Rev. Mat. Iberoam. 40 (2024), no. 1, 277–298
DOI 10.4171/RMI/1453

© 2023 Real Sociedad Matemática Española
Published by EMS Press and licensed under a CC BY 4.0 license

Free boundary minimal hypersurfaces outside of the ball

Laurent Mazet and Abraão Mendes

Abstract. In this paper, we obtain two classification theorems for free boundary minimal hypersurfaces outside of the unit ball (exterior FBMH, for short) in Euclidean space. The first result states that the only exterior stable FBMH with parallel embedded regular ends are the catenoidal hypersurfaces. To achieve this, we prove a Bôcher-type result for positive Jacobi functions on regular minimal ends in $\mathbb{R}^{n+1}$ that, after some calculations, implies the first theorem. The second theorem states that any exterior FBMH Σ with one regular end is a catenoidal hypersurface. Its proof is based on a symmetrization procedure similar to that of R. Schoen. We also give a complete description of the catenoidal hypersurfaces, including the calculation of their indices.

1. Introduction

Over the last few years, the study of free boundary minimal hypersurfaces (FBMH, for short) has occupied a prominent place in differential geometry, especially the study of FBMH in the Euclidean unit ball $\mathbb{B} \subset \mathbb{R}^{n+1}$ (see, e.g., [1,2,7–9,18,19] and the references therein).

In this paper, we deal with FBMH in $\mathbb{R}^{n+1} \setminus \mathbb{B}$ with compact boundary in $\partial\mathbb{B}$, which are called *exterior free boundary minimal hypersurfaces*. These hypersurfaces are critical for the n-volume functional with respect to deformations that let the boundary on the unit sphere. For such critical points, the second order derivative of the volume functional is given by the so-called stability operator that, here, has a contribution from the boundary. In our situation, this contribution is nonnegative due to the concavity of the unit sphere with respect to its outside. An interesting question would be to understand the geometry and the topology of these hypersurfaces in terms of their indices. In the ball, this is the study made by L. Ambrozio, A. Carlotto, and B. Sharp in [2]. A situation where non-compact FBMH have been studied is the case of Schwarzschild space: R. Montezuma [14] and E. Barbosa and J. M. Espinar [3] have looked at some properties of these hypersurfaces. One can also take a look at the work of H. Hong and A. Saturnino [12].

In $\mathbb{R}^{n+1} \setminus \mathbb{B}$, important examples of exterior FBMH are the *catenoidal hypersurfaces*, defined as exterior FBMH invariant by isometries fixing a straight line. In Section 4, we

Mathematics Subject Classification 2020: 53A10.
Keywords: classification theorems, catenoidal hypersurfaces, Euclidean space.

give a complete description of the catenoidal hypersurfaces and calculate their indices: some of them have index 0 and others have index 1.

The aim of this paper is to prove two classification results for catenoidal hypersurfaces. The first classification theorem is the following. Definitions are given in Sections 2 and 3.

Theorem 1.1. *Let Σ be an exterior free boundary minimal hypersurface in $\mathbb{R}^{n+1} \setminus \mathbb{B}$. Let us assume that Σ is stable and has parallel regular ends. Then Σ is a catenoidal hypersurface.*

In order to prove Theorem 1.1, we first prove that saying that Σ is stable is equivalent to saying that there exists a positive Jacobi function u on Σ satisfying the Robin boundary condition $\partial_\nu u + u = 0$ on $\partial\Sigma$. This is the content of Proposition 2.1. Its proof is based on the proof of a classical stability characterization due to D. Fischer-Colbrie and R. Schoen (Theorem 1 in [6]) for manifolds without boundary where, here, we make use of the Harnack inequality for positive solutions to $\Delta u + qu = 0$ on Σ with Robin boundary condition $\partial_\nu u + u = 0$ on $\partial\Sigma$ proved in Appendix A. Second, we obtain a Bôcher-type theorem for positive Jacobi functions on regular minimal ends in $\mathbb{R}^{n+1}$ that, together with Proposition 2.1, implies that Σ is invariant by isometries fixing a straight line, in other words, Σ is a catenoidal hypersurface.

As mentioned above, some of the catenoidal hypersurfaces have index 1, so it would be interesting to know if the index equal to 1 implies that the hypersurface is catenoidal. When $n = 2$, this would be a result similar to the López–Ros result [13] for boundaryless minimal surfaces. For example, it would be interesting to understand if a control on the index gives a control on the number of ends of the hypersurface. However, the positive contribution of the boundary to the stability operator seems to make this not an easy task.

The second classification theorem is the following.

Theorem 1.2. *Let Σ be an exterior free boundary minimal hypersurface. If Σ has one regular end, then Σ is a catenoidal hypersurface.*

The proof of Theorem 1.2 is based on a reflection procedure as in Schoen's paper [16]. After submitting the paper, it was brought to our attention that some versions of Theorem 1.2 were stated under different hypotheses by S.-H. Park and J. Pyo, see Theorems 3.1 and 3.2 in [15]. In fact, they proved that the conclusion of Theorem 1.2 holds for capillary (Σ meets $\partial\mathbb{B}$ at a constant contact angle) embedded exterior minimal hypersurfaces lying in a half-space. Assuming that Σ is free boundary and still embedded, they were able to drop the half-space condition. Here, we remove the embeddedness assumption. The technique of proof is similar to the one in [15]. However, let us notice that there is an omission in the argument of Park and Pyo; so we decide to keep our version of the result completing the argument (the omission is made explicit in the proof).

The paper is organized as follows. In Section 2, we present some preliminaries on FBMH and prove Proposition 2.1. In Section 3, we state and prove an auxiliary Bôcher-type result (Theorem 3.1) and present the proof of Theorem 1.1. In Section 4, we introduce the catenoidal hypersurfaces and give a complete description of them, including the calculation of their indices. In Section 5, we prove Theorem 1.2. Finally, in Appendix A, we present a proof of the Harnack inequality for positive functions satisfying a Robin-type boundary condition.

2. Free boundary minimal hypersurfaces in $\mathbb{R}^{n+1} \setminus \mathbb{B}$

Let Σ be an n-manifold with compact boundary. We say that $F\colon \Sigma \to \mathbb{R}^{n+1} \setminus \mathbb{B}$ is an *exterior proper immersion* if F is a proper immersion and $F(\Sigma) \cap \partial\mathbb{B} = F(\partial\Sigma)$. In this paper, we always consider such exterior proper immersions. We also assume that Σ is orientable so that a unit normal N is well defined along F. Besides, we will often identify Σ with its image $F(\Sigma)$ and just say that Σ is an *exterior hypersurface*.

We will consider exterior hypersurfaces that are critical for the n-volume functional with respect to any deformations keeping the boundary on $\partial\mathbb{B}$. Such a hypersurface Σ has vanishing mean curvature and meets $\partial\mathbb{B}$ orthogonally: we call them *exterior free boundary minimal hypersurfaces*.

Basic examples are given by cones over minimal hypersurfaces $S \subset \mathbb{S}^n = \partial\mathbb{B}$:

$$\Sigma = \{tp \in \mathbb{R}^{n+1}\,;\, p \in S \text{ and } t \geq 1\}.$$

One can also consider exterior FBMH that are invariant by isometries fixing a straight line: they are called *catenoidal hypersurfaces*. Their complete description is given in Section 4.

Let Σ be an exterior free boundary minimal hypersurface. The free boundary condition implies that, at $P \in \partial\Sigma$, the outgoing unit normal $\nu(P) = -P$ is a principal direction of the second fundamental form B of Σ. Indeed, for $T \in T\partial\Sigma$, we have

$$B(\nu, T) = (D_T\nu, N) = -B_{\partial\mathbb{B}}(T, N) = -(T, N) = 0, \tag{2.1}$$

where D is the covariant derivative in $\mathbb{R}^{n+1}$ and $B_{\partial\mathbb{B}}$ is the second fundamental form of $\partial\mathbb{B}$. Moreover, if $S, T \in T\partial\Sigma$, one has

$$B(S,T) = (D_T S, N) = (\nabla^{\mathbb{S}}_T S + B_{\partial\mathbb{B}}(S,T)\nu, N) = (\nabla^{\mathbb{S}}_T S, N) = B^{\mathbb{S}}_{\partial\Sigma}(S,T), \tag{2.2}$$

where $\nabla^{\mathbb{S}}$ is the Levi-Civita connection of $\mathbb{S}^n$ and $B^{\mathbb{S}}_{\partial\Sigma}$ denotes the second fundamental form of $\partial\Sigma$ as a submanifold of $\mathbb{S}^n$.

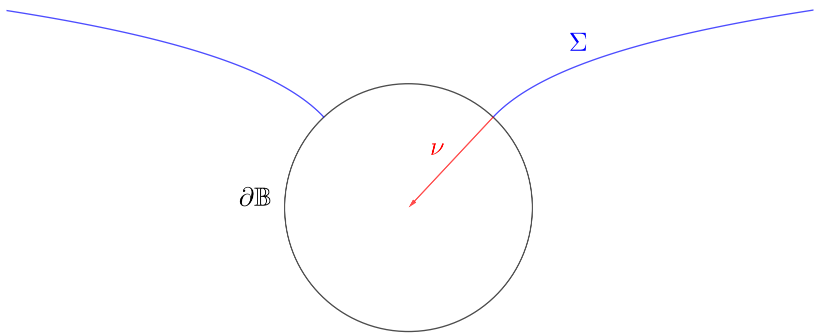

Figure 1. Exterior FBMH.

As in the boundaryless case, we also have a monotonicity formula for exterior free boundary minimal hypersurfaces:

$$\frac{|\Sigma \cap B_R|}{R^n} - \left(1 - \frac{1}{R^n}\right)\frac{|\partial\Sigma|}{n} = \int_{\Sigma \cap B_R} \frac{|X^\perp|^2}{|X|^{n+2}}, \tag{2.3}$$

where B_R is the Euclidean ball centered at the origin of radius $R \geq 1$. In fact, let

$$v(r) = |\Sigma \cap B_r|, \quad r > 1.$$

It follows from the coarea formula that

$$\frac{d}{dr}v(r) = \int_{\Sigma \cap \partial B_r} \frac{1}{|\nabla^\Sigma d|},$$

where $d(X) = |X|$ is the distance function to the origin. On the other hand, because Σ is minimal, $\operatorname{div}_\Sigma(X^\top) = n$. Therefore,

$$nv(r) = \int_{\Sigma \cap B_r} \operatorname{div}_\Sigma(X^\top) = \int_{\partial\Sigma}(X^\top, \nu) + \int_{\Sigma \cap \partial B_r}(X^\top, \nu) = -|\partial\Sigma| + \int_{\Sigma \cap \partial B_r}|X^\top|.$$

This gives

$$\begin{aligned} \frac{d}{dr}\Big(\frac{v(r)}{r^n}\Big) &= \frac{|\partial\Sigma|}{r^{n+1}} + \int_{\Sigma \cap \partial B_r} \frac{1}{r^n}\Big(\frac{1}{|\nabla^\Sigma d|} - \frac{|X^\top|}{r}\Big) \\ &= \frac{|\partial\Sigma|}{r^{n+1}} + \int_{\Sigma \cap \partial B_r} \frac{1}{|\nabla^\Sigma d|}\Big(\frac{|X^\perp|^2}{r^{n+2}}\Big), \end{aligned}$$

where above we have used that $\nabla^\Sigma d = X^\top/|X|$ and $|X^\perp|^2 = r^2 - |X^\top|^2$. Thus, integrating last equation from 1 to R and using the coarea formula, we obtain (2.3).

2.1. The stability operator

Let $\{F_t\}$ be a family of exterior proper immersions of Σ such that $F_0(\Sigma)$ is free boundary minimal and $\partial_t F_t$ has compact support. Even if the volume of $F_t(\Sigma)$ is infinite, its derivatives can be computed, since the deformation has compact support. Then the first derivative of the n-volume functional vanishes at $t = 0$, and the second derivative at $t = 0$ can be computed in terms of the function $u = (\partial_t F_t|_{t=0}, N)$ by

$$\frac{d^2}{dt^2}\operatorname{Vol}(F_t(\Sigma))\Big|_{t=0} = Q(u,u) = \int_\Sigma (|\nabla^\Sigma u|^2 - \|B\|^2u^2)\,d\mu + \int_{\partial\Sigma} u^2\,ds,$$

where ∇^Σ and $d\mu$ are the gradient and the n-volume measure on Σ, respectively, and ds is the $(n-1)$-volume measure on $\partial\Sigma$, all with respect to the metric induced by F_0 (see [2]). After integration by parts, one has

$$Q(u,u) = -\int_\Sigma u(\Delta u + \|B\|^2 u)\,d\mu + \int_{\partial\Sigma} u(u + \partial_\nu u)\,ds.$$

So the quadratic form Q is associated with the Jacobi operator defined by $\mathcal{L}u = \Delta u + \|B\|^2 u$. Then, for any bounded domain Ω in Σ, we can consider the associated spectrum of $\mathcal{L}$: a sequence of eigenvalues $\lambda_n \nearrow +\infty$ and a L^2-orthonormal sequence of functions u_n on Ω such that

$$\begin{cases} \Delta u_n + \|B\|^2 u_n = -\lambda_n u_n & \text{on } \Omega, \\ \partial_\nu u_n + u_n = 0 & \text{on } \partial\Sigma \cap \Omega, \\ u_n = 0 & \text{on } \partial\Omega \setminus \partial\Sigma. \end{cases}$$

The number of negative eigenvalues is then called the *index of Q on Ω*, and is denoted by $\mathrm{Ind}(Q,\Omega)$. If (Ω_n) is an increasing sequence of domains such that $\cup\Omega_n = \Sigma$, then the limit of the increasing sequence $(\mathrm{Ind}(Q,\Omega_n))$ is called the *index of* Σ, and is denoted by $\mathrm{Ind}(\Sigma)$.

When $\mathrm{Ind}(\Sigma) = 0$, we say that Σ is *stable* and this is equivalent to $Q(u,u) \geq 0$ for any function u with compact support on Σ. Actually, we have an alternative characterization of stability given by the following Fischer-Colbrie and Schoen-type result (also stated in Proposition 4.1 of [12]).

Proposition 2.1. *Let Σ be an exterior free boundary minimal hypersurface. Then Σ is stable if and only if there exists a positive solution u on Σ to*

$$\begin{cases} \Delta u + \|B\|^2 u = 0 & \text{on } \Sigma, \\ \partial_\nu u + u = 0 & \text{on } \partial\Sigma. \end{cases} \tag{2.4}$$

Proof. The proof is very similar to that of Theorem 1 in [6] by Fischer-Colbrie and Schoen, where they prove the equivalence between three statements analogous to the following:

(i) on any bounded domain, $\mathcal{L}$ has nonnegative first eigenvalue;

(ii) on any bounded domain, $\mathcal{L}$ has positive first eigenvalue;

(iii) there exists a positive solution to (2.4).

In fact, in order to adapt their proof, we just need to observe that, if $u > 0$ is as in (2.4), then $\partial_\nu \ln u = -1$ on $\partial\Sigma$ (this is for the part (iii) $\Longrightarrow$ (i)) and use the Harnack inequality given in Proposition A.2 in Appendix A (for the part (ii) $\Longrightarrow$ (iii)). ∎

When $n = 2$, Fischer-Colbrie's result (Theorem 2 in [5]) gives that an exterior free boundary minimal surface Σ has finite index if and only if it has finite total curvature (see also Theorem 1.4 in [12]). One difference is that, in our case, the quadratic form Q does not depend only on the Gauss map, but also on the conformal factor along the boundary $\partial\Sigma$. A second important point is that we assume $\partial\Sigma$ to be compact. For example, if Σ is stable, we have a solution u to (2.4) which can be lifted to the universal cover $\widetilde{\Sigma}$. This implies that the associated quadratic form on $\widetilde{\Sigma}$ is nonnegative. However, the universal cover may not have finite total curvature, as we are going to see below (see Example 4.2). Actually, the universal cover is not properly immersed (the boundary is not compact) and thus it is not an exterior surface according to our definition.

2.2. Regular ends

The asymptotic of an exterior free boundary minimal hypersurface can be highly complicated. A simple asymptotic is given by *regular ends* introduced by Schoen in [16].

In order to describe it, we split $P \in \mathbb{R}^{n+1}$ as $(X,z) \in \mathbb{R}^n \times \mathbb{R}$. Then an end E of an exterior free boundary minimal hypersurface is said to be *regular* if, after an isometry, a representative of E is given by the graph of a function f of bounded gradient defined on $\{|X| \geq R\}$ with the following asymptotic:

(2.5) $$f(X) = A \ln |X| + B + (C, X)|X|^{-2} + O(|X|^{-2}) \qquad \text{if } n = 2,$$

(2.6) $$f(X) = B + A|X|^{-(n-2)} + (C, X)|X|^{-n} + O(|X|^{-n}) \quad \text{if } n > 2,$$

where $A, B \in \mathbb{R}$ and $C \in \mathbb{R}^n$. We notice that the above estimate on f implies similar estimates on its derivatives (see [16]). For example, one sees that $\nabla f(X)$ goes to 0 as $|X|$ goes to $+\infty$ and, in particular, there is a well-defined unit normal at ∞ for such an end.

In the above asymptotics, if $A = 0$, we say that the end is *planar*.

If $n = 2$ and Σ has finite total curvature, then Σ is conformally equivalent to a compact Riemann surface with boundary minus a finite number of points. Moreover, a properly embedded annular end with finite total curvature is regular, see Proposition 1 in [16].

In the case $3 \leq n \leq 6$, following the arguments of J. Tysk [21], if we assume that Σ has finite index and finite volume growth in the sense that $\lim_{R\to+\infty} R^{-n}|\Sigma \cap B_R| < +\infty$, then Σ has finitely many ends, all of them being regular.

3. Stable hypersurfaces

3.1. A Bôcher-type result for the Jacobi operator

In this section, we analyze the asymptotic behavior of positive Jacobi functions (i.e., solutions to $\mathcal{L}u = 0$) on regular ends.

Theorem 3.1. *Let E be a regular minimal end in $\mathbb{R}^{n+1}$, with $X \in \mathbb{R}^n$ denoting a coordinate associated to the end as in (2.5) and (2.6), and consider a positive Jacobi function u on E. Then u has the following asymptotic: there exist $A, B \in \mathbb{R}$ such that*

$$u(X) = A \ln |X| + B + v(X) \qquad \text{if } n = 2,$$
$$u(X) = A + B|X|^{-(n-2)} + v(X) \quad \text{if } n > 2,$$

where v is such that the function $|X|^{n-1}v$ is C^2-bounded on $\mathbb{R}^n \setminus B_R$. Moreover, either $A > 0$ or $A = 0$ and $B > 0$.

Proof. Writing $X = e^t p$ with $t \in \mathbb{R}$ and $p \in \mathbb{S}^{n-1}$, a regular end can be parametrized by $[t_0, +\infty) \times \mathbb{S}^{n-1}$ with a metric g having the asymptotic $g = e^{2t}(\delta + O(e^{-2(n-1)t}))$, where δ is the product metric on $\mathbb{R} \times \mathbb{S}^{n-1}$. Moreover, the second fundamental form can be estimated by $\|B\|^2 = O(e^{-2nt})$. Thus the Jacobi operator can be computed as

$$\Delta u + \|B\|^2 u = e^{-2t}\big(u_{tt} + (n-2)u_t + \Delta^\sigma u + M(u)\big),$$

where Δ^σ is the Laplacian on $\mathbb{S}^{n-1}$ and $M(u)$ is a second order linear operator whose coefficients have $C^{0,\alpha}$-norm bounded by $Ce^{-2(n-1)t}$ for some constant $C > 0$.

Therefore a Jacobi function u satisfies

(3.1) $$u_{tt} + (n-2)u_t + \Delta^\sigma u + M(u) = 0,$$

which is a uniformly elliptic equation on $[t_0, +\infty) \times \mathbb{S}^{n-1}$.

As a consequence, by the Harnack inequality ([10], Corollary 8.21), there is a constant $C > 0$ such that, for any $p, q \in \mathbb{S}^{n-1}$ and $t, s \geq t_0 + 1$ with $|t - s| \leq 1$, and any positive Jacobi function u, we have

$$u(t, p) \leq C u(s, q).$$

By Schauder's elliptic estimates ([10], Corollary 6.3), we also have

$$\|u\|_{C^{2,\alpha}([t-1/2,t+1/2]\times\mathbb{S}^{n-1})} \leq C\|u\|_{C^0([t-1,t+1]\times\mathbb{S}^{n-1})} \quad \text{for } t \geq t_0 + 2. \tag{3.2}$$

Let us define

$$\bar{u}(t) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} u(t, p)\, d\sigma.$$

By Harnack's inequality, we obtain

$$\|u\|_{C^0([t-1,t+1]\times\mathbb{S}^{n-1})} \leq C \min_{p\in\mathbb{S}^{n-1}} u(t, p) \leq C\bar{u}(t). \tag{3.3}$$

Then, combining with (3.2), there is a constant $C > 0$ such that

$$\|M(u)\|_{C^{0,\alpha}([t-1/2,t+1/2]\times\mathbb{S}^{n-1})} \leq Ce^{-2(n-1)t}\,\bar{u}(t) \tag{3.4}$$

and

$$\overline{M}(u)(t) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} M(u)(t, p)\, d\sigma$$

satisfies

$$\|\overline{M}(u)\|_{C^{0,\alpha}([t-1/2,t+1/2])} \leq Ce^{-2(n-1)t}\,\bar{u}(t). \tag{3.5}$$

By integrating (3.1) over $\mathbb{S}^{n-1}$, we obtain that $\bar{u}$ solves

$$\bar{u}_{tt} + (n-2)\bar{u}_t + \overline{M}(u) = 0.$$

Considering first the case $n > 2$, let a and b be two functions such that

$$\begin{pmatrix} \bar{u} \\ \bar{u}' \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2-n \end{pmatrix}.$$

Then we have the system

$$\begin{cases} a' &= -\dfrac{1}{n-2}\,\overline{M}(u)(t), \\ b' &= -(n-2)b + \dfrac{1}{n-2}\,\overline{M}(u)(t). \end{cases}$$

Using the above equations, we obtain

$$\partial_t\sqrt{a^2+b^2} \leq C|\overline{M}(u)(t)| \leq Ce^{-2(n-1)t}\,\bar{u}(t) \leq Ce^{-2(n-1)t}\sqrt{a^2+b^2}.$$

Thus $\sqrt{a^2+b^2}$ and $\bar{u}$ stay bounded on $[t_0, +\infty)$. In particular,

$$|\overline{M}(u)(t)| \leq Ce^{-2(n-1)t}.$$

Also,

$$\begin{aligned}\partial_t\big(e^{nt}\sqrt{a^2+b^2}\big) &\ge n e^{nt}\sqrt{a^2+b^2}-(n-2)\,e^{nt}\,\frac{b^2}{\sqrt{a^2+b^2}}-Ce^{-(n-2)t}\sqrt{a^2+b^2}\\ &\ge (2-Ce^{-2(n-1)t})\,e^{nt}\sqrt{a^2+b^2}\ge 0\end{aligned}$$

for t sufficiently large. Therefore $e^{nt}\sqrt{a^2+b^2}$ cannot converge to 0 at t goes to $+\infty$. We can also solve the system to obtain

$$\begin{cases} a \;=\; A+\displaystyle\int_t^{+\infty}\frac{1}{n-2}\,\overline{M}(u)(s)\,ds,\\ b \;=\; Be^{-(n-2)t}-e^{-(n-2)t}\displaystyle\int_t^{+\infty}\frac{1}{n-2}\,e^{(n-2)s}\,\overline{M}(u)(s)\,ds.\end{cases}$$

We notice that if $A=B=0$, then $\lim_{t\to+\infty}e^{nt}a=\lim_{t\to+\infty}e^{nt}b=0$, which is not possible. Therefore we can be sure that either A or B is nonzero. As $\bar u=a+b$ is positive and $A=\lim_{t\to+\infty}(a+b)$, then either $A>0$ or $A=0$ and $B>0$. Observe that

$$\bar u-A-Be^{-(n-2)t}=O(e^{-2(n-1)t}).$$

If $n=2$, we notice that

$$\begin{aligned}\partial_t\sqrt{\bar u^2+\bar u_t^2} &\le \frac{\bar u\,\bar u_t}{\sqrt{\bar u^2+\bar u_t^2}}+C|\overline{M}(u)|\le\frac12\sqrt{\bar u^2+\bar u_t^2}+Ce^{-2t}\,\bar u\\ &\le\Big(\frac12+Ce^{-2t}\Big)\sqrt{\bar u^2+\bar u_t^2}.\end{aligned}$$

Thus $\bar u=O(e^{\frac12 t})$ and then $\overline{M}(u)=O(e^{-\frac32 t})$. We also have

$$\partial_t\Big(e^{\frac34 t}\sqrt{\bar u^2+\bar u_t^2}\Big)\ge\Big(\frac34-\frac12-Ce^{-2t}\Big)\,e^{\frac34 t}\sqrt{\bar u^2+\bar u_t^2}\ge 0$$

for t sufficiently large. So $e^{\frac34 t}\sqrt{\bar u^2+\bar u_t^2}$ cannot converge to 0 as t goes to $+\infty$. By integrating the equation on $\bar u$, one gets

$$\bar u(t)=At+B-\int_t^{+\infty}\Big(\int_s^{+\infty}\overline{M}(u)(r)\,dr\Big)\,ds.$$

If A and B vanish, then $\bar u,\bar u_t=O(e^{-\frac32 t})$, which is not possible. Then either $A>0$ or $A=0$ and $B>0$. Notice that $\bar u-At-B=O(te^{-2t})$. In fact, last equation gives that $\bar u=O(t)$. Then, from (3.5), we have $\overline{M}(u)=O(te^{-2t})$. Therefore, using last equation again, we obtain $\bar u-At-B=O(te^{-2t})$.

In both cases, we have $M(u)=O(te^{-2(n-1)t})$.

Now, to conclude, we need to estimate $u-\bar u$. Let v_i be a L^2-unit eigenfunction for the Laplace operator on $\mathbb{S}^{n-1}$ associated to a nonzero eigenvalue λ (in particular, $\lambda\ge n-1$). Let $u_i=\int_{\mathbb{S}^{n-1}}uv_i\,d\sigma$. Equation (3.1) implies

$$u_{i\,tt}+(n-2)u_{i\,t}-\lambda u_i=-\int_{\mathbb{S}^{n-1}}M(u)\,v_i\,d\sigma=f_i=O(te^{-2(n-1)t}).$$

Observe that $\mu^2 + (n-2)\mu - \lambda = 0$ has two roots: $\mu_+ \geq 1$ and $\mu_- \leq -(n-1)$. Then, solving the above equation, we obtain

$$u_i(t) = e^{\mu_+ t}\Big(a_i - \int_t^{+\infty} e^{-\mu_+ s} \frac{f_i(s)}{\mu_+ - \mu_-}\, ds\Big) + e^{\mu_- t}\Big(b_i - \int_{t_0}^{t} e^{-\mu_- s} \frac{f_i(s)}{\mu_+ - \mu_-}\, ds\Big) \tag{3.6}$$

for some $a_i, b_i \in \mathbb{R}$. Using (3.3) and the fact that $\bar{u} = O(t)$, we have $u_i = O(t)$ and thus $a_i = 0$. We also have

$$b_i = e^{-\mu_- t_0} u_i(t_0) + e^{(\mu_+ - \mu_-)t_0} \int_{t_0}^{+\infty} e^{-\mu_+ s} \frac{f_i(s)}{\mu_+ - \mu_-}\, ds.$$

Now, by Cauchy–Schwarz,

$$\begin{aligned}\Big(\int_t^{+\infty} e^{-\mu_+ s} f_i(s)\, ds\Big)^2 &\leq \frac{1}{\mu_+} e^{-\mu_+ t} \int_t^{+\infty} e^{-\mu_+ s} f_i^2(s)\, ds,\\ \Big(\int_{t_0}^{t} e^{-\mu_- s} f_i(s)\, ds\Big)^2 &\leq \frac{1}{-\mu_-} (e^{-\mu_- t} - e^{-\mu_- t_0}) \int_{t_0}^{t} e^{-\mu_- s} f_i^2(s)\, ds.\end{aligned}$$

Thus, by squaring (3.6), we obtain

$$\begin{aligned}u_i^2(t) \leq 16\Big(&\frac{e^{\mu_+ t}}{\mu_+(\mu_+ - \mu_-)^2} \int_t^{+\infty} e^{-\mu_+ s} f_i^2(s)\, ds + e^{2\mu_-(t-t_0)} u_i^2(t_0)\\ &+ \frac{e^{2\mu_-(t-t_0)+\mu_+ t_0}}{\mu_+(\mu_+ - \mu_-)^2} \int_{t_0}^{+\infty} e^{-\mu_+ s} f_i^2(s)\, ds + \frac{e^{\mu_- t}}{-\mu_-(\mu_+ - \mu_-)^2} \int_{t_0}^{t} e^{-\mu_- s} f_i^2(s)\, ds\Big).\end{aligned}$$

Let us define

$$\begin{aligned}\widetilde{U}(t) &= \int_{\mathbb{S}^{n-1}} (u(t,p) - \bar{u}(t))^2\, d\sigma,\\ \widetilde{M}(t) &= \int_{\mathbb{S}^{n-1}} (M(u)(t,p) - \overline{M}(u)(t))^2\, d\sigma.\end{aligned}$$

Using that $\mu_+ \geq 1$ and $\mu_- \leq -(n-1)$, we can sum the above inequalities with respect to i to obtain

$$\begin{aligned}\widetilde{U}(t) \leq 16\Big(&e^{\mu_+ t} \int_t^{+\infty} e^{-\mu_+ s} \widetilde{M}(s)\, ds + e^{2\mu_-(t-t_0)} \widetilde{U}(t_0)\\ &+ e^{2\mu_-(t-t_0)+\mu_+ t_0} \int_{t_0}^{+\infty} e^{-\mu_+ s} \widetilde{M}(s)\, ds + e^{\mu_- t} \int_{t_0}^{t} e^{-\mu_- s} \widetilde{M}(s)\, ds\Big).\end{aligned}$$

Since $\widetilde{M}(t) = O(t^2 e^{-4(n-1)t})$, it follows that

$$\|u - \bar{u}\|_{L^2([t-1,t+2]\times\mathbb{S}^{n-1})} = O(e^{-(n-1)t}).$$

Actually, $u - \bar{u}$ solves the equation

$$z_{tt} + (n-2)z_t + \Delta^\sigma z + M(z) = \overline{M}(u) - M(\bar{u}).$$

Then, combining the above L^2-estimate with (3.4) and (3.5), Schauder's estimates give

$$\|u - \bar{u}\|_{C^2([t,t+1]\times\mathbb{S}^{n-1})} = O(e^{-(n-1)t}).$$

We have then proved that

$$\begin{aligned} \|u - A - Be^{-(n-2)t}\|_{C^2([t,t+1]\times\mathbb{S}^{n-1})} &= O(e^{-(n-1)t}) \quad \text{if } n > 2,\\ \|u - At - B\|_{C^2([t,t+1]\times\mathbb{S}^{n-1})} &= O(e^{-t}) \qquad\quad \text{if } n = 2. \end{aligned}$$

This gives the expected result after going back to the original coordinate system. ■

3.2. Classification of stable hypersurfaces

If Σ is an exterior free boundary minimal hypersurface with regular ends, the unit normal to Σ has a well-defined limit at each end. Then we say that such a hypersurface has *parallel ends* if these limits coincide up to a sign. We notice that, if Σ is embedded, then its ends are always parallel.

Now, we are going to use the above Bôcher-type theorem in order to give a classification of stable exterior FBMH with parallel regular ends.

Proof of Theorem 1.1. Consider the (X,z) coordinate system on $\mathbb{R}^{n+1}$. After an isometry, we can assume that the unit normal to the ends of Σ are given by $\pm e_z$.

Now, let $M \in \mathcal{M}_n(\mathbb{R})$ be a skew-symmetric matrix and consider the Killing vector field $K(X,z) = MX$. Notice that K generates isometries fixing the z-axis (we have $\exp(tM)$ orthogonal). Then the scalar product $u = (K, N)$ is a solution to $\Delta u + \|B\|^2 u = 0$ on Σ. Moreover, since K is tangent to $\partial\mathbb{B}$, u satisfies $\partial_\nu u + u = 0$ on $\partial\Sigma$.

Each end of Σ can be parametrized by the graph of a function f with the asymptotic given by (2.5) or (2.6) (depending on n). In particular,

$$N(X, f(X)) = \pm\frac{1}{\sqrt{1 + |\nabla f(X)|^2}}(-\nabla f(X) + e_z).$$

So the asymptotic of f gives that $u = O(|X|^{-(n-1)})$.

On the other hand, since Σ is stable, there is a positive solution v to (2.4). The asymptotic of v is given by Theorem 3.1. As a consequence, we see that $u(X)/v(X)$ goes to 0 as $|X|$ goes to $+\infty$. Also, the function $w = u/v$ satisfies

$$\begin{cases} \Delta w + 2(\nabla \ln v, \nabla w) = 0 & \text{on } \Sigma,\\ \partial_\nu w = 0 & \text{on } \partial\Sigma. \end{cases}$$

Therefore, the maximum principle gives that $w = u/v$ is constant and thus equals zero. This implies that $u = 0$ and then Σ is invariant by the isometries generated by K. So Σ is a catenoidal hypersurface. ■

4. Catenoidal hypersurfaces

Theorem 1.1 gives that stable hypersurfaces are invariant by isometries fixing an axis. In this section, we describe this kind of exterior free boundary minimal hypersurfaces Σ. We fix the axis to be $\mathbb{R}e_z$.

Let ϕ be a primitive of the function $r \mapsto (r^{2(n-1)} - 1)^{-1/2}$ defined on $[1, +\infty)$. The hypersurface

$$\Sigma = \{(X, z) \in \mathbb{R}^n \times \mathbb{R}\,; |X| \geq 1 \text{ and } z^2 = \phi^2(|X|)\}$$

is a minimal hypersurface invariant by isometries fixing $\mathbb{R}e_z$. Actually, any connected piece of a minimal hypersurface invariant by isometries fixing $\mathbb{R}e_z$ is a subset of $\lambda\Sigma + \mu e_z$ for some $\lambda, \mu \in \mathbb{R}$.

Half of Σ can be parametrized by the map

$$\begin{aligned} F\,:\, [1, +\infty) \times \mathbb{S}^{n-1} &\to \mathbb{R}^{n+1} \\ (r, p) &\mapsto (rp, \phi(r)). \end{aligned}$$

Given $\alpha \in (0, \pi/2)$, we look for a rotational exterior free boundary minimal hypersurface with boundary in $\{z = \sin\alpha\}$. Let $R_\alpha = (\sin\alpha)^{-1/(n-1)}$ be such that $\phi'(R_\alpha) = \tan\alpha$. Notice that R_α decreases with α from $+\infty$ to 1. Let $\mathcal{C}_\alpha$ be the hypersurface parametrized by

$$\begin{aligned} F_\alpha\,:\, [R_\alpha, +\infty) \times \mathbb{S}^{n-1} &\to \mathbb{R}^{n+1} \\ (r, p) &\mapsto \lambda_\alpha(rp, \phi(r)) + \mu_\alpha\, e_z, \end{aligned} \tag{4.1}$$

where λ_α and μ_α are chosen such that $\mathcal{C}_\alpha$ has the expected boundary:

$$\begin{cases} \lambda_\alpha = R_\alpha^{-1}\cos\alpha = (\sin\alpha)^{\frac{1}{n-1}}\cos\alpha, \\ \mu_\alpha = \sin\alpha - \lambda_\alpha\,\phi(R_\alpha). \end{cases} \tag{4.2}$$

The hypersurface $\mathcal{C}_\alpha$ has free boundary because of the choice of R_α. Thus, for any $\alpha \in [0, \pi/2)$, there is exactly one rotational exterior free boundary minimal hypersurface: $\mathcal{C}_\alpha$ for $\alpha \neq 0$ and $\mathcal{C}_0 = \{z = 0\} \setminus \mathbb{B}$ for $\alpha = 0$.

Let us notice that the unit normal to $\mathcal{C}_\alpha$ is given by

$$N(r, p) = \frac{1}{\sqrt{1 + (\phi'(r))^2}}\,(-\phi'(r)p, 1). \tag{4.3}$$

Therefore, $\mathcal{C}_\alpha$ is an exterior free boundary minimal catenoidal hypersurface and any connected exterior free boundary minimal catenoidal hypersurface is the image of some $\mathcal{C}_\alpha$ by a linear isometry of $\mathbb{R}^{n+1}$.

Observe that

$$\partial_\alpha\lambda_\alpha = (\sin\alpha)^{-\frac{n-2}{n-1}}\Big(\frac{1}{n-1}\cos^2\alpha - \sin^2\alpha\Big). \tag{4.4}$$

Let

$$\alpha_n = \arctan\Big(\frac{1}{\sqrt{n-1}}\Big).$$

It follows that, on $[0, \alpha_n]$, λ_α is increasing from 0 to λ_{α_n} and, on $[\alpha_n, \pi/2)$, λ_α is decreasing up to 0. Moreover, $\mathcal{C}_\alpha$ converges to the hyperplane $\{z = 1\}$ as α goes to $\pi/2$.

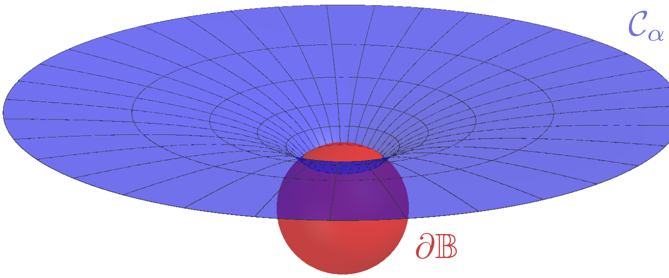

Figure 2. Catenoidal hypersurface $\mathcal{C}_\alpha$, $\alpha \approx 0.92$

The stability properties of $\mathcal{C}_\alpha$ are described by the following result, whose proof is inspired by the computation of the index of catenoids in $\mathbb{R}^{n+1}$ by L.-F. Tam and D. Zhou [20].

Proposition 4.1. *There exists $\bar{\alpha}_n \in [\alpha_n, \pi/2)$ such that $\mathcal{C}_\alpha$ is stable for $\alpha \in [0, \bar{\alpha}_n]$ and $\mathcal{C}_\alpha$ has index* 1 *for $\alpha \in (\bar{\alpha}_n, \pi/2)$.*

Actually, $\bar{\alpha}_2 = \alpha_2 = \pi/4$ and $\bar{\alpha}_n > \alpha_n$ for $n > 2$.

Proof. We first study some preliminary stability properties of $\mathcal{C}_\alpha$. Notice that $\mathcal{C}_\alpha$ is a graph over part of $\mathbb{R}^n$. Therefore, $\mathcal{C}_\alpha$ is stable as a graph with fixed boundary: the stability operator is nonnegative for any test functions that vanish on $\partial\mathcal{C}_\alpha$.

Let us consider on $\mathbb{R}^n$ coordinates $(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Let $K(x, Y, z) = (-z, 0, x)$ be the Killing vector field generating rotations around $\{x = 0, z = 0\}$ in $\mathbb{R}^{n+1}$. Then the scalar product $k = (K, N)$ defines on $\mathcal{C}_\alpha$ a solution to (2.4). The boundary condition comes from the fact that K is tangent to $\partial\mathbb{B}$. Actually, one can compute k in the (r, p) coordinates. By (4.1) and (4.3), we have

$$k = \frac{1}{\sqrt{1 + (\phi'(r))^2}} \big((\lambda_\alpha \phi(r) + \mu_\alpha)\phi'(r) + \lambda_\alpha r\big)\, p_x,$$

where p_x is the x coordinate of $p \in \mathbb{S}^{n-1} \subset \mathbb{R} \times \mathbb{R}^{n-1}$. Observe that, by (4.2),

$$\begin{aligned}(\lambda_\alpha \phi(r) + \mu_\alpha)\phi'(r) + \lambda_\alpha r &\geq (\lambda_\alpha \phi(R_\alpha) + \mu_\alpha)\phi'(r) + \lambda_\alpha R_\alpha \\ &= \sin\alpha\, \phi'(r) + \cos\alpha > 0.\end{aligned}$$

Hence k has constant sign when p_x has constant sign. This implies that the half catenoidal hypersurfaces $\mathcal{C}_\alpha \cap \{\pm x \geq 0\}$ are stable.

Let us now study the global stability of $\mathcal{C}_\alpha$. We have a one-parameter family $\{\mathcal{C}_\alpha\}$ of catenoidal hypersurfaces. Therefore the derivative with respect to α gives a deformation field whose scalar product with the unit normal to $\mathcal{C}_\alpha$ is a function u that solves (2.4). In the F_α parametrization and for the upward pointing unit normal, u can be computed as

$$u = \frac{1}{\sqrt{1 + \phi'(r)^2}} (-\partial_\alpha \lambda_\alpha r \phi'(r) + \partial_\alpha \lambda_\alpha \phi(r) + \partial_\alpha \mu_\alpha).$$

So u depends only on r and is equal to 1 on $\partial\mathcal{C}_\alpha$, i.e., at $r = R_\alpha$. In order to study the sign of u close to $r = +\infty$, let us take a look at λ_α. By (4.4), we have

$$\partial_\alpha^2 \lambda_\alpha = -\frac{\cos\alpha\ (\sin\alpha)^{-\frac{2n-3}{n-1}}\ \big((n^2+n-2)\sin^2\alpha + (n-2)\cos^2\alpha\big)}{(n-1)^2} \le 0.$$

Thus $\partial_\alpha\lambda_\alpha$ is decreasing.

If $n = 2$, we have $\lim_{r\to+\infty}\phi(r) = +\infty$. Then, for $\alpha \neq \alpha_2 = \pi/4$, we have that $\lim_{r\to+\infty} u = \pm\infty$ depending on $\operatorname{sign}(\partial_\alpha\lambda_\alpha)$: close to $r = +\infty$, u is positive for $\alpha < \alpha_2$ and negative for $\alpha > \alpha_2$.

If $n > 2$, we have $\lim_{r\to+\infty}\phi(r) < +\infty$ and, by (4.2),

$$\begin{aligned}\lim_{r\to+\infty} u &= \partial_\alpha\mu_\alpha + \partial_\alpha\lambda_\alpha \lim_{r\to+\infty}\phi(r)\\ &= \frac{n}{n-1}\cos\alpha + \partial_\alpha\lambda_\alpha\big(\lim_{r\to+\infty}\phi(r) - \phi(R_\alpha)\big).\end{aligned}$$

This limit is positive when $\alpha \le \alpha_n$. Moreover, when $\alpha \ge \alpha_n$, the limit is decreasing with α and negative for α close to $\pi/2$. Then there exists $\bar\alpha_n > \alpha_n$ such that the limit is positive for $\alpha < \bar\alpha_n$ and negative for $\alpha > \bar\alpha_n$.

Thus, for $\alpha < \bar\alpha_n$, u is positive on $\partial\mathcal{C}_\alpha$ and close to the infinity. Therefore, if u changes sign on $\mathcal{C}_\alpha$, $\{u < 0\}$ would be a precompact subdomain of $\mathcal{C}_\alpha$ with $u = 0$ on its boundary, but this would contradict the stability of $\mathcal{C}_\alpha$ as a graph. Hence, for $\alpha < \bar\alpha_n$, u is positive and then, by Proposition 2.1, $\mathcal{C}_\alpha$ is stable. The hypersurface $\mathcal{C}_{\bar\alpha_n}$ is also stable as limit of stable minimal hypersurfaces.

When $\alpha > \bar\alpha_n$, u changes sign on $\mathcal{C}_\alpha$. Thus there is $A > R_\alpha$ such that u is nonnegative on $[R_\alpha, A]\times\mathbb{S}^{n-1}$ and vanishes on $\{A\}\times\mathbb{S}^{n-1}$. This implies that $\mathcal{C}_\alpha$ has index at least 1. We notice that there is no value $B > A$ such that u vanishes on $\{B\}\times\mathbb{S}^{n-1}$. Indeed, this would contradict that $[A, +\infty)\times\mathbb{S}^{n-1}$ is stable as a graph.

Let us now prove that, for $\alpha > \bar\alpha_n$, $\mathcal{C}_\alpha$ has index 1. If it is not the case, then there is $B > R_\alpha$ such that the Jacobi operator has index at least 2 on $[R_\alpha, B]\times\mathbb{S}^{n-1}$. Let us consider u_2 the eigenfunction associated to the second eigenvalue $\lambda_2 < 0$ on $\mathcal{C}_\alpha(B) = F_\alpha([R_\alpha, B]\times\mathbb{S}^{n-1})$: u_2 is a solution to

$$\begin{cases}\Delta u_2 + \|B\|^2 u_2 = -\lambda_2 u_2 & \text{on } \mathcal{C}_\alpha(B),\\ \partial_\nu u_2 + u_2 = 0 & \text{on } \partial\mathcal{C}_\alpha,\\ u_2 = 0 & \text{on } r = B.\end{cases}$$

We are going to prove that u_2 depends only on the r variable. As above, let us consider (x, Y) coordinates on $\mathbb{R}^n$ and let S be the symmetry of $\mathbb{R}^{n+1}$ with respect to $\{x = 0\}$. As $\mathcal{C}_\alpha(B)$ is invariant by S, we can consider on it the function v defined by

$$v(p) = u_2(p) - u_2(S(p)).$$

The function v is then a solution to

$$\begin{cases}\Delta v + \|B\|^2 v = -\lambda_2 v & \text{on } \mathcal{C}_\alpha(B),\\ \partial_\nu v + v = 0 & \text{on } \partial\mathcal{C}_\alpha,\\ v = 0 & \text{on } r = B.\end{cases}$$

Moreover, we have $v = 0$ on $\mathcal{C}_\alpha(B) \cap \{x = 0\}$. If $v \neq 0$, this implies that $\mathcal{C}_\alpha(B) \cap \{x \geq 0\}$ is unstable, since $\lambda_2 < 0$, which contradicts the stability of $\mathcal{C}_\alpha \cap \{x \geq 0\}$. So $v \equiv 0$ and u_2 is invariant by S. Changing the choice of the x coordinate, we obtain that u_2 is invariant by isometries fixing the z-axis and then depends only on r.

As u_2 is associated to the second eigenvalue, u_2 must change sign. Then there is $C \in (R_\alpha, B)$ such that $u_2 = 0$ on $\{r = C\}$. As $\lambda_2 < 0$, this implies that $\{C \leq r \leq B\}$ is unstable, which contradicts the stability of $\mathcal{C}_\alpha$ as a graph. Hence $\mathcal{C}_\alpha$ has index 1 for $\alpha > \bar{\alpha}_n$. ∎

Example 4.2. Consider the universal cover of $\mathcal{C}_\alpha$ for $n = 2$:

$$\begin{aligned} F_\alpha : [R_\alpha, +\infty) \times \mathbb{R} &\to \mathcal{C}_\alpha \subset \mathbb{R}^3 \\ (r, \theta) &\mapsto \lambda_\alpha(r\cos\theta, r\sin\theta, \phi(r)) + \mu_\alpha e_z. \end{aligned}$$

Straightforward computations give that the area element and the Gaussian curvature of F_α are given by

$$d\sigma_\alpha = \lambda_\alpha^2 r(1 + (\phi')^2)^{1/2}\, dr\, d\theta \quad \text{and} \quad K_\alpha = \frac{\phi'\phi''}{\lambda_\alpha^2 r(1 + (\phi')^2)^2}.$$

Therefore,

$$K_\alpha\, d\sigma_\alpha = \frac{\phi'\phi''}{(1 + (\phi')^2)^{3/2}}\, dr\, d\theta = -\frac{dr\, d\theta}{r^2\sqrt{r^2 - 1}}.$$

Thus the total curvature of $\mathcal{C}_\alpha$ is given by

$$2\int_0^{2\pi} \Big(\int_{R_\alpha}^{+\infty} \frac{dr}{r^2\sqrt{r^2 - 1}}\Big)\, d\theta = 4\pi \int_{R_\alpha}^{+\infty} \frac{dr}{r^2\sqrt{r^2 - 1}} = 4\pi(1 - \cos\alpha),$$

while the total curvature of its universal cover is infinite. For $\alpha \leq \pi/4$, this example shows that when the boundary is not compact, even if the stability operator is nonnegative, the total curvature can be infinite.

5. Classification of one-ended examples

This section is devoted to the proof of Theorem 1.2. The idea of the proof is based on a symmetrization procedure as in Schoen's paper [16].

After a rotation, we can assume that the end of Σ is the graph of a function f over the outside of a compact set with the following asymptotic:

$$f(X) = A \ln|X| + B + O(|X|^{-1}) \quad \text{if } n = 2,$$

with $A \geq 0$ and, if $A = 0$, $B \geq 0$, and

$$f(X) = B + A|X|^{-(n-2)} + O(|X|^{-(n-1)}) \quad \text{if } n > 2,$$

with $B \geq 0$.

The first step consists in proving that either $\Sigma = \{z = 0\} \setminus \mathbb{B} = \mathcal{C}_0$ or $\Sigma \subset \{z > \varepsilon\}$ for some $\varepsilon > 0$.

Observe that $\partial\Sigma \subset \{z \geq -2\}$ and $\Sigma \setminus K \subset \{z \geq -2\}$ for some compact set $K \subset \mathbb{R}^{n+1}$. Then, by the maximum principle, $\Sigma \subset \{z \geq -2\}$. In fact, for each $t < 0$, we have $\Sigma \setminus K \subset \{z \geq t\}$ (for a possibly different compact set K). Therefore, if $\Sigma \cap \{z < 0\} \neq \varnothing$, we can start from $t = -2$ and let $t < 0$ increase up to finding a first contact point in $\Sigma \cap \{z = t_0\}$ for some $t_0 < 0$. We notice that, since Σ is free boundary, $\partial_\nu z = -(P, e_z)$ at a boundary point P. So the first contact point cannot be at $\partial\Sigma$ (indeed, in that case, we would have $\partial_\nu z \leq 0$ at that first contact point). Then the maximum principle can be applied at the first contact point in order to guarantee that $\Sigma = \{z = t_0\} \setminus \mathbb{B}$, which is not free boundary.

This shows that $\Sigma \subset \{z \geq 0\}$. Then either $\partial\Sigma \subset \{z > 0\}$ or $\partial\Sigma$ has a point in $\{z = 0\}$ and the boundary maximum principle can be applied so that $\Sigma = \{z = 0\} \setminus \mathbb{B}$.

If $\partial\Sigma \subset \{z > 0\}$, then we see that

$$-\int_{\partial\Sigma} \partial_\nu z = \int_{\partial\Sigma} (P, e_z) > 0.$$

Since z is harmonic on Σ, using the asymptotic of f, we obtain that

$$\begin{aligned} 0 < -\int_{\partial\Sigma} \partial_\nu z &= \int_{\Sigma\cap\{|X|=R\}} \partial_\nu z = \int_{\Sigma\cap\{|X|=R\}} (\nu, e_z) \\ &= \int_{\mathbb{S}^{n-1}} \Big(-(n-2)A \,\frac{1}{R^{n-1}}\, R^{n-1} + O(R^{-1}) \Big)\, d\sigma = -(n-2)\,|\mathbb{S}^{n-1}| A + o(1) \end{aligned}$$

for $n > 2$. The same estimate gives that $0 < 2\pi A + o(1)$ for $n = 2$. Therefore, if $n = 2$, we have $A > 0$ and this implies that $f(X) > 1$ for $|X|$ sufficiently large. If $n > 2$, we have $A < 0$ and, since $f \geq 0$, this implies that $B > 0$ and $f(X) > B/2$ for $|X|$ sufficiently large. In any case, we obtain that $\Sigma \subset \{z \geq \varepsilon\}$ for small positive ε, since there cannot be any first contact point with $\{z = t\}$ for $0 \leq t \leq \varepsilon$. This finishes the first step.

We fix a (x, Y) coordinate system in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. We want to prove that Σ is symmetric with respect to $\{x = 0\}$. In order to do this, we are going to follow a symmetrization procedure.

Given $\theta \in [0, \pi/2]$, let Π_θ be the hyperplane of equation $-x\sin\theta + z\cos\theta = 0$, let S_θ be the symmetry with respect to Π_θ, and let

$$\Sigma_\theta^- = \Sigma \cap \{-x\sin\theta + z\cos\theta \leq 0\}.$$

If B_ρ is the ball centered at the origin of radius $\rho > 0$, we notice that $S_\theta(B_\rho) = B_\rho$.

Lemma 5.1. *Given $\theta \in [0, \pi/2)$, there exists $\rho_\theta > 0$ such that, outside B_{ρ_θ}, $S_\theta(\Sigma_\theta^-)$ is above Σ. Moreover, ρ_θ can be chosen as an increasing function of θ.*

Proof. In $\mathbb{R}^2$, let $p = (a\cos\alpha, a\sin\alpha)$ be a point with $a > 0$ and $0 \leq \alpha \leq \theta$, and let R_t be the rotation of angle t. If $0 \leq t \leq 2(\theta - \alpha)$, then the angle between $\overrightarrow{pR_t(p)}$ and the vertical z-axis is $\alpha + t/2$, and then at most θ (see Figure 3).

Because of the asymptotic of Σ, the intersection $\Sigma \cap \partial B_\rho$ can be parametrized by $\partial B_\rho \cap \{z = 0\}$ in the following way: there is a function g such that

$$\Sigma \cap \partial B_\rho = \Big\{ \Big(\Big(1 - \frac{g^2(X)}{|X|^2}\Big)^{1/2} X, g(X) \Big) \,;\, X \in \partial B_\rho \cap \{z = 0\} \Big\},$$

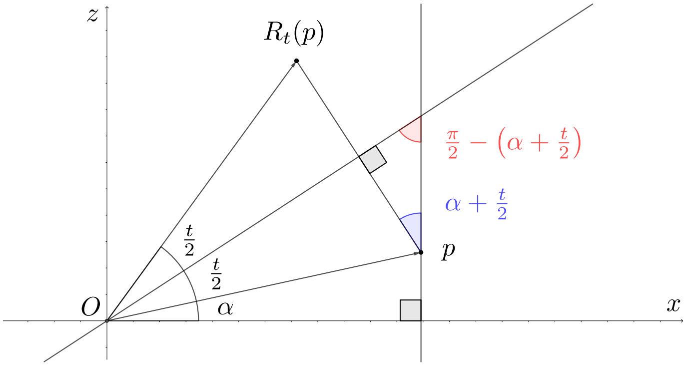

Figure 3. Angle between $\overrightarrow{pR_t(p)}$ and the z-axis.

where g satisfies

$$g(X) = A\ln|X| + B + O(|X|^{-1}) \qquad \text{if } n = 2,$$
$$g(X) = B + A|X|^{-(n-2)} + O(|X|^{-(n-1)}) \quad \text{if } n > 2,$$

and

$$dg(X)(V) = O(|X|^{-2})|V| \tag{5.1}$$

for any vector V tangent to $\partial B_{|X|}$.

In the (x, Y, z) coordinates, we can extend the rotation R_t to $\mathbb{R}^{n+1}$ by fixing the Y coordinates. Let ρ be large and let $P \in \Sigma_\theta^- \cap \partial B_\rho$. We can write

$$P = \left(\left(1 - \frac{g^2(X)}{|X|^2}\right)^{1/2} X, g(X)\right)$$

for some $X = (x, Y)$. Then $S_\theta(P)$ is above Σ if $R_t(P)$ does not meet Σ for $0 < t \le 2(\theta - \alpha)$, where $0 \le \alpha \le \theta$ is such that

$$\left(\left(1 - \frac{g^2(X)}{|X|^2}\right)^{1/2} x, g(X)\right) = a(\cos\alpha, \sin\alpha)$$

for some $a > 0$ (see Figure 4). In particular, $S_\theta(P) = R_{2(\theta-\alpha)}(P)$. We notice that $R_t(P)$ belongs to ∂B_ρ. Then, if $R_t(P) \in \Sigma$, we must have

$$R_t(P) = \left(\left(1 - \frac{g^2(X')}{|X'|^2}\right)^{1/2} X', g(X')\right)$$

for some $X' \in \partial B_\rho \cap \{z = 0\}$.

As a consequence, by integrating (5.1) along $\partial B_\rho \cap \{z = 0\}$, we have

$$|g(X) - g(X')| \le C\rho^{-2}|X - X'| \le C'\rho^{-2}\left|\left(1 - \frac{g^2(X)}{|X|^2}\right)^{1/2} X - \left(1 - \frac{g^2(X')}{|X'|^2}\right)^{1/2} X'\right|.$$

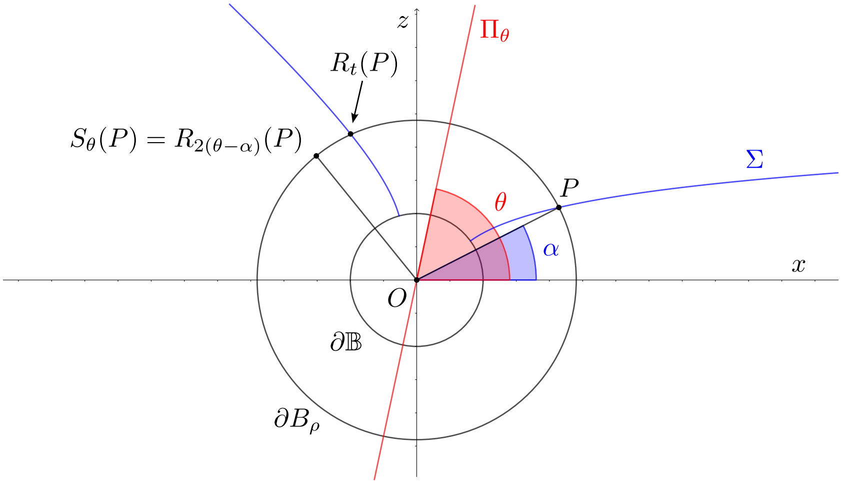

Figure 4. Configuration if $S_\theta(P)$ is not above Σ

This implies that $\overrightarrow{PR_t(P)}$ makes an angle less than $C''\rho^{-2}$ with the horizontal plane $\{z = 0\}$. Therefore, if ρ_θ is chosen such that this angle is less than $\pi/2 - \theta$, we obtain a contradiction with $R_t(P) \in \Sigma$, and the lemma is proved. ∎

We are now ready to finish the proof of Theorem 1.2. Let

$$T = \big\{\theta \in [0, \pi/2];\, S_\beta(\Sigma_\beta^-) \text{ is above } \Sigma \text{ for all } \beta \in [0, \theta]\big\}.$$

Since $\Sigma \subset \{z > 0\}$, we may choose $\theta_0 > 0$ small enough such that $\Sigma_{\theta_0}^- \subset \mathbb{R}^{n+1} \setminus B_{\rho_{\theta_0}}$. By Lemma 5.1, we have $[0, \theta_0] \subset T$. The set T is then a closed interval of the form $[0, \theta_1]$. Let us notice that when we symmetrize with respect to Π_θ, the image of a point on ∂B_ρ stays on ∂B_ρ, so that points in $\partial\Sigma$ cannot be sent to interior points of Σ and interior points of Σ cannot be sent to $\partial\Sigma$.

Then, if $\theta_1 < \pi/2$, by Lemma 5.1, there is a point $P \in \Sigma_{\theta_1}^-$ such that one of the following occurs:

- $P \notin \Pi_{\theta_1}$, $S_{\theta_1}(P) \in \Sigma$, and $S_{\theta_1}(\Sigma_{\theta_1}^-)$ is on one side of Σ;
- $P \in \Pi_{\theta_1}$, Σ is orthogonal to Π_{θ_1} at P, and $S_{\theta_1}(\Sigma_{\theta_1}^-)$ is on one side of Σ.

In the first case, if $P \notin \partial\Sigma$, then the maximum principle gives that Σ is symmetric with respect to Π_{θ_1}. If $P \in \partial\Sigma$, then, by free boundary hypothesis, Σ and $S_{\theta_1}(\Sigma_{\theta_1}^-)$ are normal to $\partial\mathbb{B}$ and thus tangent, since $S_{\theta_1}(\Sigma_{\theta_1}^-)$ is on one side of Σ. As a consequence, the boundary maximum principle implies that Σ is symmetric with respect to Π_{θ_1}.

In the second case, if $P \notin \partial\Sigma$, the boundary maximum principle implies that Σ is symmetric with respect to Π_{θ_1}. If $P \in \partial\Sigma$, then, as $S_{\theta_1}(\Sigma_{\theta_1}^-)$ is on one side of Σ, we can locally parametrize Σ and $S_{\theta_1}(\Sigma_{\theta_1}^-)$ over a quarter of the tangent plane $T_P\Sigma$ by two functions u and v such that $u \le v$, $u(P) = v(P)$, and $\nabla u(P) = \nabla v(P)$ (this last case is not considered by Park and Pyo in [15]). Moreover, at P, the tangent vector P is an eigenvector of the second fundamental form of Σ (see (2.1)). Let us also notice that, at P,

$S_{\theta_1}(\partial\Sigma \cap \Sigma^-_{\theta_1})$ is on one side of $\partial\Sigma$ in $\mathbb{S}^n$. This implies that, along $\partial\Sigma \cap \Pi_{\theta_1}$, the scalar product (N, ν_1) has a constant sign and vanishes at P, where ν_1 is the unit normal to Π_{θ_1}. Since ν_1 is parallel, this implies that $(\nabla^{\mathbb{S}}_T N, \nu_1)$ vanishes at P for any $T \in T_P\partial\Sigma \cap \Pi_{\theta_1}$. Notice that $\nu_1 \in T_P\partial\Sigma$, so $0 = (\nabla^{\mathbb{S}}_T N, \nu_1) = -B^{\mathbb{S}}_{\partial\Sigma}(T, \nu_1) = -B(T, \nu_1)$ by (2.2). This implies that, for any $V, W \in T_P\Sigma$, $B(S_{\theta_1}(V), S_{\theta_1}(W)) = B(V, W)$. Thus the Hessian of u and v at P coincide. So, applying Serrin's corner maximum principle [17] to $v - u$, we obtain that $u \equiv v$ and Σ is symmetric with respect to Π_{θ_1}.

In any case, we obtain that Π_{θ_1} is a plane of symmetry of Σ, which is not possible by Lemma 5.1. This gives that $\theta_1 = \pi/2$. Then $S_{\pi/2}(\Sigma \cap \{x \geq 0\})$ is above Σ. The same argument gives that $S_{\pi/2}(\Sigma \cap \{x \leq 0\})$ is above Σ. As a consequence, Σ is symmetric with respect to $\{x = 0\}$.

By changing the coordinate system, we obtain that Σ is symmetric with respect to any vertical hyperplane passing through the origin and then invariant by rotation around the vertical z-axis: Σ is a catenoidal surface.

Remark. In Theorem 1.2, if we know that the end is planar, then the catenoidal hypersurface is planar, since this is the only catenoidal hypersurface with a planar end. Thus we recover Theorem 3.3 in [15].

A. Harnack inequality

In this paper, we are considering solutions u to some elliptic equations on Σ under the Robin boundary condition $\partial_\nu u + u = 0$ on $\partial\Sigma$. Elliptic regularity theory for this condition can be found in Theorem 2.4 and 2.6 of [11]. Besides, one can also remark that, if d is a smooth function on Σ with $\partial_\nu d = 1$ (for example, $-d$ could be the distance function to $\partial\Sigma$) and $v = e^d u$, then $\partial_\nu v = (\partial_\nu d)e^d u + e^d \partial_\nu u = 0$ and v solves some elliptic equation. So results for Neumann boundary data can be translated to the Robin boundary condition.

In the proof of Proposition 2.1, we use a Harnack inequality up to the boundary that can be derived from the following one.

Proposition A.1. *Let Σ be a Riemannian manifold with compact boundary and let u be a positive solution to*

$$\begin{cases} \Delta u + (X, \nabla u) + qu = 0 & \text{on } \Sigma, \\ \partial_\nu u = 0 & \text{on } \partial\Sigma, \end{cases}$$

where X is a smooth vector field and q is a smooth function. Then, given a compact domain $U \subset \Sigma$, there exists a constant $C > 0$ (not depending on u) such that, for any $p, q \in U$, we have

$$\frac{u(p)}{u(q)} \leq C.$$

No such statement seems to appear in the literature. Similar results appear in [4, 22], but they are not directly applicable here because of certain hypotheses.

Proof. In order to prove such an estimate, it is enough to prove an upper bound on $|\nabla \ln u|$. Let $v = \ln u$. We have

$$\Delta v = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = -q - (X, \nabla v) - |\nabla v|^2. \tag{A.1}$$

Now, let $w = |\nabla v|^2$ and consider a nonnegative function ϕ with compact support such that $\phi = 1$ on $\partial\Sigma$ and $\phi \geq 1$ on U. Let p denote a point of maximum of ϕw. We notice that $\partial_\nu v = 0$, so ∇v is tangent to $\partial\Sigma$. Thus,

$$\begin{aligned}\partial_\nu(\phi w) &= w\,\partial_\nu\phi + 2\phi(\nabla_\nu\nabla v, \nabla v) = w\,\partial_\nu\phi + 2\phi(\nabla_{\nabla v}\nabla v, \nu)\\ &= w\,\partial_\nu\phi + 2\phi\, B_{\partial\Sigma}(\nabla v, \nabla v) \leq w\,(\partial_\nu\phi + 2H),\end{aligned}$$

where $B_{\partial\Sigma}$ is the second fundamental form of $\partial\Sigma$ and H is an upper bound for the principal curvatures of $\partial\Sigma$. So, choosing ϕ such that $\partial_\nu\phi + 2H < 0$, we can ensure that $\partial_\nu(\phi w)$ is negative. Thus the maximum cannot be on the boundary of Σ. Let us compute $\Delta(\phi w)$ by using Bochner's formula:

$$\begin{aligned}\Delta(\phi w) &= w\,\Delta\phi + 2\,(\nabla\phi, \nabla w) + \phi\,\Delta(|\nabla v|^2)\\ &= w\,\Delta\phi + 2(\nabla\phi, \nabla w) + 2\phi\left[(\nabla v, \nabla\Delta v) + |\nabla^2 v|^2 + \mathrm{Ric}(\nabla v, \nabla v)\right]\\ &\geq w(\Delta\phi - 2K\phi) + 2(\nabla\phi, \nabla w) + 2\phi|\nabla^2 v|^2 + 2\phi(\nabla v, \nabla(-q - (X, \nabla v) - w)),\end{aligned}$$

where $-K$ is a lower bound on the Ricci tensor. We also have

$$\begin{aligned}(\nabla v, \nabla(X, \nabla v)) &= (\nabla_{\nabla v} X, \nabla v) + (X, \nabla_{\nabla v}\nabla v) \leq Kw + (\nabla v, \nabla_X \nabla v)\\ &= Kw + \frac{1}{2}\,(X, \nabla w),\end{aligned}$$

where K is also chosen to be an upper bound for the tensor $(\nabla.X, \cdot)$.

At p, we have $0 = w\nabla\phi + \phi\nabla w$, that is,

$$\nabla w = -w\,\frac{\nabla\phi}{\phi}\cdot$$

Thus,

$$\begin{aligned}\Delta(\phi w) \geq{}& w(\Delta\phi - 4K\phi) + 2\phi|\nabla^2 v|^2 - 2\phi(\nabla v, \nabla q)\\ &- 2\,\frac{|\nabla\phi|^2}{\phi}\,w + w(X, \nabla\phi) + 2w(\nabla v, \nabla\phi).\end{aligned}$$

At p, $\Delta(\phi w) \leq 0$, and so

$$\begin{aligned}0 \geq{}& 2\phi\,|\nabla^2 v|^2 + w\Big(\Delta\phi - 4K\phi - 2\,\frac{|\nabla\phi|^2}{\phi} + (X, \nabla\phi)\Big)\\ &- 2|\nabla\phi|\,w^{3/2} - 2\phi\,|\nabla q|\,w^{1/2}.\end{aligned} \tag{A.2}$$

It is well known that

$$|\nabla^2 v| \geq \frac{1}{\sqrt{n}}\,|\Delta v|.$$

Then, by (A.1), we have

$$|\nabla^2 v|^2 \geq \frac{1}{n}|\Delta v|^2 = \frac{1}{n}(w^2 - aw^{3/2} - bw - cw^{1/2} - d)$$

for some constants a, b, c and d depending only on X and q. Thus, combining with (A.2) and multiplying by $\phi(p)$, we obtain, at p,

$$0 \geq \frac{2}{n}(\phi w)^2 - A(\phi w)^{3/2} - B(\phi w) - C(\phi w)^{1/2} - D$$

for some constants A, B, C and D depending only on K, X, q and ϕ. This implies that $\phi(p)w(p) \leq M$ for some constant $M = M(A, B, C, D)$, and thus $w(q) \leq M$ for $q \in U$, since $\phi \geq 1$ on U. ∎

As a consequence, we have the following Harnack inequality for the Robin boundary condition.

Proposition A.2. *Let Σ be a Riemannian manifold with compact boundary and let u be a positive solution to*

$$\begin{cases} \Delta u + qu = 0 & \text{on } \Sigma, \\ \partial_\nu u + u = 0 & \text{on } \partial\Sigma, \end{cases}$$

where q is a smooth function. Then, given a compact domain $U \subset \Sigma$, there exists a constant $C > 0$ (not depending on u) such that, for any $p, q \in U$, we have

$$\frac{u(p)}{u(q)} \leq C.$$

Proof. As explained above, let d be a smooth function on Σ such that $\partial_\nu d = 1$ and $v = e^d u$. We then have

$$\partial_\nu v = (\partial_\nu d)e^d u + e^d \partial_\nu u = 0.$$

We also have

$$\begin{aligned} \Delta v &= e^d(u|\nabla d|^2 + u\Delta d + 2(\nabla u, \nabla d) + d\Delta u) \\ &= 2(\nabla v, \nabla d) + (\Delta d - |\nabla d|^2 - dq)v. \end{aligned}$$

So the above proposition applies to v and gives the expected result for u. ∎

Acknowledgments. The authors would like to thank the anonymous referees for their good remarks on the writing of the paper.

Funding. L. M. was partially supported by the ANR-19-CE40-0014 grant. A. M. was partially supported by the National Council for Scientific and Technological Development – CNPq, Brazil (Grant 305710/2020-6) and the Fundação de Amparo à Pesquisa do Estado de Alagoas – FAPEAL, Brazil (Process E:60030.0000002254/2022). The authors were partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – CAPES, Brazil (CAPES-COFECUB 88887.143161/2017-0).

References

[1] Ambrozio, L., Carlotto, A. and Sharp, B.: Compactness analysis for free boundary minimal hypersurfaces. *Calc. Var. Partial Differential Equations* **57** (2018), no. 1, article no. 22, 39 pp.

[2] Ambrozio, L., Carlotto, A. and Sharp, B.: Index estimates for free boundary minimal hypersurfaces. *Math. Ann.* **370** (2018), no. 3-4, 1063–1078.

[3] Barbosa, E. and Espinar, J. M.: On free boundary minimal hypersurfaces in the Riemannian Schwarzschild space. *J. Geom. Anal.* **31** (2021), no. 12, 12548–12567.

[4] Chen, R.: A remark on the Harnack inequality for non-self-adjoint evolution equations. *Proc. Amer. Math. Soc.* **129** (2001), no. 7, 2163–2173.

[5] Fischer-Colbrie, D.: On complete minimal surfaces with finite Morse index in three-manifolds. *Invent. Math.* **82** (1985), no. 1, 121–132.

[6] Fischer-Colbrie, D. and Schoen, R.: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.* **33** (1980), no. 2, 199–211.

[7] Fraser, A. and Schoen, R.: Uniqueness theorems for free boundary minimal disks in space forms. *Int. Math. Res. Not. IMRN* (2015), no. 17, 8268–8274.

[8] Fraser, A. and Schoen, R.: Sharp eigenvalue bounds and minimal surfaces in the ball. *Invent. Math.* **203** (2016), no. 3, 823–890.

[9] Fraser, A. and Schoen, R.: Some results on higher eigenvalue optimization. *Calc. Var. Partial Differential Equations* **59** (2020), no. 5, article no. 151, 22 pp.

[10] Gilbarg, D. and Trudinger, N. S.: *Elliptic partial differential equations of second order*. Classics in Mathematics, Springer, Berlin, 2001.

[11] Grisvard, P.: *Elliptic problems in nonsmooth domains*. Monographs and Studies in Mathematics 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[12] Hong, H. and Saturnino, A. B.: Capillary surfaces: Stability, index and curvature estimates. *J. Reine Angew. Math.* **803** (2023), 233–265.

[13] López, F. J. and Ros, A.: Complete minimal surfaces with index one and stable constant mean curvature surfaces. *Comment. Math. Helv.* **64** (1989), no. 1, 34–43.

[14] Montezuma, R.: On free-boundary minimal surfaces in the Riemannian Schwarzschild manifold. *Bull. Braz. Math. Soc. (N.S.)* **52** (2021), no. 4, 1055–1071.

[15] Park, S.-H. and Pyo, J.: Free boundary minimal hypersurfaces with spherical boundary. *Math. Nachr.* **290** (2017), no. 5-6, 885–889.

[16] Schoen, R. M.: Uniqueness, symmetry, and embeddedness of minimal surfaces. *J. Differential Geom.* **18** (1983), no. 4, 791–809 (1984).

[17] Serrin, J.: A symmetry problem in potential theory. *Arch. Rational Mech. Anal.* **43** (1971), 304–318.

[18] Smith, G., Stern, A., Tran, H. and Zhou, D.: On the Morse index of higher-dimensional free boundary minimal catenoids. *Calc. Var. Partial Differential Equations* **60** (2021), no. 6, article no. 208, 44 pp.

[19] Smith, G. and Zhou, D.: The Morse index of the critical catenoid. *Geom. Dedicata* **201** (2019), 13–19.

[20] Tam, L.-F. and Zhou, D.: Stability properties for the higher dimensional catenoid in $\mathbb{R}^{n+1}$. *Proc. Amer. Math. Soc.* **137** (2009), no. 10, 3451–3461.

[21] Tysk, J.: Finiteness of index and total scalar curvature for minimal hypersurfaces. *Proc. Amer. Math. Soc.* **105** (1989), no. 2, 429–435.

[22] Wang, J.: Global heat kernel estimates. *Pacific J. Math.* **178** (1997), no. 2, 377–398.

Received July 4, 2022; revised September 11, 2023.

Laurent Mazet
Institut Denis Poisson, CNRS UMR 7013, Université de Tours
Parc de Grandmont, 37200 Tours, France;
laurent.mazet@univ-tours.fr

Abraão Mendes
Instituto de Matemática, Universidade Federal de Alagoas
Av. Lourival Melo Mota, s/n, 57072-900 Maceió, AL, Brazil;
abraao.mendes@im.ufal.br