Some rigidity results for charged initial data sets

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ABSTRACT

In this note, we consider some initial data rigidity results concerning marginally outer trapped surfaces (MOTS). As is well known, MOTS play an important role in the theory of black holes and, at the same time, are interesting spacetime analogues of minimal surfaces in Riemannian geometry. The main results presented here expand upon earlier works by the authors, specifically addressing initial data sets incorporating charge.

1. Introduction

Over the years, it has become clear that *marginally outer trapped surfaces* (MOTS for short) are a fundamental concept in the realm of general relativity, particularly in the study of black holes and gravitational collapse. These surfaces serve as crucial tools for understanding the complex nature of spacetime in regions of intense gravitational fields.

From a mathematical perspective, the importance of the theory of MOTS can be demonstrated through, for example, their role in proving positive mass theorems (see *e.g.* [1–4]). Positive mass results are fundamental in general relativity, as they assert that the total mass of an isolated gravitational system is nonnegative, reflecting the physical reality that energy cannot be negative.

In this article, we aim to advance our understanding of the geometry of these crucial objects (the MOTS) and, perhaps, shed some light on the intricate nature of spacetime dynamics, black holes, and gravitational collapse.

In [5], motivated in part by a result due to H. Bray, S. Brendle, and A. Neves [6], the authors considered the case of a spherical MOTS in a 3-dimensional initial data set and proved the rigidity result that we paraphrase as follows (definitions are given in Section 2):

Theorem 1.1 ([5, Theorem 3.2])**.** *Let (M, g, K) be a 3-dimensional initial data set. Assume that (M, g, K) satisfies the energy condition, $\mu - |J| \geq c$, for some constant $c > 0$. If Σ is a closed weakly outermost and outer area-minimizing MOTS in (M, g, K), then Σ is topologically S^2 and its area satisfies*

$$|\Sigma| \leq \frac{4\pi}{c}.$$

Furthermore, if equality holds, then there exists an outer neighborhood $U \cong [0, \delta) \times \Sigma$ of Σ in M such that:

(a) *(U, g) is isometric to $([0, \delta) \times \Sigma, dt^2 + g_0)$, for some $\delta > 0$, where g_0 - the induced metric on Σ - has constant Gaussian curvature $\kappa = c$;*

(b) *$K = f\, dt^2$ on U, where $f \in C^\infty(U)$ depends only on $t \in [0, \delta)$;*

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(c) $\mu = c$ and $J = 0$ on U.

Above, μ and J are the *local energy density* and the *local current density* of (M, g, K), respectively, defined in terms of the spacetime Einstein tensor G by $\mu = G(u, u)$ and $J = G(u, \cdot)|_M$, where u is the future directed timelike unit normal to M. The Gauss-Codazzi equations give that μ and J are initial data quantities.

More recently, the authors obtained a global version of Theorem 1.1, which, in particular, does not require the weakly outermost condition; see Theorem 1.2 below. This was inspired in part by their joint work with M. Eichmair [7], where, among other things, they obtained a global version of the local rigidity result in [8] in connection with J. Lohkamp's approach to the spacetime positive mass theorem in [9].

Theorem 1.2 ([10, Theorem 3.1]). *Let (M, g, K) be a 3-dimensional compact-with-boundary initial data set. Assume that (M, g, K) satisfies the energy condition, $\mu - |J| \geq c$, for some constant $c > 0$. Assume also that the boundary of M can be expressed as a disjoint union $\partial M = \Sigma_0 \cup S$ of nonempty unions of components such that the following conditions hold:*

1. *$\theta^+ \leq 0$ on Σ_0 with respect to the normal that points into M;*
2. *$\theta^+ \geq 0$ on S with respect to the normal that points out of M;*
3. *M satisfies the homotopy condition with respect to Σ_0;*
4. *the relative homology group $H_2(M, \Sigma_0)$ vanishes;*
5. *Σ_0 minimizes area.*

Then Σ_0 is topologically S^2 and its area satisfies

$$|\Sigma_0| \leq \frac{4\pi}{c}.$$

Furthermore, if equality holds, then

(a) *(M, g) is isometric to $([0, \ell] \times \Sigma_0, dt^2 + g_0)$, for some $\ell > 0$, where g_0 - the induced metric on Σ_0 - has constant Gaussian curvature $\kappa = c$;*
(b) *$K = f \, dt^2$ on M, where $f \in C^\infty(M)$ depends only on $t \in [0, \ell]$;*
(c) *$\mu = c$ and $J = 0$ on M.*

While the above result does not require the weakly outermost condition, it retains the area minimizing condition, which, of course, is an essential condition for the main result in [6], and which, from the initial data point of view, corresponds to the case $K = 0$.

The aim of this paper is to extend Theorems 1.1 and 1.2 to spherical MOTS in 3-dimensional charged initial data sets, that is, initial data for the Einstein–Maxwell equations with vanishing magnetic field. See [11] for some results related to Theorem 1.1.

The paper is organized as follows: in Section 2, we present some basic definitions; in Section 3, we prove an infinitesimal rigidity result (Proposition 3.1) for closed stable spherical MOTS in 3-dimensional charged initial data sets and use this result to obtain an extension of Theorem 1.1; this is the content of Theorem 3.2. Finally, in Section 4, we use the results obtained in the previous section to extend Theorem 1.2 to the context of MOTS in 3-dimensional compact-with-boundary initial data sets for the Einstein–Maxwell equations with vanishing magnetic field; see Theorem 4.1.

2. Preliminaries

We first discuss, based on natural physical considerations, the energy condition relevant to general non-time-symmetric initial data sets that allows for the presence of an electric field, but not a magnetic field.

Let (M, g, K) be an initial data set in a spacetime $(\bar{M}, \bar{g})$, i.e. M is a spacelike hypersurface with induced metric g and second fundamental form K with respect to the future directed timelike unit normal to M. Assume $(\bar{M}, \bar{g})$ satisfies the Einstein equation,

$$G + \Lambda \bar{g} = 8\pi(T_F + T),$$

where G is the Einstein tensor, $G = \operatorname{Ric}_{\bar{M}} - \frac{1}{2} R_{\bar{M}} \bar{g}$, T_F is the electromagnetic energy–momentum tensor, and T is the energy–momentum tensor associated to any nongravitational and nonelectromagnetic fields (*e.g.* matter fields) that may be present.

Let u be the future directed timelike unit normal vector field along M. As is standard, by the Gauss-Codazzi equations, we have

$$\mu := G(u, u) = \frac{1}{2} \left(R_M - |K|^2 + \tau^2 \right) \quad \text{and} \quad J := G(u, \cdot) = \operatorname{div}(K - \tau g),$$

where R_M is the scalar curvature of M and $\tau = \operatorname{tr} K$ is the mean curvature of M in $\bar{M}$ with respect to u.

Suppose that the fields associated to T satisfy the *dominant energy condition*, that is,

$$T(X, Y) \geq 0 \quad \text{for all future directed causal vectors} \quad X, Y.$$

Then, for all unit vectors v that are tangent to M,

$$G(u, u + v) + \Lambda \bar{g}(u, u + v) = 8\pi(T_F(u, u + v) + T(u, u + v)),$$

and hence,

$$\mu + J(v) = G(u, u) + G(u, v) \geq \Lambda + 8\pi(T_F(u, u) + T_F(u, v)).$$

Now suppose that there is no magnetic field present, $B = 0$. It then follows that $T_F(u, v) = 0$. Physically, this corresponds to the vanishing of the Poynting vector in every Lorentz orthogonal frame. Moreover, $T_F(u, u)$ represents the electromagnetic energy density and is given by the expression $\frac{1}{8\pi}|E|^2$, where E is the electric field. We thus arrive at the initial data inequality,

$$\mu + J(v) \geq \Lambda + |E|^2, \tag{2.1}$$

which holds at each point $p \in M$ and for each unit vector $v \in T_p M$. As this holds for all such vectors v, (2.1) in turn implies,

$$\mu - |J| \geq \Lambda + |E|^2,$$

which we refer to as the *charged dominant energy condition*.

Now let Σ be a closed embedded hypersurface in M.

In this paper, we assume that M and Σ are orientable; in particular, Σ is two-sided. Then, we fix a unit normal vector field v along Σ; if Σ separates M, by convention, we say that v points to the *outside* of Σ.

In the sequel, we are going to present some important definitions to our purposes.

The *charge* of Σ with respect to v is defined by

$$\mathfrak{q}(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} \langle E, v \rangle.$$

The *null second fundamental forms* χ^+ and χ^- of Σ in (M, g, K) are defined by

$$\chi^+ = K|_{\Sigma} + A \quad \text{and} \quad \chi^- = K|_{\Sigma} - A,$$

where A is the second fundamental form of Σ in (M, g) with respect to v; more precisely,

$$A(X, Y) = g(\nabla_X v, Y) \quad \text{for} \quad X, Y \in \mathfrak{X}(\Sigma),$$

where ∇ is the Levi-Civita connection of (M, g).

The *null expansion scalars* or the *null mean curvatures* θ^+ and θ^- of Σ in (M, g, K) with respect to v are defined by

$$\theta^+ = \operatorname{tr} \chi^+ = \operatorname{tr}_{\Sigma} K + H \quad \text{and} \quad \theta^- = \operatorname{tr} \chi^- = \operatorname{tr}_{\Sigma} K - H,$$

where $H = \operatorname{tr} A$ is the mean curvature of Σ in (M, g) with respect to v. Observe that $\theta^{\pm} = \operatorname{tr} \chi^{\pm}$.

After R. Penrose, Σ is said to be *trapped* if both θ^+ and θ^- are negative. Restricting our attention to one side, we say that Σ is *outer trapped* if θ^+ is negative and *marginally outer trapped* if θ^+ vanishes. In the latter case, we refer to Σ as a *marginally outer trapped surface* or a *MOTS*, for short.

Assume that Σ is a MOTS in (M, g, K) with respect to a unit normal v that is a boundary in M, i.e. v points towards a top-dimensional submanifold $M^+ \subset M$ such that $\partial M^+ = \Sigma \sqcup S$, where S (possibly $S = \varnothing$) is a union of components of ∂M (in particular, if Σ separates M). We think of M^+ as the region outside of Σ. Then we say that Σ is *outermost* (resp. *weakly outermost*) if there is no closed embedded hypersurface in M^+ with $\theta^+ \leq 0$ (resp. $\theta^+ < 0$) that is homologous to and different from Σ.

We say that Σ *minimizes area* in M if Σ has the least area in its homology class in M, i.e. $|\Sigma| \leq |\Sigma'|$ for every closed embedded hypersurface Σ' in M that is homologous to Σ. Similarly, Σ is said to be *outer area-minimizing* if Σ minimizes area in M^+.

An important notion that we are going to recall now is the notion of stability for MOTS.

Let Σ be a MOTS in (M, g, K) with respect to v and $t \to \Sigma_t$ be a variation of $\Sigma = \Sigma_0$ in M with variation vector field $\frac{\partial}{\partial t}|_{t=0} = \phi v$, for some $\phi \in C^{\infty}(\Sigma)$. Denote by $\theta^{\pm}(t)$ the null expansion scalars of Σ_t with respect to the unit normal v_t, where $v = v_t|_{t=0}$. It is well known that

$$\frac{\partial \theta^+}{\partial t}\Big|_{t=0} = -\Delta\phi + 2\langle X, \nabla\phi \rangle + (Q - |X|^2 + \operatorname{div} X)\phi,$$

where Δ and div are the Laplace and divergent operators of Σ with respect to the induced metric $\langle\,,\,\rangle$, respectively; $X \in \mathfrak{X}(\Sigma)$ is the vector field that is dual to the 1-form $K(v, \cdot)|_{\Sigma}$, and

$$Q = \frac{1}{2} R_{\Sigma} - (\mu + J(v)) - \frac{1}{2}|\chi^+|^2.$$

Here R_{Σ} represents the scalar curvature of Σ.

The operator

$$L\phi = -\Delta\phi + 2\langle X, \nabla\phi \rangle + (Q - |X|^2 + \operatorname{div} X)\phi, \quad \phi \in C^{\infty}(\Sigma),$$

is called the *MOTS stability operator*. It can be proved that L has a real eigenvalue λ_1, called the *principal eigenvalue* of L, such that $\operatorname{Re} \lambda \geq \lambda_1$ for any complex eigenvalue λ. Furthermore, the associated eigenfunction ϕ_1, $L\phi_1 = \lambda_1\phi_1$, is unique up to scale and can be chosen to be everywhere positive.

We say that Σ is *stable* if $\lambda_1 \geq 0$; this is equivalent to saying that $L\phi \geq 0$ for some positive function $\phi \in C^{\infty}(\Sigma)$. It is not difficult to see that if Σ is weakly outermost (in particular, if Σ is outermost), then Σ is stable.

Before finishing this section, we would like to point out that, in general (when Σ is not necessarily a MOTS), the first variation of θ^+ is given by

$$\frac{\partial \theta^+}{\partial t}\Big|_{t=0} = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2}\theta^+ + \tau\theta^+\right)\phi.$$

3. Infinitesimal rigidity and local splitting

The aim of this section is to generalize the local splitting result, Theorem 1.1 in the introduction, to initial data sets with charge. The first step is to establish an *infinitesimal rigidity* result; *cf.* [11, Proposition 1.1].

Proposition 3.1 (*Infinitesimal Rigidity*). *Let* (M, g, K, E) *be a 3-dimensional initial data set for the Einstein–Maxwell equations with vanishing magnetic field,* $B = 0$. *Assume that* (M, g, K, E) *satisfies the charged dominant energy condition,* $\mu - |J| \geq \Lambda + |E|^2$, *for some constant* $\Lambda > 0$. *If* Σ *is a closed stable MOTS in* (M, g, K), *then* Σ *is topologically* S^2 *and its charge and area satisfy*

$$4\Lambda q(\Sigma)^2 \leq 1, \tag{3.1}$$

$$2\pi\left(1 - \sqrt{1 - 4\Lambda q(\Sigma)^2}\right) \leq \Lambda|\Sigma| \leq 2\pi\left(1 + \sqrt{1 - 4\Lambda q(\Sigma)^2}\right). \tag{3.2}$$

Furthermore, if equality holds in one of the above inequalities, then

(a) $E = a\nu$ *on* Σ, *for some constant* a;
(b) Σ *is isometric to the round 2-sphere of Gaussian curvature* $\kappa = \Lambda + a^2$;
(c) *the null second fundamental form* χ^+ *of* Σ *vanishes*;
(d) $\mu + J(\nu) = \mu - |J| = \Lambda + a^2$ *on* Σ *and, in particular,* $J|_\Sigma = 0$;
(e) *the principal eigenvalue* λ_1 *of* L *equals zero.*

Observe that, if equality in (3.1) holds, then $\Lambda|\Sigma| = 2\pi$ and equalities in (3.2) also hold.

Proof. Let $\phi > 0$ be an eigenfunction associated to the principal eigenvalue λ_1 of L,

$$\lambda_1\phi = L\phi = -\Delta\phi + 2\langle X, \nabla\phi\rangle + (Q - |X|^2 + \operatorname{div} X)\phi.$$

Thus, dividing by ϕ, we get

$$0 \leq \lambda_1 = \operatorname{div} Y - |Y|^2 + Q \leq \operatorname{div} Y + Q, \tag{3.3}$$

where $Y = X - \nabla \ln \phi$. Therefore, integrating over Σ and using the Divergence Theorem, we obtain

$$0 \leq \int_\Sigma Q = \int_\Sigma \left(\kappa - (\mu + J(\nu)) - \frac{1}{2}|\chi^+|^2\right) \leq 4\pi(1 - \mathbf{g}(\Sigma)) - \int_\Sigma (\Lambda + |E|^2), \tag{3.4}$$

where above we have used the Gauss–Bonnet Theorem and inequalities

$$\mu + J(\nu) \geq \mu - |J| \geq \Lambda + |E|^2. \tag{3.5}$$

Here $\mathbf{g}(\Sigma)$ represents the genus of Σ. Inequalities in (3.4) give that $\mathbf{g}(\Sigma) = 0$, since $\Lambda > 0$.

On the other hand,

$$q(\Sigma)^2 = \left(\frac{1}{4\pi}\int_\Sigma \langle E, \nu\rangle\right)^2 \leq \frac{1}{(4\pi)^2}\left(\int_\Sigma |E|\right)^2 \leq \frac{|\Sigma|}{(4\pi)^2}\int_\Sigma |E|^2. \tag{3.6}$$

Then, using (3.6) into (3.4), we obtain

$$\Lambda|\Sigma| + \frac{(4\pi q(\Sigma))^2}{|\Sigma|} \leq 4\pi(1 - \mathbf{g}(\Sigma)) = 4\pi, \tag{3.7}$$

that is,

$$\Lambda|\Sigma|^2 - 4\pi|\Sigma| + (4\pi q(\Sigma))^2 \leq 0.$$

Completing the square in the last inequality, we get

$$(\Lambda|\Sigma| - 2\pi)^2 \leq (2\pi)^2(1 - 4\Lambda q(\Sigma)^2). \tag{3.8}$$

This gives (3.1) and (3.2).

If equality in (3.1) or (3.2) holds, then we have equality in (3.8), which gives that all above inequalities must be equalities. Therefore,

- equalities in (3.3) give that $\lambda_1 = 0$, $Y = 0$ and, in particular, $Q = 0$;
- equalities in (3.4) and (3.5) give that $\chi^+ = 0$;
- equalities in (3.6) give that E is parallel to ν and $|E|$ is constant; thus $E = a\nu$, for some constant a (here we have used that Σ is connected);

- equalities in (3.5), together with $E = av$, give that $\mu + J(v) = \mu - |J| = \Lambda + a^2$.

Finally,

$$0 = Q = \kappa - (\mu + J(v)) - \frac{1}{2}|\chi^+|^2 = \kappa - (\Lambda + a^2), \quad \text{that is,} \quad \kappa = \Lambda + a^2. \quad \square$$

We will now make use of Infinitesimal Rigidity to prove the following Local Splitting. A key feature of the latter result (in comparison with the $\Lambda > 0$ case of Theorem 1.2 in [11]) is that it does not assume a convexity condition on K, specifically that K be 2-convex; this is the requirement that the sum of the two smallest eigenvalues of K be nonnegative.

Theorem 3.2 (*Local Splitting*). *Let (M, g, K, E) be a 3-dimensional initial data set for the Einstein–Maxwell equations with vanishing magnetic field, $B = 0$. Assume that (M, g, K, E) satisfies the charged dominant energy condition, $\mu - |J| \geq \Lambda + |E|^2$, for some constant $\Lambda > 0$, and $\operatorname{div} E = 0$. If Σ is a closed weakly outermost and outer area-minimizing MOTS in (M, g, K), then Σ is topologically S^2 and its charge and area satisfy*

$$4\Lambda \mathfrak{q}(\Sigma)^2 \leq 1, \tag{3.9}$$

$$\Lambda |\Sigma| \leq 2\pi \left(1 + \sqrt{1 - 4\Lambda \mathfrak{q}(\Sigma)^2}\right). \tag{3.10}$$

Furthermore, if equality holds in one of the above inequalities, then there exists an outer neighborhood $U \cong [0, \delta) \times \Sigma$ of Σ in M such that

(a) $E = av_t$, for some constant a, where v_t is the unit normal to $\Sigma_t \cong \{t\} \times \Sigma$ in direction of the foliation;
(b) (U, g) is isometric to $([0, \delta) \times \Sigma, dt^2 + g_0)$, for some $\delta > 0$, where g_0 - the induced metric on Σ - has constant Gaussian curvature $\kappa = \Lambda + a^2$;
(c) $K = f dt^2$ on U, where $f \in C^\infty(U)$ depends only on $t \in [0, \delta)$;
(d) $\mu = \Lambda + a^2$ and $J = 0$ on U.

By Proposition 3.1, if equality in (3.9) holds, then $\Lambda |\Sigma| = 2\pi$ and equality in (3.10) also holds.

Proof. Since Σ is weakly outermost (in particular, stable), we may apply the Infinitesimal Rigidity (Proposition 3.1) to obtain that Σ is topologically S^2 and its charge and area satisfy (3.9) and (3.10). Furthermore, if equality in (3.9) or (3.10) holds, then conditions (a)–(e) in the Infinitesimal Rigidity also hold. In particular, as a consequence of the fact that the principal eigenvalue λ_1 of L equals zero, there exists a foliation of an outer neighborhood $U \cong [0, \delta) \times \Sigma$ of Σ in M by constant null mean curvature surfaces $\Sigma_t \cong \{t\} \times \Sigma$ (see [8, Lemma 2.3]), with $\Sigma_0 = \Sigma$, such that

$$g = \phi^2 dt^2 + g_t \quad \text{on} \quad U,$$

where g_t is the induced metric on Σ_t. Recall that, on Σ_t,

$$\frac{d\theta}{dt} = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2}\theta^2 + \tau\theta\right)\phi,$$

where $\theta = \theta(t)$ is the null mean curvature of Σ_t with respect to $v_t = \phi^{-1}\partial_t$. Therefore, dividing both sides of the above equation by ϕ and integrating over Σ_t, we obtain

$$\theta' \int_{\Sigma_t} \frac{1}{\phi} - \theta \int_{\Sigma_t} \tau = \int_{\Sigma_t} \left(\operatorname{div} Y - |Y|^2 + Q - \frac{1}{2}\theta^2\right) \leq \int_{\Sigma_t} Q$$

$$= \int_{\Sigma_t} \left(\kappa_t - (\mu + J(v_t)) - \frac{1}{2}|\chi_t^+|^2\right) \tag{3.11}$$

$$\leq 4\pi - \int_{\Sigma_t} (\Lambda + |E|^2),$$

where $Y = X - \nabla \ln\phi$ on Σ_t, and κ_t and χ_t^+ represent the Gaussian curvature and the null second fundamental form of Σ_t, respectively. Above, we have used the Divergence and the Gauss–Bonnet Theorems and inequalities

$$\mu + J(v_t) \geq \mu - |J| \geq \Lambda + |E|^2.$$

As in (3.6), we can see that

$$\mathfrak{q}(\Sigma_t)^2 \leq \frac{|\Sigma_t|}{(4\pi)^2} \int_{\Sigma_t} |E|^2. \tag{3.12}$$

Therefore, using (3.12) into (3.11), we get

$$\theta' \int_{\Sigma_t} \frac{1}{\phi} - \theta \int_{\Sigma_t} \tau \leq 4\pi - \left(\Lambda |\Sigma_t| + \frac{(4\pi\mathfrak{q}(\Sigma_t))^2}{|\Sigma_t|}\right).$$

On the other hand, the Divergence Theorem gives that $\mathfrak{q}(\Sigma_t) = \mathfrak{q}(\Sigma_0) =: \mathfrak{q}_0$ for all $t \in [0, \delta)$, since $\operatorname{div} E = 0$. Then, defining the function

$$h(t) = \Lambda |\Sigma_t| + \frac{(4\pi\mathfrak{q}_0)^2}{|\Sigma_t|},$$

we have that $h(0) = 4\pi$ (this follows from equality in (3.7)) and

$$\theta' \int_{\Sigma_t} \frac{1}{\phi} - \theta \int_{\Sigma_t} \tau \le h(0) - h(t).$$

Also, because $|\Sigma_0| = 2\pi \left(1 + \sqrt{1 - 4\Lambda \mathfrak{q}_0}\right)/\Lambda$ is the largest root of the polynomial

$$p(x) = \Lambda x^2 - 4\pi x + (4\pi \mathfrak{q}_0)^2$$

and $|\Sigma_0| \le |\Sigma_t|$, since $\Sigma_0 = \Sigma$ is outer area-minimizing, we have

$$0 \le p(|\Sigma_t|) = \Lambda |\Sigma_t|^2 - 4\pi |\Sigma_t| + (4\pi \mathfrak{q}_0)^2,$$

that is,

$$h(0) = 4\pi \le \Lambda |\Sigma_t| + \frac{(4\pi \mathfrak{q}_0)^2}{|\Sigma_t|} = h(t).$$

This gives that

$$\theta' \int_{\Sigma_t} \frac{1}{\phi} - \theta \int_{\Sigma_t} \tau \le h(0) - h(t) \le 0.$$

Taking $\alpha(t) = \int_{\Sigma_t} \tau / \int_{\Sigma_t} \frac{1}{\phi}$, we get

$$\left(\theta(t) e^{-\int_0^t \alpha(s) ds}\right)' \le 0,$$

which implies

$$\theta(t) e^{-\int_0^t \alpha(s) ds} \le \theta(0) = 0 \quad \text{for all} \quad t \in [0, \delta).$$

Since we are assuming that $\Sigma_0 = \Sigma$ is weakly outermost, we obtain that $\theta(t) = 0$ for all $t \in [0, \delta)$. In particular, all above inequalities must be equalities. Computations as in the proof of [5, Theorem 3.2] give conclusions (b)–(d) of the theorem. By equality in (3.12), we get that $E = av_t$ on Σ_t, for some function a that depends only on $t \in [0, \delta)$. Finally, using that Σ_t is minimal and $\text{div } E = 0$, we can see that a is constant. This finishes the proof of the theorem. □

3.1. The charged Nariai spacetime

The charged Nariai spacetime $(\bar{M}, \bar{g})$ is an exact solution to the (source-free) Einstein–Maxwell equations,

$$\begin{cases} G + \Lambda \bar{g} = 8\pi T_F, \\ dF = 0, \quad \text{div}_{\bar{g}} F = 0, \end{cases}$$

where F is a differential 2-form on $\bar{M}$ and T_F is the electromagnetic energy–momentum tensor given by

$$T_F = \frac{1}{4\pi} \left(F \circ F - \frac{1}{4} |F|_{\bar{g}}^2 \bar{g}\right), \quad (F \circ F)_{\alpha\beta} = \bar{g}^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}.$$

This is what is called an *electrovacuum solution* or *electrovacuum spacetime* (see Section 3 of [12] for a list of electrovacuum solutions, including the charged Nariai spacetime and the ultracold black hole system). In static coordinates,

$$\bar{g} = -V^2 dt^2 + ds^2 + \rho^2 d\Omega^2, \quad \bar{M} = \mathbb{R} \times (0, \pi/\alpha) \times S^2,$$

where $V(s) = \sin(\alpha s)$, $d\Omega^2$ is the round metric on S^2 of constant Gaussian curvature one, and $\alpha > 0$ and $\rho^2 > 0$ are suitable constants. More precisely, given parameters $m > 0$, $\Lambda > 0$ and $\mathfrak{q} \in \mathbb{R}$, representing the mass, the cosmological constant and the electric charge, respectively, satisfying

$$4\Lambda \mathfrak{q}^2 < 1 \quad \text{and} \quad m^2 = \frac{1}{18\Lambda} \left[1 + 12\Lambda \mathfrak{q}^2 + (1 - 4\Lambda \mathfrak{q}^2)^{3/2}\right],$$

ρ^2 is defined as the only solution to

$$\rho^2(1 - \Lambda \rho^2) = \mathfrak{q}^2 \quad \text{such that} \quad \frac{1}{2\Lambda} < \rho^2 \le \frac{1}{\Lambda},$$

that is,

$$\rho^2 = \frac{1 + \sqrt{1 - 4\Lambda \mathfrak{q}^2}}{2\Lambda}, \quad \text{and} \quad \alpha := \sqrt{\Lambda - \frac{\mathfrak{q}^2}{\rho^4}}.$$

The 2-form F is given by

$$F = -\frac{\mathfrak{q} \sin(\alpha s)}{\rho^2} dt \wedge ds$$

and, on a t-slice M of $\bar{M}$, the electric field is given by

$$E = \frac{\mathfrak{q}}{\rho^2} \partial_s.$$

Straightforward computations show that M is totally geodesic in $(\bar{M}, \bar{g})$, *i.e.* $K = 0$.

Now, defining $a = \mathfrak{q}/\rho^2$, we have

$$a^2 = \frac{\mathfrak{q}^2}{\rho^4} = \frac{1}{\rho^2} - \Lambda, \quad \text{that is,} \quad \rho^2 = \frac{1}{\Lambda + a^2}.$$

Therefore, the metric $\rho^2 d\Omega^2$ equals to metric g_0 of constant Gaussian curvature $\kappa = \Lambda + a^2$. Thus, the induced metric g on $M = (0, \pi/\alpha) \times S^2$ can be written as

$$g = ds^2 + g_0.$$

On the other hand, the area of $\Sigma = (S^2, g_0) = (S^2, \rho^2 d\Omega^2)$ satisfies

$$\Lambda|\Sigma| = 4\pi\Lambda\rho^2 = 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}^2} \right).$$

This shows that the t-slices of the charged Nariai spacetime satisfy the hypotheses of Theorem 3.2 when $4\Lambda\mathfrak{q}^2 < 1$ and (3.10) is saturated. When $\mathfrak{q} = 0$, we have the standard Nariai spacetime.

In a similar fashion, the t-slices of the ultracold black hole system satisfy the hypotheses of Theorem 3.2 when $4\Lambda\mathfrak{q}^2 = 1$ and $\Lambda|\Sigma| = 2\pi$.

4. Compact-with-boundary initial data sets

We now establish conditions under which the local splitting result can be extended to a global splitting.

Theorem 4.1. *Let (M, g, K, E) be a 3-dimensional compact-with-boundary initial data set for the Einstein–Maxwell equations with vanishing magnetic field, $B = 0$. Assume that (M, g, K, E) satisfies the charged dominant energy condition, $\mu - |J| \geq \Lambda + |E|^2$, for some constant $\Lambda > 0$, and $\operatorname{div} E = 0$. Assume also that the boundary of M can be expressed as a disjoint union $\partial M = \Sigma_0 \cup S$ of nonempty unions of components such that the following conditions hold:*

1. *$\theta^+ \leq 0$ on Σ_0 with respect to the normal that points into M;*
2. *$\theta^+ \geq 0$ on S with respect to the normal that points out of M;*
3. *M satisfies the homotopy condition with respect to Σ_0;*
4. *the relative homology group $H_2(M, \Sigma_0)$ vanishes;*
5. *Σ_0 minimizes area.*

Then Σ_0 is topologically S^2 and its charge and area satisfy

$$4\Lambda\mathfrak{q}(\Sigma_0)^2 \leq 1, \tag{4.1}$$

$$\Lambda|\Sigma_0| \leq 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}(\Sigma_0)^2} \right). \tag{4.2}$$

Furthermore, if equality holds in one of the above inequalities, then $M \cong [0, \ell] \times \Sigma_0$, for some $\ell > 0$, and

(a) *$E = a\nu_t$, for some constant a, where ν_t is the unit normal to $\Sigma_t \cong \{t\} \times \Sigma_0$ in direction of the foliation;*
(b) *(M, g) is isometric to $([0, \ell] \times \Sigma_0, dt^2 + g_0)$, where g_0 - the induced metric on Σ_0 - has constant Gaussian curvature $\kappa = \Lambda + a^2$;*
(c) *$K = f \, dt^2$ on M, where $f \in C^\infty(M)$ depends only on $t \in [0, \ell]$;*
(d) *$\mu = \Lambda + a^2$ and $J = 0$ on M.*

If equality in (4.1) holds, then $\Lambda|\Sigma_0| = 2\pi$ and equality in (4.2) also holds.

As in Theorem 1.2 in the introduction, the weakly outermost assumption is not required. We do, however, impose the same topological assumptions (3) and (4). By definition, M satisfies the *homotopy condition* with respect to $\Sigma \subset M$ provided there exists a continuous map $\rho : M \to \Sigma$ such that $\rho \circ i : \Sigma \to \Sigma$ is homotopic to id_Σ, where $i : \Sigma \hookrightarrow M$ is the inclusion map (for example, if Σ is a retract of M).

The following is the key lemma that enables us to apply the local splitting result.

Lemma 4.2. *Under the assumptions of Theorem 4.1, we have that*

(a) *$4\Lambda\mathfrak{q}(\Sigma_0)^2 \leq 1$ and*
(b) *Σ_0 is a weakly outermost MOTS whose area satisfies*

$$\Lambda|\Sigma_0| = 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}(\Sigma_0)^2} \right) \tag{4.3}$$

unless

$$\Lambda|\Sigma_0| < 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}(\Sigma_0)^2} \right).$$

Furthermore, the above inequality cannot happen if $4\Lambda\mathfrak{q}(\Sigma_0)^2 = 1$.

Proof. Assume, for the moment, that (a) is true. Suppose

$$\Lambda|\Sigma_0| \geq 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}(\Sigma_0)^2}\right).$$

We are going to prove that, in this case, Σ_0 is a weakly outermost MOTS whose area satisfies (4.3). Suppose, by contradiction, that Σ_0 is not a MOTS, *i.e.* $\theta_K^+ \leq 0$ and $\theta_K^+ \not\equiv 0$ on Σ_0. Therefore, it follows from [13, Lemma 5.2] that a small perturbation Σ of Σ_0 in M is such that $\theta_K^+ < 0$ on Σ; in particular, Σ is homologous to Σ_0 in M.

Now denote by W the compact region bounded by Σ and S in M and consider the initial data set $(W, g, -K)$. It follows that the null expansion scalars of Σ and S in $(W, g, -K)$ satisfy

- $\theta_{-K}^+ \leq 0$ on S with respect to the normal that points into W;
- $\theta_{-K}^+ > 0$ on Σ with respect to the normal that points out of W.

Therefore, by the MOTS Existence Theorem due to L. Andersson and J. Metzger [13] in dimension $n = 3$ and M. Eichmair [14,15] in dimensions $3 \leq n \leq 7$ (see [7, Theorem 2.3] for the version that we are using here), there exists an outermost MOTS Σ' in $(W, g, -K)$ that is homologous to Σ in W; in particular, Σ' is homologous to Σ_0 in M. Without loss of generality, we may assume that Σ' has only homologically nontrivial components.

Now observe that Σ_0 is connected, since M is connected and it satisfies the homotopy condition with respect to Σ_0. Because $H_2(M, \Sigma_0) = 0$ and Σ' is homologous to Σ_0, we have that Σ' is also connected. Thus, by the Infinitesimal Rigidity (Proposition 3.1),

$$\Lambda|\Sigma'| \leq 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}(\Sigma')^2}\right).$$

On the other hand, $|\Sigma_0| \leq |\Sigma'|$, since Σ_0 minimizes area. Also, because $\mathrm{div}\, E = 0$, the Divergence Theorem gives that $\mathfrak{q}(\Sigma') = \mathfrak{q}(\Sigma_0) =: \mathfrak{q}_0$. Therefore,

$$2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}_0^2}\right) \leq \Lambda|\Sigma_0| \leq \Lambda|\Sigma'| \leq 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}_0^2}\right),$$

that is,

$$\Lambda|\Sigma'| = \Lambda|\Sigma_0| = 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}_0^2}\right)$$

and the Local Splitting Theorem (Theorem 3.2) implies that an outer neighborhood of Σ' in $(W, g, -K)$ is foliated by MOTS (here we have used that Σ' minimizes area, since $|\Sigma'| = |\Sigma_0|$ and Σ_0 minimizes area), which contradicts the fact that Σ' is outermost. This proves that Σ_0 is a MOTS.

Now we are going to prove that Σ_0 is weakly outermost. Suppose, by contradiction, that Σ_0 is not weakly outermost, that is, suppose that there exists a surface Σ, homologous to Σ_0 in M, such that $\theta_K^+ < 0$ on Σ. Without loss of generality, we may assume that each component of Σ is homologically nontrivial. Denote by W the compact region bounded by Σ and S in M and consider the initial data set $(W, g, -K)$ as before. Then, repeating exactly the same arguments as above, we have a contradiction. Thus Σ_0 is a weakly outermost MOTS. To get that (4.3) holds and finish the proof of the first part of (b), we may apply the Infinitesimal Rigidity to obtain that

$$\Lambda|\Sigma_0| \leq 2\pi \left(1 + \sqrt{1 - 4\Lambda\mathfrak{q}(\Sigma_0)^2}\right),$$

since Σ_0 is weakly outermost and, in particular, stable.

Now, to finish the proof of (b), we are going to prove that, if $4\Lambda\mathfrak{q}(\Sigma_0)^2 = 1$, then $\Lambda|\Sigma_0| = 2\pi$. We claim that, in this case, Σ_0 is a weakly outermost (in particular, stable) MOTS. Therefore, by the Infinitesimal Rigidity, we have $\Lambda|\Sigma_0| = 2\pi$. In fact, suppose that Σ_0 is not a MOTS. Then, the same arguments as before can be applied to ensure the existence of a connected outermost MOTS Σ' in $(W, g, -K)$ that is homologous to Σ_0 in M. Then, since $\mathrm{div}\, E = 0$, we have $\mathfrak{q}(\Sigma') = \mathfrak{q}(\Sigma_0)$, and thus $4\Lambda\mathfrak{q}(\Sigma')^2 = 1$. Therefore, by the Local Splitting Theorem, an outer neighborhood of Σ' in $(W, g, -K)$ is foliated by MOTS, which is a contradiction. Analogously we can prove that Σ_0 is weakly outermost. This finishes the proof of (b).

Now let us prove (a). Assume, by contradiction, that $4\Lambda\mathfrak{q}(\Sigma_0)^2 > 1$. The above arguments give that Σ_0 is a weakly outermost MOTS as, otherwise, there exists a connected outermost (in particular, stable) MOTS Σ' in $(W, g, -K)$ that is homologous to Σ_0 in M. Then, by the Infinitesimal Rigidity,

$$1 \geq 4\Lambda\mathfrak{q}(\Sigma')^2 = 4\Lambda\mathfrak{q}(\Sigma_0)^2 > 1,$$

which is a contradiction. Thus, if $4\Lambda\mathfrak{q}(\Sigma_0)^2 > 1$, then Σ_0 is a weakly outermost MOTS, which also contradicts (3.1) in the Infinitesimal Rigidity. This finishes the proof of (a).

Proof of Theorem 4.1. First, we observe that Σ_0 is topologically S^2. In fact, if that is not the case, then Σ_0 is homeomorphic to a torus or to a connected sum of tori. As such, Σ_0 satisfies what is called the *cohomology condition* in [7]. The rigidity result Theorem 1.2 in [7] then applies so that $0 = \mu - |J| \geq \Lambda + |E|^2$, which is a contradiction. Hence Σ_0 is topologically S^2.

On the other hand, Lemma 4.2 gives inequalities (4.1) and (4.2); furthermore, if equality holds in (4.1) or (4.2), then Σ_0 is a weakly outermost MOTS whose area satisfies

$$\Lambda|\Sigma_0| = 2\pi\left(1 + \sqrt{1 - 4\Lambda q_0^2}\right),$$

where $q_0 := q(\Sigma_0)$. Therefore, we may apply the Local Splitting Theorem to obtain an outer neighborhood $U \cong [0, \delta) \times \Sigma_0$ of Σ_0 in M such that conclusions (a)–(d) of the theorem hold on U. Clearly, $\Sigma_t \cong \{t\} \times \Sigma_0$ converges to a closed embedded MOTS Σ_δ as $t \nearrow \delta$. If $\Sigma_\delta \cap S \neq \varnothing$, the Strong Maximum Principle implies that $\Sigma_\delta = S$. Otherwise, if $\Sigma_\delta \cap S = \varnothing$, we replace Σ_0 by Σ_δ and M by the complement of U and run the process again (this can be done since $q(\Sigma_\delta) = q_0$ and $|\Sigma_\delta| = |\Sigma_0|$). The theorem follows by a continuity argument. $\square$

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Data availability

No data was used for the research described in the article.

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