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Area-charge inequality and local rigidity in charged initial data sets

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Abstract

This paper investigates the geometric consequences of equality in area-charge inequalities for spherical minimal surfaces and, more generally, for marginally outer trapped surfaces, within the framework of the Einstein–Maxwell equations. We show that, under appropriate energy and curvature conditions, saturation of the inequality $\mathcal{A} \geqslant 4\pi(\mathcal{Q}_{\mathrm{E}}^2 + \mathcal{Q}_{\mathrm{M}}^2)$ imposes a rigid geometric structure in a neighborhood of the surface. In particular, the electric and magnetic fields must be normal to the foliation, and the local geometry is isometric to a Riemannian product. We establish two main rigidity theorems: one in the time-symmetric case and another for initial data sets that are not necessarily time-symmetric. In both cases, equality in the area-charge bound leads to a precise characterization of the intrinsic and extrinsic geometry of the initial data near the critical surface.

Keywords: area-charge inequality, minimal surfaces, marginally outer trapped surfaces, local rigidity results

1. Introduction

In his influential 1999 paper, Gibbons [21] explores the profound interplay between geometry and gravitation, with particular emphasis on the role of inverse mean curvature flow (IMCF) in the understanding of gravitational entropy. Among the key results discussed is the derivation of an *area-charge inequality*, which asserts that, under natural energy conditions, the area $\mathcal{A}$ of a closed, stable minimal surface enclosing an electric or magnetic charge $\mathcal{Q}$ in a time-symmetric initial data set must satisfy

$$\mathcal{A} \geqslant 4\pi \mathcal{Q}^2. \tag{1.1}$$

As noted in [21], this inequality also extends to maximal initial data sets that are not necessarily time-symmetric.

Inequality (1.1) expresses a fundamental geometric constraint imposed by general relativity: the area of a black hole horizon cannot be arbitrarily small for a given charge. In other words, if a black hole were to have charge $\mathcal{Q}$ but an area smaller than $4\pi\mathcal{Q}^2$, it would contradict physical expectations.

More recently, Dain *et al* [13] extended inequality (1.1) to the setting of dynamical black holes without making any symmetry assumptions. They showed that, if Σ is an orientable, closed, marginally outer trapped surface satisfying the *spacetime stably outermost condition*[1], in a spacetime that obeys the Einstein equations

$$G + \Lambda h = 8\pi \left(T^{\mathrm{EM}} + T^{\mathrm{matter}}\right),$$

with a non-negative cosmological constant Λ, and where the non-electromagnetic matter field T^{matter} satisfies the dominant energy condition DEC, then the following area-charge inequality holds:

$$\mathcal{A} \geqslant 4\pi \left(\mathcal{Q}_{\mathrm{E}}^2 + \mathcal{Q}_{\mathrm{M}}^2\right), \tag{1.2}$$

where $\mathcal{A}$, $\mathcal{Q}_{\mathrm{E}}$, and $\mathcal{Q}_{\mathrm{M}}$ denote the area, electric charge, and magnetic charge of Σ, respectively. Notably, no assumption is made that the matter fields are electrically neutral.

The aim of this paper is to investigate the geometric consequences of equality in (1.1) or (1.2), formulated in terms of initial data. More precisely, we show that, under suitable conditions, equality in either (1.1) or (1.2) implies that the initial data set containing Σ exhibits a specific, expected geometric structure in a vicinity of Σ.

Our first result is the following (see section 2 for definitions):

Theorem 1.1. *Let (M^3, g) be a Riemannian three-manifold with scalar curvature R satisfying*

$$\frac{1}{2}R \geqslant \Lambda + |E|^2 + |B|^2, \tag{1.3}$$

where Λ is a non-negative constant representing the cosmological constant, and E and B are divergence-free vector fields on M representing the electric and magnetic fields, respectively.

If Σ is an area-minimizing two-sphere embedded in (M, g), then the area, electric charge, and magnetic charge of Σ satisfy

$$\mathcal{A} \geqslant 4\pi \left(\mathcal{Q}_{\mathrm{E}}^2 + \mathcal{Q}_{\mathrm{M}}^2\right).$$

Moreover, if equality holds, then there exists a neighborhood $U \cong (-\delta, \delta) \times \Sigma$ of Σ in M such that:

(1) *The electric and magnetic fields are normal to the foliation; more precisely,*

$$E = a\nu_t, \quad B = b\nu_t,$$

for some constants a and b, where ν_t is the unit normal to $\Sigma_t \cong \{t\} \times \Sigma$ along the foliation, pointing to the outside of Σ_t.

[1] See definition 3.2 in [13].

(2) *(U, g) is isometric to* $((-\delta,\delta)\times\Sigma, dt^2+g_0)$ *for some* $\delta>0$*, where the induced metric* g_0 *on* Σ *has constant Gaussian curvature*

$$\kappa_\Sigma = a^2+b^2.$$

(3) *The cosmological constant* Λ *equals zero.*

Our second result is a generalization of theorem 1.1 to initial data sets that are not necessarily time-symmetric. It reads as follows (see section 2 for definitions):

Theorem 1.2. *Let* (M^3,g,K,E,B) *be a three-dimensional initial data set for the Einstein–Maxwell equations satisfying the charged DEC*

$$\mu+J(v)\geqslant\Lambda+|E|^2+|B|^2-2\langle E\times B,v\rangle \tag{1.4}$$

for every unit vector $v\in T_pM$*, every point* $p\in M$*, and some constant* $\Lambda\geqslant 0$*. Assume that E and B are divergence-free and that K is two-convex.*

Let Σ *be a weakly outermost, spherical MOTS in* (M,g,K)*. Then the area, electric charge, and magnetic charge of* Σ *satisfy*

$$\mathcal{A}\geqslant 4\pi\left(\mathcal{Q}_{\mathrm{E}}^2+\mathcal{Q}_{\mathrm{M}}^2\right).$$

Moreover, if equality holds, then there exists an outer neighborhood $U\cong[0,\delta)\times\Sigma$ *of* Σ *in M such that:*

(1) *The electric and magnetic fields are normal to the foliation; more precisely,*

$$E=a\nu_t,\quad B=b\nu_t,$$

for some constants a and b, where ν_t *is the unit normal to* $\Sigma_t\cong\{t\}\times\Sigma$ *along the foliation, pointing to the outside of* Σ_t*.*

(2) *(U, g) is isometric to* $([0,\delta)\times\Sigma,\mathrm{d}t^2+g_0)$ *for some* $\delta>0$*, where the induced metric* g_0 *on* Σ *has constant Gaussian curvature*

$$\kappa_\Sigma=a^2+b^2.$$

(3) *The second fundamental form satisfies* $K=f dt^2$ *on U, where* $f\in C^\infty(U)$ *depends only on* $t\in[0,\delta)$*.*

(4) *The energy and momentum densities satisfy*

$$\mu=a^2+b^2,\quad J=0\quad\text{on}\quad U.$$

(5) *The cosmological constant* Λ *equals zero.*

The norms and inner products in theorems 1.1 and 1.2 are computed with respect to the metric g.

In section 2, we derive inequalities (1.3) and (1.4) from the DEC for the energy-momentum tensor T^{matter}.

It is worth noting that a version of theorem 1.1 remains valid when Σ is assumed to be weakly outermost rather than area-minimizing. However, in this case, if equality holds, then an *outer* neighborhood of Σ must split. This result corresponds to theorem 1.2 in the time-symmetric setting.

The paper is organized as follows: In section 2, we present some preliminaries necessary for a proper understanding of this work. In section 3, we provide the proofs of theorems 1.1 and 1.2. Finally, section 4 offers a model illustrating these results.

2. Preliminaries

In this paper, we work in the C^∞ category. Thus, unless otherwise stated, all manifolds, vector fields, functions, etc are assumed to be smooth.

Let (M^3, g, K) be a three-dimensional initial data set in a four-dimensional spacetime (V^4, h); that is, M is a spacelike hypersurface in (V, h) with induced metric g and second fundamental form K, taken with respect to the future-directed timelike unit normal to M. Assume that (V, h) satisfies the Einstein equations with cosmological constant Λ:

$$G + \Lambda h = 8\pi \left(T^{\mathrm{EM}} + T^{\mathrm{matter}}\right),$$

where $G = \mathrm{Ric}_h - \frac{1}{2}R_h h$ is the Einstein tensor of (V, h), T^{EM} is the electromagnetic energy-momentum tensor, and T^{matter} is the energy-momentum tensor associated with non-gravitational and non-electromagnetic matter fields.

The electromagnetic energy-momentum tensor T^{EM} is given by

$$T^{\mathrm{EM}}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}F_{cd}F^{cd}h_{ab}\right),$$

where F is the electromagnetic 2-form, which is also referred to as the *Faraday tensor*. Here Latin indices correspond to those of the spacetime coordinate system.

Let u be the future-directed timelike unit normal vector field along M. As is standard, by the Gauss–Codazzi equations,

$$\mu := G(u,u) = \frac{1}{2}\left(R - |K|^2 + \tau^2\right),$$
$$J := G(u,\cdot) = \mathrm{div}(K - \tau g),$$

where R is the scalar curvature of (M, g) and $\tau = \mathrm{tr}\,K$ is the mean curvature of M in (V, h) with respect to u.

The *electric* and *magnetic vector fields* E and B on M are defined in such a way that

$$E_a = F_{ab}u^b,$$
$$B_a = \frac{1}{2}\epsilon_{abc}F^{bc},$$

where ϵ_{abc} is the induced volume form associated with the metric g. Specifically, if $\hat{\epsilon}$ denotes the volume form of the spacetime metric h, then $\epsilon_{abc} = u^d\hat{\epsilon}_{dabc}$. In the main results of this paper, we assume the absence of charged matter, that is, we assume that $\mathrm{div}\,E = \mathrm{div}\,B = 0$.

We refer to (M, g, K, E, B) as initial data for the Einstein–Maxwell equations.

Standard calculations give that

$$T^{\mathrm{EM}}(u,u) = \frac{1}{8\pi}\left(|E|^2 + |B|^2\right),$$
$$T^{\mathrm{EM}}(u,v) = -\frac{1}{4\pi}\langle E \times B, v\rangle,$$

for any vector v tangent to M, where $(E\times B)_a=\epsilon_{abc}E^bB^c$ defines the cross product of E and B, which is known in the literature as the *Poynting vector*.

Now assume that T^{matter} satisfies the *DEC*:

$$T^{\text{matter}}(X,Y)\geqslant 0 \text{ for all future-directed causal vectors } X,Y.$$

Therefore,

$$G(u,u)+\Lambda h(u,u)=8\pi\left(T^{\text{EM}}(u,u)+T^{\text{matter}}(u,u)\right)\geqslant 8\pi T^{\text{EM}}(u,u),$$

and thus

$$\mu\geqslant\Lambda+|E|^2+|B|^2.$$

In this case, if M is maximal (in particular, if M is time-symmetric), then

$$\frac{1}{2}R\geqslant\Lambda+|E|^2+|B|^2. \tag{2.1}$$

More generally, when M is not necessarily maximal, it holds that

$$G(u,u+v)+\Lambda h(u,u+v)\geqslant 8\pi T^{\text{EM}}(u,u+v),$$

and so

$$\mu+J(v)\geqslant\Lambda+|E|^2+|B|^2-2\langle E\times B,v\rangle, \tag{2.2}$$

for every unit vector v tangent to M.

Inequalities (2.1) and (2.2) are commonly referred to as the *charged DEC* and have been considered in numerous situations (see, e.g. [1, 8, 11, 13, 19, 21–23, 29]).

Now let Σ^2 be a closed embedded surface in M^3.

In this paper, we assume that Σ and M are orientable; in particular, Σ is two-sided. Then we fix a unit normal vector field ν along Σ; if Σ separates M, by convention, we say that ν points to the *outside* of Σ.

In the sequel, we are going to present some important definitions to our purposes.

The *electric* and *magnetic charges* of Σ are defined, respectively, by

$$\mathcal{Q}_{\text{E}}=\frac{1}{4\pi}\int_\Sigma\langle E,\nu\rangle,\quad \mathcal{Q}_{\text{M}}=\frac{1}{4\pi}\int_\Sigma\langle B,\nu\rangle.$$

The *null second fundamental forms* χ^+ and χ^- of Σ in (M,g,K) are defined by

$$\chi^+=K|_\Sigma+A,\quad \chi^-=K|_\Sigma-A,$$

where A is the second fundamental form of Σ in (M,g) with respect to ν; more precisely,

$$A(X,Y)=g(\nabla_X\nu,Y)\quad\text{for}\quad X,Y\in\mathfrak{X}(\Sigma),$$

where ∇ is the Levi–Civita connection of (M,g).

The *null expansion scalars* or the *null mean curvatures* θ^+ and θ^- of Σ in (M,g,K) with respect to ν are defined by

$$\theta^+=\operatorname{tr}\chi^+=\operatorname{tr}_\Sigma K+H^\Sigma,\quad \theta^-=\operatorname{tr}\chi^-=\operatorname{tr}_\Sigma K-H^\Sigma,$$

where $H^\Sigma=\operatorname{tr}A$ is the mean curvature of Σ in (M,g) with respect to ν, and $\operatorname{tr}_\Sigma K$ is the partial trace of K on Σ. Observe that $\theta^\pm=\operatorname{tr}\chi^\pm$.

After Penrose, Σ is said to be *trapped* if both θ^+ and θ^- are negative.

Restricting our attention to one side, we say that Σ is *outer trapped* if θ^+ is negative and *marginally outer trapped* if θ^+ vanishes. In the latter case, we refer to Σ as a *marginally outer trapped surface* or a *MOTS*, for short.

Assume that Σ is a MOTS in (M,g,K), with respect to a unit normal ν, that is a boundary in M; more precisely, ν points towards a top-dimensional submanifold $M^+ \subset M$ such that $\partial M^+ = \Sigma$. Then we say that Σ is *outermost* (resp. *weakly outermost*) if there is no closed embedded surface in M^+ with $\theta^+ \leq 0$ (resp. $\theta^+ < 0$) that is homologous to and different from Σ.

We say that Σ *minimizes area* in M if Σ has the least area in its homology class in M; *id est*, $\mathcal{A}(\Sigma) \leqslant \mathcal{A}(\Sigma')$ for every closed embedded surface Σ' in M that is homologous to Σ. Similarly, Σ is said to be *outer area-minimizing* if Σ minimizes area in M^+.

An important notion that we are going to recall now is the notion of stability for MOTS introduced by Andersson *et al* [5, 6].

Let Σ be a MOTS in (M,g,K) with respect to ν and $t \to \Sigma_t$ be a variation of $\Sigma = \Sigma_0$ in M with variation vector field $\frac{\partial}{\partial t}|_{t=0} = \phi\nu$, for some $\phi \in C^\infty(\Sigma)$. Denote by $\theta^\pm(t)$ the null expansion scalars of Σ_t with respect to the unit normal ν_t, where $\nu = \nu_t|_{t=0}$. It is well known that (see [6])

$$\frac{\partial \theta^+}{\partial t}\Big|_{t=0} = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - |X|^2 + \operatorname{div} X\right)\phi,$$

where Δ and div are the Laplace and divergence operators of Σ with respect to the induced metric, respectively; $X \in \mathfrak{X}(\Sigma)$ is the vector field that is dual to the 1-form $K(\nu,\cdot)|_\Sigma$, and

$$Q = \kappa_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|\chi^+|^2.$$

Here κ_Σ represents the Gaussian curvature of Σ.

At this point, it is important to emphasize that, in the general case (i.e. when Σ is not necessarily a MOTS), the first variation of θ^+ (see, e.g. [4]) is given by

$$\frac{\partial \theta^+}{\partial t}\Big|_{t=0} = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2}\theta^+ + \tau\theta^+\right)\phi.$$

The operator

$$L\phi = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - |X|^2 + \operatorname{div} X\right)\phi, \quad \phi \in C^\infty(\Sigma),$$

is referred to as the *MOTS stability operator*. It can be proved that L has a real eigenvalue λ_1, called the *principal eigenvalue* of L, such that $\operatorname{Re}\lambda \geq \lambda_1$ for any complex eigenvalue λ. Furthermore, the associated eigenfunction ϕ_1, $L\phi_1 = \lambda_1\phi_1$, is unique up to scale and can be chosen to be everywhere positive.

The principal eigenvalue $\lambda_1(\mathcal{L})$ of the symmetrized operator $\mathcal{L} = -\Delta + Q$ is characterized by the Rayleigh formula (see Lemma 1.34 and its proof in [10]):

$$\lambda_1(\mathcal{L}) = \min_{u \in C^\infty(\Sigma)\setminus\{0\}} \frac{\int_\Sigma \left(|\nabla u|^2 + Qu^2\right)}{\int_\Sigma u^2}. \tag{2.3}$$

Furthermore, the eigenfunctions of $\mathcal{L}$ associated with $\lambda_1(\mathcal{L})$ are the only functions that attain the minimum in (2.3).

It was proved by Galloway and Schoen (see [16, 20]) through direct estimates, and by Andersson *et al* [6] using a different method, that $\lambda_1(L) \leqslant \lambda_1(\mathcal{L})$.

We say that Σ is *stable* if $\lambda_1(L) \geqslant 0$; this is equivalent to saying that $L\phi \geq 0$ for some positive function $\phi \in C^\infty(\Sigma)$. It is not difficult to see that, if Σ is weakly outermost (in particular, if Σ is outermost), then Σ is stable.

Before concluding this section, let us recall the notion of *2-convexity*. The tensor K is said to be *2-convex* if, at every point, the sum of its two smallest eigenvalues is non-negative. In particular, if K is 2-convex, then $\mathrm{tr}_\Sigma K \geqslant 0$ along Σ. This convexity condition has been employed by the author in related contexts [14, 15, 18, 25, 26] (see also [24]).

3. Proofs

This section is devoted to the proofs of the main results of the paper, namely theorems 1.1 and 1.2. We begin by proving theorem 1.2. The proof of theorem 1.1 follows a similar structure.

As a first step, we establish an auxiliary *infinitesimal rigidity result*, which plays a crucial role in the argument.

For convenience, we define the *total charge* of Σ as

$$\mathcal{Q}_{\mathrm{T}} = \sqrt{\mathcal{Q}_{\mathrm{E}}^2 + \mathcal{Q}_{\mathrm{M}}^2}.$$

Proposition 3.1. *Let (M^3, g, K, E, B) be a three-dimensional initial data set for the Einstein–Maxwell equations satisfying the charged DEC*

$$\mu + J(\nu) \geqslant \Lambda + |E|^2 + |B|^2 - 2\langle E \times B, \nu\rangle$$

for every unit vector $v \in T_pM$, every point $p \in M$, and some constant $\Lambda \geqslant 0$.

Let Σ be a stable, spherical MOTS in (M, g, K). Then the area and total charge of Σ satisfy

$$\mathcal{A} \geqslant 4\pi\, \mathcal{Q}_{\mathrm{T}}^2. \tag{3.1}$$

Moreover, if equality holds, then the following conditions are satisfied:

(1) *The normal components of the electric and magnetic fields along Σ are constant, say $\langle E, \nu\rangle = a$ and $\langle B, \nu\rangle = b$.*
(2) *Σ is a round two-sphere with constant Gaussian curvature $\kappa_\Sigma = a^2 + b^2$.*
(3) *The constants $\lambda_1(L)$, $\lambda_1(\mathcal{L})$, and Λ equal zero.*

Proof. Since Σ is stable and $\lambda_1(L) \leqslant \lambda_1(\mathcal{L})$, we have the following inequality for every $u \in C^\infty(\Sigma)$:

$$0 \leqslant \lambda_1(\mathcal{L}) \int_\Sigma u^2 \leqslant \int_\Sigma \left(|\nabla u|^2 + Qu^2\right).$$

Taking $u \equiv 1$, we obtain

$$\begin{aligned} 0 \leqslant \lambda_1(\mathcal{L})\mathcal{A} \leqslant \int_\Sigma Q &= \int_\Sigma \left(\kappa_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|\chi^+|^2\right) \\ &\leqslant 4\pi - \int_\Sigma (\mu + J(\nu)), \end{aligned} \tag{3.2}$$

where we have used the Gauss–Bonnet theorem.

Now observe that

$$\begin{aligned}\mu+J(\nu) &\geqslant \Lambda+|E|^2+|B|^2-2\langle E\times B,\nu\rangle \\ &\geqslant |E^\top|^2+|B^\top|^2-2\langle E^\top\times B^\top,\nu\rangle+\langle E,\nu\rangle^2+\langle B,\nu\rangle^2,\end{aligned} \tag{3.3}$$

where $E^\top$ and $B^\top$ are the tangent components to Σ:

$$E^\top = E-\langle E,\nu\rangle\nu, \quad B^\top = B-\langle B,\nu\rangle\nu.$$

On the other hand, it is not difficult to see that

$$|E^\top|^2+|B^\top|^2-2\langle E^\top\times B^\top,\nu\rangle \geqslant \left(|E^\top|-|B^\top|\right)^2 \geqslant 0. \tag{3.4}$$

Using these estimates, we conclude that

$$0\leqslant 4\pi-\int_\Sigma\left(\langle E,\nu\rangle^2+\langle B,\nu\rangle^2\right).$$

Applying the Cauchy–Schwarz inequality, we obtain

$$(4\pi\,\mathcal{Q}_{\mathrm{E}})^2=\left(\int_\Sigma\langle E,\nu\rangle\right)^2\leqslant\mathcal{A}\int_\Sigma\langle E,\nu\rangle^2. \tag{3.5}$$

Similarly,

$$(4\pi\,\mathcal{Q}_{\mathrm{M}})^2\leqslant\mathcal{A}\int_\Sigma\langle B,\nu\rangle^2. \tag{3.6}$$

Therefore,

$$0\leqslant\mathcal{A}-4\pi\left(\mathcal{Q}_{\mathrm{E}}^2+\mathcal{Q}_{\mathrm{M}}^2\right)=\mathcal{A}-4\pi\,\mathcal{Q}_{\mathrm{T}}(\Sigma)^2,$$

proving the desired inequality.

If equality in (3.1) holds, then all inequalities above must also be equalities. In particular:

- Second equality in (3.5) implies that $\langle E,\nu\rangle$ is constant, say $\langle E,\nu\rangle=a$. Similarly, equality in (3.6) gives that $\langle B,\nu\rangle=b$ is constant.
- Equalities in (3.3) and (3.4) imply

$$\Lambda=0,\quad \mu+J(\nu)=\langle E,\nu\rangle^2+\langle B,\nu\rangle^2=a^2+b^2.$$

- Equalities in (3.2) furnish that $\lambda_1(\mathcal{L})=0$, $\chi^+=0$, and $u\equiv 1$ is an eigenfunction of $\mathcal{L}$ associated with $\lambda_1(\mathcal{L})$. Therefore,

$$0=Q=\kappa_\Sigma-(\mu+J(\nu)),$$

and thus

$$\kappa_\Sigma=\mu+J(\nu)=a^2+b^2.$$

Finally, since $0\leqslant\lambda_1(L)\leqslant\lambda_1(\mathcal{L})=0$, we conclude that $\lambda_1(L)=0$. □

It is worth noting that proposition 3.1 is a quasi-local statement, in the sense that it depends only on the intrinsic and extrinsic geometric data on Σ, not on the behavior of the initial data set in a neighborhood of the surface.

Proof of theorem 1.2. Since Σ is weakly outermost and, in particular, stable, it follows from the infinitesimal rigidity (proposition 3.1) that

$$\mathcal{A} \geqslant 4\pi\, \mathcal{Q}_{\mathrm{T}}^2.$$

Furthermore, if equality holds, then $\lambda_1(L) = 0$ (and $\Lambda = 0$). Thus, an outer neighborhood $U \cong [0,\delta) \times \Sigma$ of Σ in M is foliated by constant outward null mean curvature surfaces $\Sigma_t \cong \{t\} \times \Sigma$ (see [17, lemma 2.3]), with $\Sigma_0 = \Sigma$ and

$$g = \phi^2 \mathrm{d}t^2 + g_t \quad \text{on} \quad U,$$

where g_t is the induced metric on Σ_t.

On Σ_t, we recall that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2}\theta^2 + \tau\theta\right)\phi,$$

where $\theta = \theta(t)$ is the null mean curvature of Σ_t with respect to $\nu_t = \phi^{-1}\partial_t$.

Dividing both sides of last equation by ϕ and integrating over Σ_t, we obtain

$$\begin{aligned}
\theta' \int_{\Sigma_t} \frac{1}{\phi} - \theta \int_{\Sigma_t} \tau &= \int_{\Sigma_t} \left(\operatorname{div} Y - |Y|^2 + Q - \frac{1}{2}\theta^2\right) \leqslant \int_{\Sigma_t} Q \\
&= \int_{\Sigma_t} \left(\kappa_{\Sigma_t} - (\mu + J(\nu_t)) - \frac{1}{2}|\chi_t^+|^2\right) \\
&\leqslant 4\pi - \int_{\Sigma_t} (\mu + J(\nu_t)),
\end{aligned} \tag{3.7}$$

where $Y = X - \nabla \ln\phi$.

Using the proof strategy from proposition 3.1, we have

$$\mu + J(\nu_t) \geqslant |E|^2 + |B|^2 - 2\langle E \times B, \nu_t\rangle \geqslant \langle E, \nu_t\rangle^2 + \langle B, \nu_t\rangle^2. \tag{3.8}$$

Thus,

$$\begin{aligned}
\theta' \int_{\Sigma_t} \frac{1}{\phi} - \theta \int_{\Sigma_t} \tau &\leqslant 4\pi - \int_{\Sigma_t} \left(\langle E, \nu_t\rangle^2 + \langle B, \nu_t\rangle^2\right) \\
&\leqslant 4\pi - \frac{\left(\int_{\Sigma_t} \langle E, \nu_t\rangle\right)^2 + \left(\int_{\Sigma_t} \langle B, \nu_t\rangle\right)^2}{\mathcal{A}(t)} \\
&= 4\pi\left(1 - \frac{4\pi\, \mathcal{Q}_{\mathrm{T}}(t)^2}{\mathcal{A}(t)}\right),
\end{aligned} \tag{3.9}$$

where $\mathcal{A}(t)$ and $\mathcal{Q}_{\mathrm{T}}(t)$ are the area and total charge of Σ_t, respectively.

Because we are assuming $4\pi\,\mathcal{Q}_{\mathrm{T}}(0)^2 = \mathcal{A}(0)$ and $\operatorname{div} E = \operatorname{div} B = 0$ (implying $\mathcal{Q}_{\mathrm{T}}(t) = \mathcal{Q}_{\mathrm{T}}(0)$), we find

$$\begin{aligned} \theta'\left(\frac{\mathcal{A}(t)}{4\pi}\int_{\Sigma_t}\frac{1}{\phi}\right) - \theta\left(\frac{\mathcal{A}(t)}{4\pi}\int_{\Sigma_t}\tau\right) &\leqslant \mathcal{A}(t) - 4\pi\,\mathcal{Q}_{\mathrm{T}}(t)^2 \\ &= \mathcal{A}(t) - \mathcal{A}(0) \\ &= \int_0^t\left(\int_{\Sigma_s} H^{\Sigma_s}\phi\right)\mathrm{d}s, \end{aligned} \tag{3.10}$$

where we have used the fundamental theorem of calculus along with the first variation of area formula.

Since, by hypothesis, K is 2-convex, it follows that

$$H^{\Sigma_s} \leqslant \operatorname{tr}_{\Sigma_s} K + H^{\Sigma_s} = \theta(s)\,. \tag{3.11}$$

Therefore,

$$\theta'(t)\left(\frac{\mathcal{A}(t)}{4\pi}\int_{\Sigma_t}\frac{1}{\phi}\right) - \theta(t)\left(\frac{\mathcal{A}(t)}{4\pi}\int_{\Sigma_t}\tau\right) \leqslant \int_0^t \theta(s)\left(\int_{\Sigma_s}\phi\right)\mathrm{d}s.$$

Since Σ is weakly outermost, we have $\theta(t) \geqslant 0$ for each t. Then, using lemma 3.2 in [25], we conclude that $\theta(t) = 0$, forcing all inequalities above to be equalities.

Thus:

- Equalities in (3.11) give that $\operatorname{tr}_{\Sigma_t} K = H^{\Sigma_t} = 0$ along Σ_t. In particular,

  $$\theta^-(t) = \operatorname{tr}_{\Sigma_t} K - H^{\Sigma_t} = 0$$

  for every $t \in [0, \delta)$.
- Equalities in (3.10) imply that all surfaces Σ_t have the same area as Σ: $\mathcal{A}(t) = \mathcal{A}(0)$.
- Equalities in (3.8) hold, that is,

  $$\mu + J(\nu_t) = |E|^2 + |B|^2 - 2\langle E\times B, \nu_t\rangle = \langle E, \nu_t\rangle^2 + \langle B, \nu_t\rangle^2 \tag{3.12}$$

  along Σ_t for every $t \in [0, \delta)$.
- Finally, equalities in (3.7) imply $Y = X - \nabla \ln\phi = 0$ and $\chi_t^+ = 0$ along Σ_t. Indeed, equalities in (3.7), together with $\theta' = \theta = 0$, yield

  $$\begin{aligned} \int_{\Sigma_t}\left(-|Y|^2 + Q\right) = \int_{\Sigma} Q &= \int_{\Sigma_t}\left(\kappa_{\Sigma_t} - (\mu + J(\nu_t)) - \frac{1}{2}|\chi_t^+|^2\right) \\ &= 4\pi - \int_{\Sigma_t}(\mu + J(\nu_t))\,. \end{aligned}$$

  The first equality implies $Y = 0$, and the last one gives $\chi_t^+ = 0$. (Recall that $\int_{\Sigma_t}\kappa_{\Sigma_t} = 4\pi$ by the Gauss–Bonnet theorem).

Now, taking the first variation of $\theta^-(t) = 0$, with $\phi^- = -\phi$ instead of ϕ, we obtain

$$0 = \frac{\mathrm{d}\theta^-}{\mathrm{d}t} = -\Delta\phi^- + 2\langle X^-, \nabla\phi^-\rangle + \left(Q^- - |X^-|^2 + \operatorname{div} X^-\right)\phi^-, \tag{3.13}$$

where $X^- = (K(-\nu_t,\cdot)|_{\Sigma_t})^\sharp = -X = -\nabla \ln\phi$, and

$$Q^- = \kappa_{\Sigma_t} - (\mu - J(\nu_t)) - \frac{1}{2}|\chi_t^-|^2.$$

Thus, dividing both sides of (3.13) by $\phi^- = -\phi$ and integrating over Σ_t, we get

$$0 = \int_{\Sigma_t} \left(\operatorname{div} Y^- - |Y^-|^2 + Q^-\right) \leqslant \int_{\Sigma_t} Q^- \leqslant 4\pi - \int_{\Sigma_t} (\mu - J(\nu_t)), \tag{3.14}$$

where $Y^- = X^- - \nabla\ln\phi = -2\nabla\ln\phi$. Above we have used the Gauss–Bonnet theorem.

Observe that

$$\mu - J(\nu_t) \geqslant |E|^2 + |B|^2 + 2\langle E\times B, \nu_t\rangle \geqslant \langle E,\nu_t\rangle^2 + \langle B,\nu_t\rangle^2. \tag{3.15}$$

Therefore,

$$\begin{aligned} 0 &\leqslant 4\pi - \int_{\Sigma_t} (\mu - J(\nu_t)) \leqslant 4\pi - \int_{\Sigma_t} \left(\langle E,\nu_t\rangle^2 + \langle B,\nu_t\rangle^2\right) \\ &\leqslant 4\pi\left(1 - \frac{4\pi\,\mathcal{Q}_{\mathrm{T}}(t)^2}{\mathcal{A}(t)}\right) = 4\pi\left(1 - \frac{\mathcal{A}(0)}{\mathcal{A}(t)}\right) = 0, \end{aligned}$$

thus all inequalities above must be equalities.

Then:

- From (3.12) and equalities in (3.15), we have

$$\begin{aligned} |E|^2 + |B|^2 - 2\langle E\times B,\nu_t\rangle &= \langle E,\nu_t\rangle^2 + \langle B,\nu_t\rangle^2 \\ &= |E|^2 + |B|^2 + 2\langle E\times B,\nu_t\rangle. \end{aligned}$$

  Therefore, $\langle E\times B,\nu_t\rangle = 0$ and $|E|^2 + |B|^2 = \langle E,\nu_t\rangle^2 + \langle B,\nu_t\rangle^2$. Thus, E and B are parallel to ν_t, say $E = a\nu_t$ and $B = b\nu_t$. Furthermore, from the second equality in (3.9), we obtain that $a = a(t)$ and $b = b(t)$ are constant on Σ_t.
- Equalities in (3.14) imply $Y^- = -2\nabla\ln\phi = 0$ and $\chi_t^- = 0$ along Σ_t. Therefore, $\phi = \phi(t)$ is constant on Σ_t for each $t\in[0,\delta)$. In this case, after a change of variable if necessary, we may assume that $\phi\equiv 1$. Moreover, $\chi_t^+ = K|_{\Sigma_t} + A_t = 0$ and $\chi_t^- = K|_{\Sigma_t} - A_t = 0$ imply that $K|_{\Sigma_t} = 0$ and Σ_t is totally geodesic in (M,g). This gives that

$$g = \mathrm{d}t^2 + g_0 \quad \text{on} \quad U \cong [0,\delta)\times\Sigma,$$

  where g_0 is the induced metric on Σ.
- Because $\operatorname{div} E = \operatorname{div} B = 0$, we can see that a e b are constant.
- Looking at (3.12) and equalities in (3.15) again, we get

$$\mu + J(\nu_t) = a^2 + b^2 = \mu - J(\nu_t).$$

  Therefore,

$$\mu = a^2 + b^2, \quad J(\nu_t) = 0.$$

- It follows from (3.13) that

$$0 = Q^{-} = \kappa_{\Sigma_t} - \mu \quad \therefore \quad \kappa_{\Sigma_t} = \mu = a^2 + b^2.$$

- Given a unit vector v tangent to M, we are assuming that

$$\mu + J(v) \geqslant |E|^2 + |B|^2 - 2\langle E \times B, v\rangle = a^2 + b^2 = \mu.$$

  Therefore, $J(v) \geqslant 0$ for every v, that is, $J = 0$.
- Finally, $K|_{\Sigma_t} = 0$, $K(\nu_t, \cdot)|_{\Sigma_t} = X^\flat = 0$, and $J = \mathrm{div}(K - \tau g) = 0$ give that $K = f\mathrm{d}t^2$ on $U \cong [0,\delta) \times \Sigma$, where f depends only on $t \in [0,\delta)$.

This concludes the proof of theorem 1.2. □

We now proceed with the proof of theorem 1.1. To this end, we first present the following auxiliary result:

Proposition 3.2. *Let (M^3, g) be a three-dimensional Riemannian manifold whose scalar curvature R satisfies*

$$\frac{1}{2}R \geqslant \Lambda + |E|^2 + |B|^2,$$

where Λ is a non-negative constant, and E and B are vector fields on M.

If Σ is a stable, minimal two-sphere embedded in (M, g), then the area and total charge of Σ satisfy

$$\mathcal{A} \geqslant 4\pi\, \mathcal{Q}_{\mathrm{T}}^2. \tag{3.16}$$

Moreover, if equality holds, then the following conditions are satisfied:

(1) *The electric and magnetic fields are parallel to ν; more precisely,*

$$E = a\nu, \quad B = b\nu,$$

for some constants a and b.

(2) *Σ is a round two-sphere with constant Gaussian curvature $\kappa_\Sigma = a^2 + b^2$.*

(3) *Σ is totally geodesic, $\Lambda = 0$, and $R = 2(a^2 + b^2)$ on Σ.*

It is worth noting that inequality (3.16) was originally derived by Gibbons [21] (see also [13, theorem 4.4]). Our contribution lies in establishing the infinitesimal rigidity statement.

Proof. Since Σ is a stable minimal surface, the stability inequality says that (see, e.g. [10, section 1.8])

$$\frac{1}{2}\int_\Sigma \left(R + |A|^2\right) u^2 \leqslant \int_\Sigma |\nabla u|^2 + \int_\Sigma \kappa_\Sigma u^2$$

for every $u \in C^\infty(\Sigma)$. Taking $u \equiv 1$, we obtain

$$\frac{1}{2}\int_\Sigma R \leqslant 4\pi, \tag{3.17}$$

where we have used the Gauss–Bonnet theorem.

Next, using the estimate

$$\frac{1}{2}R \geqslant \Lambda + |E|^2 + |B|^2 \geqslant \langle E, \nu\rangle^2 + \langle B, \nu\rangle^2, \tag{3.18}$$

we conclude that

$$\int_\Sigma \left(\langle E, \nu\rangle^2 + \langle B, \nu\rangle^2\right) \leqslant 4\pi.$$

Finally, applying the Cauchy–Schwarz inequality yields

$$(4\pi\, \mathcal{Q}_{\mathrm{T}})^2 = \left(\int_\Sigma \langle E, \nu\rangle\right)^2 + \left(\int_\Sigma \langle B, \nu\rangle\right)^2 \leqslant \mathcal{A}\int_\Sigma \left(\langle E, \nu\rangle^2 + \langle B, \nu\rangle^2\right) \leqslant 4\pi\,\mathcal{A}, \tag{3.19}$$

which proves inequality (3.16).

Now suppose that equality holds in (3.16). Then equality must also hold in each of the steps above.

Equality in (3.17) implies that Σ is totally geodesic and that $u_0 \equiv 1$ is a Jacobi function on Σ (see [10, lemma 1.34]):

$$\Delta u_0 + \frac{1}{2}\left(R - 2\kappa_\Sigma + |A|^2\right) u_0 = 0 \quad \text{on} \quad \Sigma.$$

Therefore, $R = 2\kappa_\Sigma$.

Equality in (3.18) implies $\Lambda = 0$, and that E and B are parallel to ν along Σ, i.e. $E = a\nu$ and $B = b\nu$ for some functions a, b.

Finally, second equality in (3.19) implies that $\langle E, \nu\rangle$ and $\langle B, \nu\rangle$ are constant, hence a and b are constant functions. □

Proof of theorem 1.1. Since Σ is area-minimizing (in particular, stable minimal), it follows from proposition 3.2 that $\mathcal{A} \geqslant 4\pi\, \mathcal{Q}_{\mathrm{T}}^2$. Furthermore, if equality holds, then the Jacobi operator of Σ reduces to $-\Delta$.

Therefore, as in the proof of theorem 1.2, by a classical argument in the literature (see, e.g. [3, 7, 27, 28]), a neighborhood $U \cong (-\delta, \delta) \times \Sigma$ of Σ in M can be foliated by constant mean curvature surfaces $\Sigma_t \cong \{t\} \times \Sigma$, with $\Sigma_0 = \Sigma$ and

$$g = \phi^2 \mathrm{d}t^2 + g_t \quad \text{on} \quad U.$$

The first variation of $H(t) := H^{\Sigma_t}$ gives (see, for instance, the appendix of [2])

$$H' = -\Delta\phi - \frac{1}{2}\left(R - 2\kappa_{\Sigma_t} + |A_t|^2 + H^2\right)\phi.$$

Thus,

$$\begin{aligned}
H'\int_{\Sigma_t}\frac{1}{\phi} &= -\int_{\Sigma_t}\frac{\Delta\phi}{\phi} - \frac{1}{2}\int_{\Sigma_t}\left(R + |A_t|^2 + H^2\right) + \int_{\Sigma_t}\kappa_{\Sigma_t} \\
&\leqslant -\int_{\Sigma_t}\frac{|\nabla\phi|^2}{\phi^2} - \frac{1}{2}\int_{\Sigma_t} R + 4\pi \\
&\leqslant -\frac{1}{2}\int_{\Sigma_t} R + 4\pi.
\end{aligned}$$

Using the estimates

$$\frac{1}{2}R \geqslant |E|^2 + |B|^2 \geqslant \langle E, \nu_t \rangle^2 + \langle B, \nu_t \rangle^2,$$

and applying the Cauchy–Schwarz inequality, we obtain

$$H' \int_{\Sigma_t} \frac{1}{\phi} \leqslant 4\pi \left(1 - \frac{4\pi\, \mathcal{Q}_{\mathrm{T}}(t)^2}{\mathcal{A}(t)} \right).$$

On the other hand, $4\pi\, \mathcal{Q}_{\mathrm{T}}(t)^2 = 4\pi\, \mathcal{Q}_{\mathrm{T}}(0)^2 = \mathcal{A}(0)$, since $\operatorname{div} E = \operatorname{div} B = 0$. Therefore,

$$H'(t) \int_{\Sigma_t} \frac{1}{\phi} \leqslant \frac{4\pi}{\mathcal{A}(t)} \left(\mathcal{A}(t) - \mathcal{A}(0) \right) = \frac{4\pi}{\mathcal{A}(t)} \int_0^t H(s) \left(\int_{\Sigma_s} \phi \right) \mathrm{d}s,$$

that is,

$$H'(t)\, \eta(t) \leqslant \int_0^t H(s)\, \xi(s)\, \mathrm{d}s, \quad \eta(t) := \frac{\mathcal{A}(t)}{4\pi} \int_{\Sigma_t} \frac{1}{\phi}, \quad \xi(t) := \int_{\Sigma_t} \phi,$$

where we have used the fundamental theorem of calculus together with the first variation of area formula. This holds for every $t \in (-\delta, \delta)$.

It follows directly from lemma 3.2 in [25] that $H(t) \leqslant 0$ for every $t \in [0, \delta)$. Similarly, by applying the same strategy as in the proof of lemma 3.2 in [25] for $\rho(t) = 0$, it is not difficult to show that $H(t) \geqslant 0$ for every $t \in (-\delta, 0]$. Therefore,

$$\mathcal{A}'(t) = \int_{\Sigma_t} H(t)\, \phi \begin{cases} \leqslant 0 & \text{for} \quad t \in [0, \delta), \\ \geqslant 0 & \text{for} \quad t \in (-\delta, 0]. \end{cases} \tag{3.20}$$

In any case, $\mathcal{A}(t) \leqslant \mathcal{A}(0)$ for every $t \in (-\delta, \delta)$. This implies that $\mathcal{A}(t) = \mathcal{A}(0)$ for all $t \in (-\delta, \delta)$, since $\Sigma_0 = \Sigma$ is area-minimizing. Using this in (3.20), we obtain that $H(t) = 0$ for every $t \in (-\delta, \delta)$. Therefore, all inequalities above must be equalities.

Thus, each Σ_t is an area-minimizing surface satisfying $\mathcal{A}(t) = 4\pi\, \mathcal{Q}_{\mathrm{T}}(t)^2$. Then, by proposition 3.2,

- $E = a\nu_t$ and $B = b\nu_t$, where $a = a(t)$ and $b = b(t)$ are constant on Σ_t;
- Each Σ_t is a totally geodesic round two-sphere with constant Gaussian curvature $\kappa_{\Sigma_t} = a^2 + b^2$;
- $R = 2(a^2 + b^2)$ on Σ_t for each $t \in (-\delta, \delta)$.

Finally, since $\operatorname{div} E = \operatorname{div} B = 0$, we conclude that a and b are constant functions. Standard calculations guarantee theorem 1.1. □

Remark 3.3. An important result in the theory of marginally outer trapped surfaces(MOTS) is theorem 3.1 in [17], which states that if (M^{n+1}, g, K) is an initial data set satisfying the DEC, $\mu \geqslant |J|$, and if $\Sigma^n \subset M^{n+1}$ is a closed weakly outermost MOTS that does not admit a metric of positive scalar curvature, then an outer neighborhood of Σ can be foliated by MOTS. In fact, each leaf Σ_t of this foliation has vanishing outward null second fundamental form ($\chi_t^+ = 0$) and is Ricci flat. Example 4.2 in [15] shows that this result is sharp, in the sense that (M, g) does not necessarily split as a Riemannian product in a neighborhood of Σ unless additional assumptions are made on Σ and/or on the initial data (for instance, that Σ is outer volume-minimizing and K is n-convex; see [15, theorem 5.2]).

In our setting, although we assume that Σ^2 is weakly outermost and (M^3,g,K) satisfies the CDEC (or the DEC when $B=0$), we cannot *a priori* guarantee the existence of such a foliation by MOTS without imposing additional conditions—for instance, that $\mathcal{A}=4\pi\,\mathcal{Q}_{\mathrm{T}}^2$. This is because, in our case, Σ admits a metric of positive scalar curvature, as we assume that Σ is topologically a two-sphere.

4. The model

Let $q>0$ and consider the dyonic Bertotti–Robinson spacetime (V^4,h) defined by

$$V^4=\mathbb{R}\times\mathbb{R}\times S^2,\quad h=q^2\left(-\cosh^2 r\,\mathrm{d}t^2+\mathrm{d}r^2+\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right).$$

Note that (V,h) is the direct product of a two-dimensional anti-de Sitter space of curvature $-1/q^2$ and a round two-sphere of curvature $1/q^2$. Consequently, one can verify that

$$\mathrm{Ric}_h=\frac{1}{q^2}\,\mathrm{diag}\left(-h_{tt},-h_{rr},h_{\theta\theta},h_{\phi\phi}\right).$$

In particular, the scalar curvature of (V,h) vanishes.

Now let q_e and q_m be constants and define the Faraday tensor F by

$$F=-q_e\cosh r\,\mathrm{d}t\wedge\mathrm{d}r+q_m\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi.$$

A direct computation shows that the associated electromagnetic energy-momentum tensor T^{EM} takes the form

$$T^{\mathrm{EM}}=\frac{1}{8\pi}\frac{q_e^2+q_m^2}{q^4}\,\mathrm{diag}\left(-h_{tt},-h_{rr},h_{\theta\theta},h_{\phi\phi}\right).$$

Therefore, by choosing q_e and q_m such that $q^2=q_e^2+q_m^2$, the spacetime (V,h) satisfies the Einstein equations with zero cosmological constant:

$$\mathrm{Ric}_h=8\pi\,T^{\mathrm{EM}}.$$

Observe that each t-slice $M=\{t\}\times\mathbb{R}\times S^2$ is time-symmetric and isometric to the Riemannian product of a line with a round two-sphere of Gaussian curvature $1/q^2$. Furthermore, the electric and magnetic vector fields on M are given by

$$E=\frac{q_e}{q^2}\nu,\quad B=\frac{q_m}{q^2}\nu,\quad \nu:=\frac{1}{q}\partial_r.$$

The coordinates of E are

$$E^r=\frac{q_e}{q^3},\quad E^\theta=E^\phi=0.$$

Thus, its divergence with respect to the induced metric g on M is

$$\mathrm{div}\,E=\frac{1}{\sqrt{\det g}}\,\partial_i\left(\sqrt{\det g}\,E^i\right)=\frac{1}{q^3\sin\theta}\,\partial_r\left(q_e\sin\theta\right)=0,$$

since $E^\theta = E^\phi = 0$ and q_e is constant. Analogously, $\operatorname{div} B = 0$.

Finally, consider the 2-sphere $\Sigma = \{t\} \times \{r\} \times S^2$. The electric charge enclosed by Σ is

$$\mathcal{Q}_{\mathrm{E}} = \frac{1}{4\pi}\int_\Sigma \langle E, \nu\rangle = \frac{1}{4\pi}\frac{q_e}{q^2}\mathcal{A} = q_e.$$

Similarly, the magnetic charge is

$$\mathcal{Q}_{\mathrm{M}} = q_m.$$

Clearly, Σ and M satisfy all the assumptions of theorems 1.1 and 1.2 with $\mathcal{A} = 4\pi\mathcal{Q}_{\mathrm{T}}^2$.

For a detailed discussion of the Bertotti–Robinson spacetime with $q_m = 0$, as well as other notable spacetimes in dimension $D \geqslant 4$ with vanishing magnetic field, see [9].

5. Conclusions

In this work, we have investigated the geometric consequences of equality in the area-charge inequalities for both time-symmetric minimal surfaces and more general marginally outer trapped surface MOTS within the Einstein–Maxwell framework. Our results show that, under suitable energy and curvature conditions, saturation of the inequality $\mathcal{A} \geqslant 4\pi(\mathcal{Q}_{\mathrm{E}}^2 + \mathcal{Q}_{\mathrm{M}}^2)$ imposes strong rigidity on the local geometry: the local metric splits as a Riemannian product, the electric and magnetic fields must be normal to the foliation, and various geometric and physical quantities are fully determined.

These rigidity results provide a precise characterization of the local geometric and electromagnetic structure around weakly outermost MOTS in initial data sets satisfying the charged DEC. In particular, they highlight the intimate connection between area-charge equality and the underlying geometric and electromagnetic fields, offering insight into the rigidity of extremal configurations in general relativity. In [12], in a joint work with Cruz, we extend these rigidity phenomena to both compact and noncompact time-symmetric initial data sets, establishing sharp area-charge inequalities and analyzing the corresponding rigidity of the boundary and ambient geometries under suitable assumptions, including cases with different boundary topologies.

Data availability statement

No new data were created or analysed in this study.

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Conflict of interest

We know of no conflicts of interest associated with this publication, and there has been no significant financial support for this work that could have influenced its outcome.

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