Consequently, the right-hand side of (15) becomes

$$k + 2 + \frac{2}{k\,\mathrm{Vol}(\Sigma)} \int_\Sigma k\, d\Sigma = k + 4.$$

Thus the standard complex projective subspaces also realize equality in the sharp estimate (15).

Case 3: real projective subspaces. Let $\Sigma = \mathbb{R}P^k \subset \mathbb{H}P^m$ be a standard totally real and totally geodesic subspace. Its induced metric has constant sectional curvature 1, and $\sigma = 0$. The first nonconstant eigenvalue of Δ on the unit $\mathbb{R}P^k$ is $\mu_1 = 2(k + 1)$. Therefore

$$\lambda_2(L) = 2(k + 1) - k = k + 2.$$

Since the tangent space is totally real and isotropic with respect to each of the three quaternionic structures, $\Theta_{\mathbb{H}}(T_p\Sigma) = 0$. Hence the right-hand side of (15) is exactly $k+2$, and the real projective subspaces realize equality in the sharp estimate.

Case 4: the spherical covering of $\mathbb{R}P^k$. Consider the totally geodesic immersion

$$S^k \longrightarrow \mathbb{R}P^k \subset \mathbb{H}P^m$$

given by the antipodal covering. Although its image is totally geodesic, the induced domain is the unit round sphere, whose first nonzero Laplace eigenvalue is $\mu_1(S^k) = k$. Since $\sigma = 0$, we obtain

$$\lambda_2(L) = k - k = 0.$$

Thus, as in the complex-projective setting, the value of $\lambda_2(L)$ also depends on the parametrization of the totally geodesic image.

Case 5: extrinsic spheres. Let $S^k(r) \subset \mathbb{R}P^{k+1} \subset \mathbb{H}P^m$, $0 < r < \frac{\pi}{2}$. Because $\mathbb{R}P^{k+1}$ is totally geodesic in $\mathbb{H}P^m$, the second fundamental form of $S^k(r)$ in $\mathbb{H}P^m$ agrees with its second fundamental form in $\mathbb{R}P^{k+1}$.

The induced metric is round with radius $\sin r$, and therefore

$$\mu_1(\Delta) = \frac{k}{\sin^2 r}.$$

Moreover, all principal curvatures are equal to $\cot r$, so $|\sigma|^2 = k\cot^2 r$. It follows that

$$\lambda_2(L) = \frac{k}{\sin^2 r} - k\cot^2 r - k = 0.$$

Since $T_pS^k(r) \subset T_p\mathbb{R}P^{k+1}$, one has $\Theta_{\mathbb{H}}(T_pS^k(r)) = 0$. Therefore the right-hand side of (15) is $k+2$, while $\lambda_2(L) = 0$. This shows that non-totally-geodesic extrinsic spheres do not realize equality in (15). ∎

Corollary 5.5. *Let $\iota : \Sigma^k \longrightarrow \mathbb{H}P^m$, $k > 4$, be a closed connected immersion. If equality holds in* (15)*, then the image of Σ is totally geodesic. More precisely, up to an ambient isometry, the possible images are*

$$\mathbb{R}P^k, \qquad \mathbb{C}P^{k/2}, \qquad \mathbb{H}P^{k/4},$$

whenever the corresponding divisibility and dimensional conditions are satisfied.

Proof. By Theorem 4.1, equality in (15) implies that Σ is totally umbilical. Since $k > 4$, Chen's classification [7] applies. On the other hand, Proposition 5.4 shows that the non-totally-geodesic extrinsic spheres satisfy $\lambda_2(L) = 0$ and hence cannot realize equality. Therefore the image is one of the totally geodesic real, complex, or quaternionic projective subspaces. ∎

Remark 5.6. The three totally geodesic equality models are distinguished by the value of the quaternionic tangent-plane invariant:

model	$\Theta_{\mathbb{H}}(T_p\Sigma)$	$\lambda_2(L)$
$\mathbb{R}P^k$	0	$k+2,$
$\mathbb{C}P^{k/2}$	k	$k+4,$
$\mathbb{H}P^{k/4}$	$3k$	$k+8.$

Thus $\Theta_{\mathbb{H}}$ detects whether the tangent space is totally real isotropic, complex, or quaternionic and gives the exact correction required in the sharp estimate.

5.3. **Totally umbilical submanifolds of the Cayley plane.** We conclude the discussion of totally umbilical models with the Cayley projective plane. The classification in this setting is more involved than in the real, complex, and quaternionic projective cases. By the results of Chen [6] and Nikolaevskij [23], a totally umbilical submanifold of $\mathbb{O}P^2$ has real dimension at most eight. Moreover, every totally umbilical submanifold that is maximal with respect to inclusion is, up to an ambient isometry, one of the following:

(1) a totally geodesic quaternionic projective plane

$$\mathbb{H}P^2 \subset \mathbb{O}P^2;$$

(2) a totally geodesic Cayley line

$$\mathbb{O}P^1 \cong S^8(1/2) \subset \mathbb{O}P^2;$$

(3) an extrinsic sphere contained in a totally geodesic Cayley line;
(4) a non-totally-geodesic totally umbilical submanifold contained in a totally geodesic copy of $\mathbb{H}P^2$.

Here, $\mathbb{O}P^2$ is equipped with its standard metric g_0, normalized so that its sectional curvatures lie between 1 and 4. With this normalization, the Cayley line $\mathbb{O}P^1$ is isometric to the round sphere $S^8(1/2)$ of constant sectional curvature 4.

Unlike the complex and quaternionic projective cases, total umbilicity in $\mathbb{O}P^2$ does not immediately reduce the equality analysis to totally geodesic projective subspaces and real extrinsic spheres. We next compute $\lambda_2(L)$ for the explicit models in the first three classes. The models in the fourth class will not be treated, since their induced geometry and spectral data are not determined by the preceding description alone.

Proposition 5.7. *For the operator $L = \Delta + |\sigma|^2 + k$, the following statements hold.*

(1) *On the totally geodesic quaternionic projective plane $\mathbb{H}P^2 \subset \mathbb{O}P^2$, one has $\lambda_2(L) = 16$.*
(2) *On the totally geodesic Cayley line $\mathbb{O}P^1 \cong S^8(1/2) \subset \mathbb{O}P^2$, one has $\lambda_2(L) = 24$.*
(3) *Let $S^k(r) \subset S^{k+1}(1/2) \subset \mathbb{O}P^1 \subset \mathbb{O}P^2$, $0 < r < \frac{\pi}{2}$, be a geodesic k-sphere of radius r. Then $\lambda_2(L) = 3k$. For $r = \pi/4$, this is a totally geodesic great sphere; for $r \neq \pi/4$, it is a non-totally-geodesic extrinsic sphere.*

Proof. We consider the three cases separately.

Case 1: the quaternionic projective plane. The standard inclusion $\mathbb{H}P^2 \subset \mathbb{O}P^2$ is totally geodesic. Hence $\sigma = 0$, $k = 8$, $L = \Delta + 8$. For the metric of quaternionic sectional curvature 4, the Laplace spectrum of $\mathbb{H}P^\ell$ is

$$\mu_j = 4j(j + 2\ell + 1), \qquad j = 0, 1, 2, \dots;$$

see [3, 5]. In particular, $\mu_1 = 8(\ell + 1)$. Taking $\ell = 2$, we obtain $\mu_1 = 24$. Consequently,

$$\lambda_2(L) = \mu_1 - k = 24 - 8 = 16.$$

Case 2: the Cayley line. The Cayley line is a totally geodesic submanifold satisfying $\mathbb{O}P^1 \cong S^8(1/2)$. The first nonzero Laplace eigenvalue of the round sphere $S^k(\rho)$ is k/ρ^2.

Therefore $\mu_1\Big(S^8(1/2)\Big) = \frac{8}{(1/2)^2} = 32$. Since $\sigma = 0$ and $k = 8$, it follows that

$$\lambda_2(L) = 32 - 8 = 24.$$

Case 3: geodesic spheres in a Cayley line. Consider a totally geodesic round subspace $S^{k+1}(1/2) \subset S^8(1/2) \cong \mathbb{O}P^1$ and a geodesic sphere $S^k(r) \subset S^{k+1}(1/2)$. Since all the successive ambient inclusions are totally geodesic, the second fundamental form of $S^k(r)$ in $\mathbb{O}P^2$ agrees with its second fundamental form in $S^{k+1}(1/2)$.

The round sphere $S^{k+1}(1/2)$ has constant sectional curvature 4. The metric induced on the geodesic sphere of radius r is the round metric of radius $\frac{1}{2}\sin(2r)$. Its first nonzero Laplace eigenvalue is therefore

$$\mu_1 = \frac{k}{\frac{1}{4}\sin^2(2r)} = \frac{4k}{\sin^2(2r)}.$$

The geodesic sphere is totally umbilical, and all its principal curvatures are equal to $2\cot(2r)$. It follows that $|\sigma|^2 = 4k\cot^2(2r)$. Hence

$$\lambda_2(L) = \frac{4k}{\sin^2(2r)} - 4k\cot^2(2r) - k = 4k\left(\frac{1-\cos^2(2r)}{\sin^2(2r)}\right) - k = 3k.$$

Thus $\lambda_2(L)$ is independent of the radius. Notice that $r = \pi/4$ corresponds to a totally geodesic great sphere, whereas $r \neq \pi/4$ gives a non-totally-geodesic extrinsic sphere. □

Remark 5.8. Proposition 5.7 treats the explicit totally umbilical models for which the induced metric and the second fundamental form are directly available. The general classification also allows non-totally-geodesic totally umbilical submanifolds contained in a totally geodesic copy of $\mathbb{H}P^2$. These remaining models are not covered by Proposition 5.7, since their spectral contribution and the corresponding value of $\mathcal{A}_{\mathbb{O}}(T_p\Sigma)$ require a separate analysis.

Consequently, the preceding calculations determine the second eigenvalue on the standard totally geodesic models and on the geodesic spheres contained in a Cayley line, but they do not provide a complete spectral classification of all totally umbilical submanifolds of $\mathbb{O}P^2$. For this reason, in the equality statement of Theorem 4.5, we retain the conclusion that equality forces Σ to be totally umbilical, without claiming that it must be totally geodesic.

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