



Rigidity of free boundary MOTS

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ABSTRACT

The aim of this work is to present an initial data version of Hawking's theorem on the topology of black hole spacetimes in the context of manifolds with boundary. More precisely, we generalize the results of G.J. Galloway and R. Schoen (Galloway et al., 2006) and G.J. Galloway (Galloway, 2008; Galloway, 2018) by proving that a compact free boundary stable marginally outer trapped surface (MOTS) Σ in an initial data set with boundary satisfying natural dominant energy conditions (DEC) is of positive Yamabe type, i.e. Σ admits a metric of positive scalar curvature with minimal boundary, provided Σ is outermost. To do so, we prove that if Σ is a compact free boundary stable MOTS which does not admit a metric of positive scalar curvature with minimal boundary in an initial data set satisfying the interior and the boundary DEC, then an outer neighborhood of Σ can be foliated by free boundary MOTS Σ_t , assuming that Σ is weakly outermost. Moreover, each Σ_t has vanishing outward null second fundamental form, is Ricci flat with totally geodesic boundary, and the dominant energy conditions saturate on Σ_t .

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1. Introduction

In a seminal paper, S.W. Hawking [14] showed, among other things, that closed cross-sections of the event horizon in $(3 + 1)$ -dimensional asymptotically flat stationary black hole spacetimes obeying the *spacetime* dominant energy condition (DEC) are topologically 2-spheres. Hawking's theorem extends to outer apparent horizons, i.e. *weakly outermost* marginally outer trapped surfaces (MOTS) in more general (not necessarily stationary) black hole spacetimes.

In [13], G.J. Galloway and R. Schoen extended Hawking's theorem to higher dimensional black hole spacetimes by proving that cross-sections of the event horizon (in the stationary case) and outer apparent horizons (in the general case) are of positive Yamabe type, i.e. they admit metrics of positive scalar curvature. Their result is consistent with the example presented by R. Emparan and H.S. Reall [10] of a $(4 + 1)$ -dimensional stationary vacuum black hole spacetime with horizon topology $S^2 \times S^1$.

We paraphrase Galloway–Schoen's result as follows. Definitions are given in Section 2.

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Theorem 1.1 ([13]). *Let (M, g, K) be an $(n + 1)$ -dimensional, $n \geq 2$, initial data set satisfying the DEC. If Σ is a closed stable MOTS in (M, g, K) , then either:*

- (1) Σ admits a metric of positive scalar curvature or
- (2) Σ is Ricci flat, the DEC saturates on Σ , and Σ has vanishing outward null second fundamental form.

Assuming the initial data set (M, g, K) can be embedded into a spacetime obeying the spacetime DEC and Σ is a closed outermost MOTS in (M, g, K) , Galloway [11] was able to rule out the ‘exceptional circumstances’ (item (2)) of Theorem 1.1; that is, under these hypotheses, Σ in fact admits a metric of positive scalar curvature.

A few years ago, Galloway [12] was able to drop the embeddedness assumption on the initial data set and prove a purely initial data version of the results presented in [11] and [13]. Among other things, he proved the following rigidity result.

Theorem 1.2 ([12]). *Let (M, g, K) be an $(n + 1)$ -dimensional, $n \geq 2$, initial data set satisfying the DEC. Suppose Σ is a closed weakly outermost MOTS in (M, g, K) that does not admit a metric of positive scalar curvature. Then there exists an outer neighborhood $U \cong [0, \epsilon) \times \Sigma$ of Σ in M such that, for each $t \in [0, \epsilon)$, the following hold:*

- (1) $\Sigma_t \cong \{t\} \times \Sigma$ is a MOTS. In fact, Σ_t has vanishing outward null second fundamental form.
- (2) Σ_t is Ricci flat with respect to the induced metric.
- (3) The DEC saturates on Σ_t .

As an immediate consequence of Theorem 1.2, he obtained the following:

Theorem 1.3 ([12]). *Let (M, g, K) be an $(n + 1)$ -dimensional, $n \geq 2$, initial data set satisfying the DEC. If Σ is a closed outermost MOTS in (M, g, K) , then Σ is of positive Yamabe type, i.e. Σ admits a metric of positive scalar curvature.*

Our main goal in the present work is to extend Theorems 1.2 and 1.3 to the context of compact free boundary MOTS in initial data sets with boundary satisfying the interior and the boundary dominant energy conditions.

Our first result is the following:

Theorem 1.4. *Let (M, g, K) be an $(n + 1)$ -dimensional, $n \geq 2$, initial data set with boundary and Σ be a compact free boundary stable MOTS in (M, g, K) . Suppose (M, g, K) satisfies the interior and the boundary DEC. If Σ does not admit a metric of positive scalar curvature with minimal boundary and is weakly outermost in (M, g, K) , then there exists an outer neighborhood $U \cong [0, \epsilon) \times \Sigma$ of Σ in M such that, for each $t \in [0, \epsilon)$, the following hold:*

- (1) $\Sigma_t \cong \{t\} \times \Sigma$ is a free boundary MOTS. In fact, Σ_t has vanishing outward null second fundamental form.
- (2) Σ_t is Ricci flat and has totally geodesic boundary with respect to the induced metric.
- (3) The interior DEC saturates on U and $J|_{\Sigma_t} = 0$.
- (4) The boundary DEC saturates along $\partial\Sigma_t$ and $(\iota_{\partial}\pi)^\top|_{\partial\Sigma_t} = 0$.

The 1-forms J and $(\iota_{\partial}\pi)^\top$ are defined in Section 2.

As an immediate consequence of Theorem 1.4, we obtain the following topological obstruction result for the existence of free boundary outermost MOTS in initial data sets under natural energy hypotheses, which can be seen as an initial data version of Hawking’s theorem in the context of manifolds with boundary.

Theorem 1.5. *Let (M, g, K) be an $(n + 1)$ -dimensional, $n \geq 2$, initial data set with boundary and Σ be a compact free boundary stable MOTS in (M, g, K) . Suppose (M, g, K) satisfies the interior and the boundary DEC. If Σ is outermost in (M, g, K) , then Σ admits a metric of positive scalar curvature with minimal boundary.*

It is important to mention that the compact-with-boundary orientable manifolds which admit metrics of positive scalar curvature with minimal boundary are completely characterized in $n = 2, 3$ dimensions. In $n = 2$, this follows directly from the Gauss–Bonnet theorem. In particular, they are topologically a disk. In $n = 3$, the characterization of such manifolds was done by A. Carlotto and C. Li [6].

This paper is organized as follows. In Section 2, we review some background material on MOTS. In Section 3, we state and prove several auxiliary results. In Section 4, we present the proof of Theorem 1.4. Finally, in Section 5, we establish a splitting result under natural convexity and volume-minimizing conditions.

2. Marginally outer trapped surfaces

An initial data set (M, g, K) consists of a Riemannian manifold (M, g) and a symmetric $(0, 2)$ -tensor K on M . We assume that M is oriented throughout this paper.

The local energy density μ and the local current density J of (M, g, K) are given by

$$\mu = \frac{1}{2} \left(R - |K|^2 + (\text{tr } K)^2 \right) \quad \text{and} \quad J = \text{div}(K - (\text{tr } K)g),$$

where R is the scalar curvature of (M, g) . The initial data set (M, g, K) is said to satisfy the dominant energy condition (DEC) if

$$\mu \geq |J| \quad \text{on } M. \tag{1}$$

The momentum tensor π of (M, g, K) is given by

$$\pi = K - (\text{tr } K)g.$$

Then μ and J can be written in terms of π by

$$\mu = \frac{1}{2} \left(R - |\pi|^2 + \frac{1}{n}(\text{tr } \pi)^2 \right) \quad \text{and} \quad J = \text{div } \pi,$$

where $n + 1$ is the dimension of M , $n \geq 2$.

When M is a manifold with boundary, we denote the second fundamental form of ∂M in (M, g) by $\Pi^{\partial M}$. More precisely,

$$\Pi^{\partial M}(Y, Z) = \langle \nabla_Y \varrho, Z \rangle, \quad Y, Z \in \mathfrak{X}(\partial M),$$

where ϱ is the outward unit normal of ∂M in (M, g) . The mean curvature $H^{\partial M}$ of ∂M in (M, g) is given by

$$H^{\partial M} = \text{tr } \Pi^{\partial M} = \text{div}_{\partial M} \varrho.$$

We define $(\iota_{\varrho}\pi)^{\top}$ as the restriction of $\iota_{\varrho}\pi = \pi(\varrho, \cdot)$ to tangent vector fields of ∂M and say that (M, g, K) satisfies the boundary DEC if

$$H^{\partial M} \geq |(\iota_{\varrho}\pi)^{\top}| \quad \text{along } \partial M.$$

The above boundary DEC was introduced by S. Almaraz, L.L. Lima, and L. Mari [2] in the context of proving spacetime positive mass inequalities for asymptotically flat and asymptotically hyperbolic initial data sets with noncompact boundary (see [2, Remark 2.7] for a motivation behind this definition).

In order to avoid ambiguity, we call (1) the interior DEC.

Let Σ be a connected two-sided hypersurface in (M, g) with unit normal N and

$$H = \operatorname{div}_\Sigma N$$

be its associated mean curvature. The null mean curvatures θ^+ and θ^- of Σ in (M, g, K) are defined by

$$\theta^+ = \operatorname{tr}_\Sigma K + H \quad \text{and} \quad \theta^- = \operatorname{tr}_\Sigma K - H.$$

Then the hypersurface Σ is said to be *outer trapped* if $\theta^+ < 0$, *weakly outer trapped* if $\theta^+ \leq 0$, and *marginally outer trapped* if $\theta^+ = 0$. In the latter case, we refer to Σ as a *marginally outer trapped surface* (MOTS).

The null second fundamental forms χ^+ and χ^- of Σ in (M, g, K) are defined by

$$\chi^+ = K|_\Sigma + A \quad \text{and} \quad \chi^- = K|_\Sigma - A,$$

where A is the second fundamental form of Σ in (M, g) . Our sign convention is such that $H = \operatorname{tr} A$ and, in particular, $\theta^\pm = \operatorname{tr} \chi^\pm$.

Remark 2.1. Initial data sets arise naturally in general relativity. In fact, let (\bar{M}, \bar{g}) be a spacetime, i.e. a time-oriented Lorentzian manifold. Consider a spacelike hypersurface M in (\bar{M}, \bar{g}) and let g be the induced Riemannian metric on M and K be the second fundamental form of M with respect to the future-pointing timelike unit normal u of M in (\bar{M}, \bar{g}) . Then (M, g, K) is an initial data set. As before, let Σ be a two-sided hypersurface in (M, g) with unit normal N . In this setting, χ^+ and χ^- are the second fundamental forms of Σ in (\bar{M}, \bar{g}) with respect to the null normal fields

$$\ell^+ = u|_\Sigma + N \quad \text{and} \quad \ell^- = u|_\Sigma - N,$$

respectively. Therefore $\theta^+ = \operatorname{tr} \chi^+ = \operatorname{div}_\Sigma \ell^+$ is the *future outgoing null expansion scalar* and $\theta^- = \operatorname{tr} \chi^- = \operatorname{div}_\Sigma \ell^-$ is the *future ingoing null expansion scalar* of Σ in (\bar{M}, \bar{g}) .

Let $(\Sigma_t)_{|t|<\epsilon}$ be a variation of Σ in M , $\Sigma = \Sigma_0$, with variation vector field

$$\mathcal{V} = \left. \frac{\partial}{\partial t} \right|_{t=0} = \phi N \quad \text{for some } \phi \in C^\infty(\Sigma).$$

We may view the null mean curvature θ^+ of Σ_t in (M, g, K) as a parameter-dependent function on Σ . Computations as in [4] show that

$$\left. \frac{d}{dt} \right|_{t=0} \theta^+(t, \cdot) = L\phi + \left(-\frac{1}{2}(\theta^+)^2 + \theta^+ \tau \right) \phi, \tag{2}$$

where

$$L\phi \triangleq -\Delta\phi + 2\langle X, \nabla\phi \rangle + (Q - |X|^2 + \operatorname{div} X)\phi$$

and

$$Q \triangleq \frac{1}{2}R^\Sigma - (\mu + J(N)) - \frac{1}{2}|\chi^+|^2.$$

Here, Δ is the nonpositive Laplace–Beltrami operator, ∇ the gradient, div the divergence, and R^Σ the scalar curvature of Σ with respect to the induced metric. Moreover, X is the tangent vector field of Σ that is dual to the 1-form $K(N, \cdot)|_\Sigma$, and $\tau = \operatorname{tr} K$. In this paper, the symbol \triangleq means “is defined to be equal to”.

When Σ is a MOTS, L is called the *MOTS stability operator* of Σ . This is because in the Riemannian case ($K = 0$), saying that Σ is a MOTS is equivalent to saying that Σ is a minimal hypersurface and, in this case, L reduces to the classical stability operator of minimal surfaces theory.

It is possible to show that, when Σ is a closed MOTS, the operator L admits a real eigenvalue $\lambda_1 = \lambda_1(L)$, called its *principal eigenvalue*, such that $\lambda_1 \leq \text{Re } \lambda$ for any other eigenvalue λ . Moreover, the associated eigenfunction ϕ_1 , $L\phi_1 = \lambda_1\phi_1$, is unique up to a multiplicative constant, and ϕ_1 can be chosen to be positive. Also, λ_1 is nonnegative if, and only if, there exists a positive function $\phi \in C^\infty(\Sigma)$ such that $L\phi \geq 0$ (see [5]). In analogy with the minimal surface case, a closed MOTS Σ is said to be *stable* provided $\lambda_1(L) \geq 0$.

Now, we are going to present a notion of *stability* for *capillary* MOTS in initial data sets with boundary introduced by A. Alaae, M. Lesourd, and S.-T. Yau [1]. We restrict our attention to the *free boundary* case.

Assume that M is a manifold with boundary and Σ is a compact-with-boundary *properly embedded* hypersurface in M , which means that Σ is embedded in M and $\partial\Sigma = \Sigma \cap \partial M$. We say that Σ is *free boundary* in (M, g) if Σ meets ∂M orthogonally, that is, $\varrho = \nu$ along $\partial\Sigma$, where ν is the outward unit normal of $\partial\Sigma$ in Σ with respect to the induced metric.

Remark 2.2. The mean curvature $H^{\partial\Sigma}$ of $\partial\Sigma$ in Σ with respect to the induced metric is given by

$$H^{\partial\Sigma} = \text{div}_{\partial\Sigma} \nu.$$

In particular, if Σ is free boundary (equivalently, N is tangent to ∂M along $\partial\Sigma$), then

$$H^{\partial M} = \text{div}_{\partial M} \varrho = \text{div}_{\partial\Sigma} \nu + \langle \nabla_N \varrho, N \rangle = H^{\partial\Sigma} + \text{II}^{\partial M}(N, N).$$

Let $(\Sigma_t)_{|t|<\epsilon}$ be a variation of $\Sigma = \Sigma_0$ in (M, g, K) by compact-with-boundary properly embedded hypersurfaces and consider the functional

$$\mathcal{F}[\Sigma_t] = \int_{\Sigma_t} \theta^+(t) \langle \mathcal{V}, N_t \rangle dv_t + \int_{\partial\Sigma_t} \langle \mathcal{V}, \nu_t \rangle ds_t,$$

where

$$\mathcal{V} = \mathcal{V}_t = \frac{\partial}{\partial t}.$$

If Σ is a free boundary MOTS in (M, g, K) , then

$$\frac{d}{dt} \Big|_{t=0} \mathcal{F}[\Sigma_t] = \int_{\Sigma} \phi L\phi dv - \int_{\partial\Sigma} \phi \left(\frac{\partial\phi}{\partial\nu} - \text{II}^{\partial M}(N, N)\phi \right) ds, \tag{3}$$

where $\phi = \langle \mathcal{V}, N \rangle$ (see [1, Section 5]). The corresponding eigenvalue problem to functional (3) is the following:

$$\begin{aligned} L\phi &= -\Delta\phi + 2\langle X, \nabla\phi \rangle + (Q - |X|^2 + \text{div } X)\phi = \lambda\phi \quad \text{on } \Sigma, \\ B\phi &\triangleq \frac{\partial\phi}{\partial\nu} - \text{II}^{\partial M}(N, N)\phi = 0 \quad \text{along } \partial\Sigma. \end{aligned}$$

Then a compact free boundary MOTS Σ in (M, g, K) is said to be *stable* provided there exists a nonnegative function $\phi \in C^\infty(\Sigma)$, $\phi \not\equiv 0$, satisfying the Robin boundary condition $B\phi = 0$ such that $L\phi \geq 0$. Without loss of generality, by the maximum principle for nonpositive functions, we may assume that $\phi > 0$.

Before finishing this section, let us present some important definitions.

Suppose Σ is a separating MOTS in (M, g, K) . By definition, the unit normal N is said to be *outward-pointing*. We say that Σ is *weakly outermost* if there is no outer trapped ($\theta^+ < 0$) surface outside of, and homologous to, Σ . Analogously, we say that Σ is *outermost* if there is no weakly outer trapped ($\theta^+ \leq 0$) surface outside of, and homologous to, Σ .

3. Auxiliary results

The first auxiliary result we are going to prove is the following.

Lemma 3.1. *Let (M, g, K) be an $(n + 1)$ -dimensional, $n \geq 2$, initial data set with boundary and Σ be a compact free boundary MOTS in (M, g, K) . If Σ is stable, then the first eigenvalue $\lambda_1(\mathcal{L}_0)$ of $\mathcal{L}_0 \triangleq -\Delta + Q$ on Σ with Robin boundary condition $B_0u \triangleq Bu + \langle X, \nu \rangle u = 0$ is nonnegative.*

The proof of Lemma 3.1 is essentially contained in the proof of Lemma 5.4 in [1]. But, for the sake of completeness, we include it here. The inspiring closed case was first proved by Galloway and Schoen [13] (see also [5, Section 4] and [11, Lemma 2.2]).

Proof of Lemma 3.1. Let $\phi \in C^\infty(\Sigma)$ be a positive function satisfying the boundary condition $B\phi = 0$ such that $L\phi \geq 0$. Observe that

$$\begin{aligned} \frac{L\phi}{\phi} &= -\frac{\Delta\phi}{\phi} + 2\langle X, \nabla \ln \phi \rangle + Q - |X|^2 + \operatorname{div} X \\ &= -\frac{\Delta\phi}{\phi} + |\nabla \ln \phi|^2 - |X - \nabla \ln \phi|^2 + Q + \operatorname{div} X \\ &= \operatorname{div} Y - |Y|^2 + Q, \end{aligned}$$

where $Y = X - \nabla \ln \phi$. Therefore

$$\begin{aligned} u^2 \frac{L\phi}{\phi} &= \operatorname{div}(u^2 Y) - 2u\langle Y, \nabla u \rangle - |uY|^2 + Qu^2 \\ &= \operatorname{div}(u^2 Y) - |uY + \nabla u|^2 + |\nabla u|^2 + Qu^2 \\ &\leq \operatorname{div}(u^2 Y) + |\nabla u|^2 + Qu^2, \end{aligned}$$

for any function $u \in C^\infty(\Sigma)$.

Then, taking u as a first eigenfunction of \mathcal{L}_0 , $\mathcal{L}_0u = \lambda_1(\mathcal{L}_0)u$, with Robin boundary condition $B_0u = 0$ and using that $L\phi \geq 0$, we obtain

$$\begin{aligned} 0 &\leq \int_\Sigma \left(|\nabla u|^2 + Qu^2 + \operatorname{div}(u^2 Y) \right) dv \\ &= \int_\Sigma \left(|\nabla u|^2 + Qu^2 \right) dv - \int_{\partial\Sigma} u^2 \left(\frac{1}{\phi} \frac{\partial\phi}{\partial\nu} - \langle X, \nu \rangle \right) ds \\ &= \int_\Sigma \left(|\nabla u|^2 + Qu^2 \right) dv - \int_{\partial\Sigma} u^2 \left(\Pi^{\partial M}(N, N) - \langle X, \nu \rangle \right) ds \\ &= \int_\Sigma \left(|\nabla u|^2 + Qu^2 \right) dv - \int_{\partial\Sigma} u \frac{\partial u}{\partial\nu} ds \\ &= \int_\Sigma u \mathcal{L}_0 u dv = \lambda_1(\mathcal{L}_0) \int_\Sigma u^2 dv, \end{aligned}$$

where above we have used that $B\phi = 0$ and $B_0u = 0$. Thus $\lambda_1(\mathcal{L}_0) \geq 0$. \square

Our second auxiliary result is the following.

Lemma 3.2. *Let (Σ, γ) be an n -dimensional, $n \geq 2$, compact-with-boundary, connected, orientable, Riemannian manifold. Suppose there exists $u \in C^\infty(\Sigma)$, $u > 0$, such that*

$$\begin{aligned} \mathcal{L}u &\triangleq -\Delta u + \left(\frac{1}{2} R^\Sigma - P \right) u \geq 0 \text{ on } \Sigma, \\ \frac{\partial u}{\partial\nu} + H^{\partial\Sigma} u &\geq 0 \text{ along } \partial\Sigma, \end{aligned}$$

where R^Σ is the scalar curvature of (Σ, γ) , $H^{\partial\Sigma}$ is the mean curvature of $\partial\Sigma$ in (Σ, γ) , and P is a nonnegative function on Σ . Then either:

- (1) Σ admits a metric of positive scalar curvature with minimal boundary or
- (2) (Σ, γ) is Ricci flat with totally geodesic boundary, $P = 0$, and u is constant.

Proof. Suppose Σ does not admit a metric of positive scalar curvature with minimal boundary. Consider the metric $\hat{\gamma} = u^{2/(n-1)}\gamma$. The scalar curvature $R_{\hat{\gamma}}^\Sigma$ of $(\Sigma, \hat{\gamma})$ and the mean curvature $H_{\hat{\gamma}}^{\partial\Sigma}$ of $\partial\Sigma$ in $(\Sigma, \hat{\gamma})$ are given by

$$R_{\hat{\gamma}}^\Sigma = u^{-\frac{n+1}{n-1}} \left(-2\Delta u + R^\Sigma u + \frac{n}{n-1} \frac{|\nabla u|^2}{u} \right) = u^{-\frac{n+1}{n-1}} \left(2(\mathcal{L}u + Pu) + \frac{n}{n-1} \frac{|\nabla u|^2}{u} \right), \tag{4}$$

$$H_{\hat{\gamma}}^{\partial\Sigma} = u^{-\frac{n}{n-1}} \left(\frac{\partial u}{\partial \nu} + H^{\partial\Sigma} u \right). \tag{5}$$

In particular, $R_{\hat{\gamma}}^\Sigma$ and $H_{\hat{\gamma}}^{\partial\Sigma}$ are nonnegative.

If $n = 2$, by the Gauss–Bonnet theorem, $\chi(\Sigma) \geq 0$. On the other hand, in dimension $n = 2$, saying that Σ does not admit a metric of positive scalar curvature (positive Gaussian curvature) with minimal boundary (geodesic boundary) is equivalent to saying that $\chi(\Sigma) \leq 0$ (see [8, Theorem 1.2]). Therefore $\chi(\Sigma) = 0$ and, by the Gauss–Bonnet theorem, $R_{\hat{\gamma}}^\Sigma$ and $H_{\hat{\gamma}}^{\partial\Sigma}$ vanish, since they are nonnegative. From (4) and (5), we obtain that u is constant and the functions P , R^Σ , and $H^{\partial\Sigma}$ vanish. This finishes the proof in case $n = 2$.

Assume that $n \geq 3$.

Claim 1. $R_{\hat{\gamma}}^\Sigma$ and $H_{\hat{\gamma}}^{\partial\Sigma}$ vanish. If not, consider the eigenvalue problem

$$\begin{aligned} -c_n \Delta_{\hat{\gamma}} \psi + R_{\hat{\gamma}}^\Sigma \psi &= \lambda \psi \quad \text{on } \Sigma, \\ c_n \frac{\partial \psi}{\partial \nu_{\hat{\gamma}}} + 2H_{\hat{\gamma}}^{\partial\Sigma} \psi &= 0 \quad \text{along } \partial\Sigma, \end{aligned} \tag{6}$$

where $c_n = \frac{4(n-1)}{n-2}$, and let ψ_1 be an eigenfunction associated with the first eigenvalue λ_1 for this problem. We may assume that $\psi_1 > 0$. Multiplying (6) by $\psi = \psi_1$ and integrating over $(\Sigma, \hat{\gamma})$, we obtain

$$\lambda_1 = \frac{\int_\Sigma \left(c_n |\nabla \psi_1|_{\hat{\gamma}}^2 + R_{\hat{\gamma}}^\Sigma \psi_1^2 \right) dv_{\hat{\gamma}} + 2 \int_{\partial\Sigma} H_{\hat{\gamma}}^{\partial\Sigma} \psi_1^2 ds_{\hat{\gamma}}}{\int_\Sigma \psi_1^2 dv_{\hat{\gamma}}}.$$

Since $R_{\hat{\gamma}}^\Sigma$ and $H_{\hat{\gamma}}^{\partial\Sigma}$ are nonnegative and, by hypothesis, at least one of them is not identically zero, we have $\lambda_1 > 0$. Therefore the scalar curvature of $(\Sigma, \gamma_1 = \psi_1^{4/(n-2)}\hat{\gamma})$ and the mean curvature of $\partial\Sigma$ in (Σ, γ_1) satisfy

$$\begin{aligned} R_{\gamma_1}^\Sigma &= \psi_1^{-\frac{n+2}{n-2}} (-c_n \Delta_{\hat{\gamma}} \psi_1 + R_{\hat{\gamma}}^\Sigma \psi_1) = \psi_1^{-\frac{4}{n-2}} \lambda_1 > 0, \\ H_{\gamma_1}^{\partial\Sigma} &= \frac{1}{2} \psi_1^{-\frac{n}{n-2}} \left(c_n \frac{\partial \psi_1}{\partial \nu} + 2H_{\hat{\gamma}}^{\partial\Sigma} \psi_1 \right) = 0, \end{aligned}$$

which is a contradiction, because we are assuming that Σ does not admit a metric of positive scalar curvature with minimal boundary. Thus $R_{\hat{\gamma}}^\Sigma$ and $H_{\hat{\gamma}}^{\partial\Sigma}$ vanish.

Using $R_{\hat{\gamma}}^\Sigma = 0$ and $H_{\hat{\gamma}}^{\partial\Sigma} = 0$ into (4) and (5), we obtain that u is constant and the functions P , R^Σ , and $H^{\partial\Sigma}$ vanish. Then we claim that (Σ, γ) is Ricci flat with totally geodesic boundary. If not, it follows from [7, Lemma 2.2] that Σ admits a metric of positive scalar curvature with mean convex boundary. In this case, acting exactly as in the proof of Claim 1, we get a contradiction. This finishes the proof. \square

Now, we use [Lemmas 3.1](#) and [3.2](#) to obtain an *infinitesimal rigidity* result.

Proposition 3.3. *Let (M, g, K) be an $(n+1)$ -dimensional, $n \geq 2$, initial data set with boundary and Σ be a compact free boundary stable MOTS in (M, g, K) . Suppose (M, g, K) satisfies the interior and the boundary DEC. If Σ does not admit a metric of positive scalar curvature with minimal boundary, then*

- (1) Σ has vanishing outward null second fundamental form, i.e. $\chi^+ = 0$.
- (2) Σ is Ricci flat and has totally geodesic boundary with respect to the induced metric.
- (3) The interior DEC saturates on Σ and $J|_{\Sigma} = 0$.
- (4) The boundary DEC saturates along $\partial\Sigma$ and $(\iota_{\varrho}\pi)^{\top}|_{\partial\Sigma} = 0$.

Proof. Let $u > 0$ be an eigenfunction of $\mathcal{L}_0 = -\Delta + Q$ on Σ , with Robin boundary condition $B_0u = 0$, associated with the first eigenvalue $\lambda_1(\mathcal{L}_0)$, that is,

$$\begin{aligned} \mathcal{L}_0u &= -\Delta u + \left(\frac{1}{2}R^{\Sigma} - P\right)u = \lambda_1(\mathcal{L}_0)u \text{ on } \Sigma, \\ \frac{\partial u}{\partial \nu} &= \left(\Pi^{\partial M}(N, N) - \langle X, \nu \rangle\right)u \text{ along } \partial\Sigma, \end{aligned}$$

where

$$P \triangleq \mu + J(N) + \frac{1}{2}|\chi^+|^2.$$

In order to use [Lemma 3.2](#), we need to check that \mathcal{L}_0u , P , and $\frac{\partial u}{\partial \nu} + H^{\partial\Sigma}u$ are nonnegative. The first condition follows from [Lemma 3.1](#), since Σ is stable. The second one follows from the interior DEC. In fact,

$$P \geq \mu + J(N) \geq \mu - |J| \geq 0. \tag{7}$$

For the third, observe that

$$H^{\partial M} = H^{\partial\Sigma} + \Pi(N, N)$$

and

$$(\iota_{\varrho}\pi)^{\top}(N) = K(\varrho, N) - (\text{tr } K)\langle \varrho, N \rangle = K(\nu, N) = \langle X, \nu \rangle,$$

since Σ is free boundary. Therefore, using the boundary DEC, we obtain

$$\begin{aligned} \frac{\partial u}{\partial \nu} + H^{\partial\Sigma}u &= \left(H^{\partial\Sigma} + \Pi^{\partial M}(N, N) - \langle X, \nu \rangle\right)u \\ &= \left(H^{\partial M} - (\iota_{\varrho}\pi)^{\top}(N)\right)u \\ &\geq \left(|(\iota_{\varrho}\pi)^{\top}| - (\iota_{\varrho}\pi)^{\top}(N)\right)u \\ &\geq 0. \end{aligned} \tag{8}$$

Then, from [Lemma 3.2](#), we have:

- Σ is Ricci flat and has totally geodesic boundary with respect to the induced metric.
- $P = 0$. In particular, from (7), we obtain: $\chi^+ = 0$, $\mu = |J|$, and $J(N) = -|J|$. Also, observing that $|J|^2 = |J|_{\Sigma}|^2 + J(N)^2$, we get $J|_{\Sigma} = 0$.
- u is constant. In this case, using that Σ has totally geodesic boundary and, in particular, $H^{\partial\Sigma} = 0$, inequality (8) must saturate, that is, $|(\iota_{\varrho}\pi)^{\top}| = (\iota_{\varrho}\pi)^{\top}(N)$. Finally, $|(\iota_{\varrho}\pi)^{\top}|^2 = |(\iota_{\varrho}\pi)^{\top}|_{\partial\Sigma}|^2 + ((\iota_{\varrho}\pi)^{\top}(N))^2$ gives $(\iota_{\varrho}\pi)^{\top}|_{\partial\Sigma} = 0$.

This finishes the proof. \square

Consider the operator

$$L^*\psi \triangleq -\Delta\psi - 2\langle X, \nabla\psi \rangle + (Q - |X|^2 - \operatorname{div} X)\psi, \quad \psi \in C^\infty(\Sigma),$$

called the *formal adjoint* of L . Straightforward computations show that

$$\int_\Sigma \psi L\phi \, dv + \int_{\partial\Sigma} \psi B\phi \, ds = \int_\Sigma \phi L^*\psi \, dv + \int_{\partial\Sigma} \phi B^*\psi \, ds$$

for every $\phi, \psi \in C^\infty(\Sigma)$, where

$$B^*\psi \triangleq \frac{\partial\psi}{\partial\nu} - \left(\Pi^{\partial M}(N, N) - 2\langle X, \nu \rangle \right) \psi.$$

Remark 3.4. It is important to note that, because L is not necessarily symmetric and Σ is a manifold with boundary, there is no *standard* general existence result for the principal eigenvalue $\lambda_1(L)$ of L (with Robin boundary condition $B\phi = 0$) as in the closed case. Then, in order to guarantee the existence of $\lambda_1(L)$ with similar properties to those of the closed case, Alaae, Lesourd, and Yau [1] assumed that $\Pi^{\partial M}(N, N) \leq 0$ along $\partial\Sigma$. But here, we succeed in circumventing this technical issue and do not make such an assumption.

Lemma 3.5. *Under the assumptions of Proposition 3.3, zero is a simple eigenvalue of L on Σ with Robin boundary condition $B\phi = 0$ whose associated eigenfunctions can be chosen to be positive. The same holds for the formal adjoint operator L^* on Σ with Robin boundary condition $B^*\phi^* = 0$.*

Proof. From Proposition 3.3 (and its proof), we know that $Q = 0$ and $\Pi^{\partial M}(N, N) = \langle X, \nu \rangle$. Since Σ is stable, we can take $\phi \in C^\infty(\Sigma)$, $\phi > 0$, satisfying $L\phi \geq 0$ on Σ and $B\phi = 0$ along $\partial\Sigma$. Then

$$0 \leq \frac{L\phi}{\phi} = -|X - \nabla \ln \phi|^2 + \operatorname{div}(X - \nabla \ln \phi)$$

provides, via the divergence theorem,

$$\int_\Sigma |X - \nabla \ln \phi|^2 \, dv \leq \int_{\partial\Sigma} \left(\langle X, \nu \rangle - \frac{1}{\phi} \frac{\partial\phi}{\partial\nu} \right) \, ds = \int_{\partial\Sigma} \left(\langle X, \nu \rangle - \Pi^{\partial M}(N, N) \right) \, ds = 0.$$

Thus $X = \nabla \ln \phi$. In particular, $L\phi = 0$. Therefore $\lambda = 0$ is an eigenvalue of L with boundary condition $B\phi = 0$. To show that $\lambda = 0$ is simple, let $\psi \in C^\infty(\Sigma)$ be such that $L\psi = 0$ on Σ with $B\psi = 0$ along $\partial\Sigma$ and define $u = \psi/\phi$. Simple computations show that

$$\begin{aligned} \Delta u &= 0 \quad \text{on } \Sigma, \\ \frac{\partial u}{\partial\nu} &= 0 \quad \text{along } \partial\Sigma. \end{aligned}$$

Therefore $u = \psi/\phi$ is constant. Thus $\lambda = 0$ is a simple eigenvalue whose associated eigenfunctions are given by $\psi = c\phi$, $c \in \mathbb{R}$.

Now, observe that

$$\begin{aligned} L^*\phi^* &= -\Delta\phi^* - 2\langle X, \nabla\phi^* \rangle - (|X|^2 + \operatorname{div} X)\phi^* \\ &= -\Delta\phi^* - 2\frac{\langle \nabla\phi, \nabla\phi^* \rangle}{\phi} - \left(\frac{|\nabla\phi|^2}{\phi^2} + \Delta \ln \phi \right) \phi^* \\ &= -\frac{\phi \Delta\phi^* + 2\langle \nabla\phi, \nabla\phi^* \rangle + \phi^* \Delta\phi}{\phi} \\ &= -\frac{\Delta(\phi^*\phi)}{\phi} \end{aligned}$$

and

$$\begin{aligned} B^* \phi^* &= \frac{\partial \phi^*}{\partial \nu} - \left(\Pi^{\partial M}(N, N) - 2\langle X, \nu \rangle \right) \phi^* \\ &= \frac{\partial \phi^*}{\partial \nu} + \Pi^{\partial M}(N, N) \phi^* \\ &= \frac{\partial \phi^*}{\partial \nu} + \frac{\phi^*}{\phi} \frac{\partial \phi}{\partial \nu} \\ &= \frac{1}{\phi} \frac{\partial}{\partial \nu} (\phi^* \phi). \end{aligned}$$

Then $\phi^* \in C^\infty(\Sigma)$ satisfies $L^* \phi^* = 0$ on Σ with boundary condition $B^* \phi^* = 0$ if, and only if, $\phi^* \phi$ is constant. Thus $\lambda^* = 0$ is an eigenvalue of L^* with Robin boundary condition $B^* \phi^* = 0$ whose associated eigenfunctions are given by $\phi^* = c/\phi$, $c \in \mathbb{R}$. In particular, $\lambda^* = 0$ is simple. \square

The last auxiliary result of this section is a foliation lemma which we are going to use in the proof of [Theorem 1.4](#). It is similar to [Lemma 2.3](#) in [\[12\]](#).

Lemma 3.6. *Under the assumptions of [Proposition 3.3](#), there exist a neighborhood $V \cong (-\epsilon, \epsilon) \times \Sigma$ of $\Sigma \cong \{0\} \times \Sigma$ in M and a positive function $\varphi : V \rightarrow \mathbb{R}$ such that:*

(1) $g|_V$ has the orthogonal decomposition,

$$g|_V = \varphi^2 dt^2 + \gamma_t,$$

where γ_t is the induced metric on $\Sigma_t \cong \{t\} \times \Sigma$.

- (2) Each Σ_t is a free boundary hypersurface in (M, g, K) with constant null mean curvature $\theta^+(t)$ with respect to the outward unit normal $N_t = \varphi^{-1} \frac{\partial}{\partial t}$, where $N_0 = N$.
- (3) $\frac{\partial \varphi}{\partial \nu_t} = \Pi^{\partial M}(N_t, N_t) \varphi$ along $\partial \Sigma_t$, where ν_t is the outward unit normal of $\partial \Sigma_t$ in (Σ_t, γ_t) .

To prove [Lemma 3.6](#), we use ideas presented in [\[1,3\]](#), and [\[11\]](#).

Proof. Let Z be a smooth vector field on M such that $Z_x = N_x$ for $x \in \Sigma$ and $Z_p \in T_p \partial M$ for $p \in \partial M$. Denote by $\Phi = \Phi(p, t)$ the flow of Z . Since Σ is compact, there exists $\delta > 0$ such that $\Phi(x, t)$ is well-defined for $(x, t) \in \Sigma \times (-\delta, \delta)$.

Fix $0 < \alpha < 1$ and define $B_\delta(0) = \{u \in C^{2,\alpha}(\Sigma); \|u\|_{2,\alpha} < \delta\}$. Given $u \in B_\delta(0)$, consider $\Sigma_u = \{\Phi(x, u(x)); x \in \Sigma\}$ and let θ_u^+ denote the null mean curvature of Σ_u with respect to the (suitably chosen) outward unit normal N_u of Σ_u . (Taking a smaller $\delta > 0$ if necessary, we may assume that Σ_u is a properly embedded hypersurface in M for each $u \in B_\delta(0)$.)

Define $\Theta : B_\delta(0) \times \mathbb{R} \rightarrow C^{0,\alpha}(\Sigma) \times C^{1,\alpha}(\partial \Sigma) \times \mathbb{R}$ by

$$\Theta(u, k) = \left(\theta_u^+ - k, -\langle N_u, \varrho_u \rangle, \int_\Sigma u dv \right),$$

where ϱ_u is the restriction of ϱ to $\partial \Sigma_u$. It follows from [\(2\)](#) and [\[3, Proposition 17\]](#) that the linearization of Θ at $(0, 0)$ is given by

$$D\Theta_{(0,0)}(u, k) = \frac{d}{ds} \Big|_{s=0} \Theta(su, sk) = \left(Lu - k, Bu, \int_\Sigma u dv \right), \quad (u, k) \in C^{2,\alpha}(\Sigma) \times \mathbb{R}.$$

Claim 1. $D\Theta_{(0,0)}$ is an isomorphism. First, observe that if $(u, k) \in \ker(D\Theta_{(0,0)})$, then

$$Lu = k \text{ on } \Sigma, \tag{9}$$

$$Bu = 0 \text{ along } \partial\Sigma,$$

$$\int_{\Sigma} u dv = 0.$$

Now, let $\phi^* > 0$ be an eigenfunction of L^* on Σ , with boundary condition $B^*\phi^* = 0$, associated with the eigenvalue $\lambda^* = 0$, which exists by Lemma 3.5. Then, multiplying (9) by ϕ^* and integrating over Σ , we obtain

$$k \int_{\Sigma} \phi^* dv = \int_{\Sigma} \phi^* L u dv = \int_{\Sigma} u L^* \phi^* dv + \int_{\partial\Sigma} u B^* \phi^* ds - \int_{\partial\Sigma} \phi^* B u ds = 0.$$

Thus $k = 0$. In this case, $Lu = 0$ on Σ , $Bu = 0$ along $\partial\Sigma$, and $\int_{\Sigma} u dv = 0$. From Lemma 3.5, $u = c\phi$, $c \in \mathbb{R}$, where $\phi > 0$ is an eigenfunction of L on Σ , with Robin boundary condition $B\phi = 0$, associated with the eigenvalue $\lambda = 0$. Since $\int_{\Sigma} u dv = 0$, we have $u = 0$. This shows that $D\Theta_{(0,0)}$ is injective.

To see that $D\Theta_{(0,0)}$ is onto, observe that, by the Fredholm alternative, the problem

$$\begin{cases} Lu = f \text{ on } \Sigma \\ Bu = h \text{ along } \partial\Sigma \end{cases}$$

has a solution if, and only if,

$$\int_{\Sigma} \phi^* f dv + \int_{\partial\Sigma} \phi^* h ds = 0.$$

Therefore, given $(f, h, c) \in C^{0,\alpha}(\Sigma) \times C^{1,\alpha}(\partial\Sigma) \times \mathbb{R}$, we can take

$$k_0 = - \frac{\int_{\Sigma} \phi^* f dv + \int_{\partial\Sigma} \phi^* h ds}{\int_{\Sigma} \phi^* dv},$$

$u_0 \in C^{2,\alpha}(\Sigma)$ such that $Lu_0 = k_0 + f$ on Σ and $Bu_0 = h$ along $\partial\Sigma$, which exists by the Fredholm alternative, and

$$t_0 = \frac{c - \int_{\Sigma} u_0 dv}{\int_{\Sigma} \phi dv}.$$

Straightforward computations show that

$$D\Theta_{(0,0)}(u_0 + t_0\phi, k_0) = (f, h, c).$$

This finishes the proof of Claim 1.

It follows from the inverse function theorem that, for $s \in \mathbb{R}$ sufficiently small, say $|s| < \varepsilon$, there exist $u(s) \in B_{\delta}(0)$ and $k(s) \in \mathbb{R}$ such that $\Theta(u(s), k(s)) = (0, 0, s)$ with $u(0) = 0$ and $k(0) = 0$. By the chain rule,

$$\left(Lu'(0) - k'(0), Bu'(0), \int_{\Sigma} u'(0) dv \right) = D\Theta_{(0,0)}(u'(0), k'(0)) = \frac{d}{ds} \Big|_{s=0} \Theta(u(s), k(s)) = (0, 0, 1).$$

Therefore, using the same arguments as before, we have $k'(0) = 0$ and $u'(0) = c\phi$, $c \in \mathbb{R}$. Observe that $c > 0$ since $\int_{\Sigma} u'(0) dv = 1$. Then, taking a smaller $\varepsilon > 0$ if necessary, we obtain that $(\Sigma_{u(s)})_{|s| < \varepsilon}$ forms a foliation of a neighborhood of Σ in M by free boundary hypersurfaces with constant null mean curvature.

Finally, we can introduce coordinates (t, x^i) in a neighborhood V of Σ such that, with respect to these coordinates, $V \cong (-\varepsilon, \varepsilon) \times \Sigma$, each slice $\Sigma_t \cong \{t\} \times \Sigma$ is a free boundary hypersurface in M with constant null mean curvature $\theta^+(t)$ with respect to the outward unit normal N_t , and $\Sigma = \Sigma_0$. Furthermore, these coordinates can be chosen in a such way that $\frac{\partial}{\partial t} = \varphi N_t$ on Σ_t for some positive function $\varphi : V \rightarrow \mathbb{R}$. Item (3) follows from the free boundary condition (see [3, Proposition 17]). \square

4. Proof of Theorem 1.4

Let φ , Σ_t , and $\theta^+(t)$, $t \in (-\epsilon, \epsilon)$, be as in Lemma 3.6. From (2),

$$\frac{d\theta^+}{dt} = -\Delta\varphi + 2\langle X, \nabla\varphi \rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2}(\theta^+)^2 + \theta^+\tau \right) \varphi, \tag{10}$$

where $\Delta = \Delta_t$, $X = X_t$, $\nabla = \nabla_t$, $Q = Q_t$, and $\operatorname{div} = \operatorname{div}_t$ are the respective entities associated with Σ_t for each $t \in (-\epsilon, \epsilon)$. Taking a smaller $\epsilon > 0$ if necessary, we may assume that $\tau\varphi \leq c$ on $U \cong [0, \epsilon) \times \Sigma$ for some constant $c > 0$. Observe that $\theta^+(t) \geq 0$ for each $t \in [0, \epsilon)$, since Σ is weakly outermost. Then, from (10), we have

$$L_t\varphi \triangleq -\Delta\varphi + 2\langle X, \nabla\varphi \rangle + (Q - |X|^2 + \operatorname{div} X)\varphi \geq \frac{d\theta^+}{dt} - c\theta^+ = e^{ct} \frac{d}{dt}(e^{-ct}\theta^+) \text{ on } \Sigma_t$$

for each $t \in [0, \epsilon)$.

Claim 1. $\frac{d}{dt}(e^{-ct}\theta^+) \leq 0$ for $t \in [0, \epsilon)$. Suppose, by contradiction, that $\frac{d}{dt}(e^{-ct}\theta^+) > 0$ for some $t \in [0, \epsilon)$. In particular, $L_t\varphi > 0$ on Σ_t . Then, acting as in the proof of Lemma 3.1, we obtain

$$\frac{L_t\varphi}{\varphi} = -|Y|^2 + \operatorname{div} Y + Q, \tag{11}$$

where $Y = X - \nabla \ln \varphi$, and

$$u^2 \frac{L_t\varphi}{\varphi} \leq \operatorname{div}(u^2 Y) + |\nabla u|^2 + Qu^2$$

for any smooth function u on Σ_t .

Let $u = u_t > 0$ be an eigenfunction of $\mathcal{L}_t \triangleq -\Delta + Q$ on Σ_t , with boundary condition $\frac{\partial u_t}{\partial \nu_t} = \left(\Pi^{\partial M}(N_t, N_t) - \langle X, \nu_t \rangle \right) u_t$, associated with the first eigenvalue $\lambda_1(\mathcal{L}_t)$. Then, acting as in the proof of Lemma 3.1 and using that $L_t\varphi > 0$, we have

$$0 < \int_{\Sigma_t} u_t^2 \frac{L_t\varphi}{\varphi} dv_t \leq \int_{\Sigma_t} \left(|\nabla u_t|^2 + Qu_t^2 + \operatorname{div}(u_t^2 Y) \right) dv_t = \lambda_1(\mathcal{L}_t) \int_{\Sigma_t} u_t^2 dv_t.$$

In particular, $\lambda_1(\mathcal{L}_t) > 0$. Therefore

$$\mathcal{L}_t u_t = -\Delta u_t + \left(\frac{1}{2} R^{\Sigma_t} - P_t \right) u_t = \lambda_1(\mathcal{L}_t) u_t > 0 \text{ on } \Sigma_t,$$

where

$$P_t \triangleq \mu + J(N_t) + \frac{1}{2} |\chi_t^+|^2.$$

Using the interior and the boundary DEC, and acting as in the proof of Proposition 3.3, we obtain

$$P_t \geq 0 \text{ and } \frac{\partial u_t}{\partial \nu_t} + H^{\partial \Sigma_t} u_t \geq 0.$$

Then, from Lemma 3.2, (Σ_t, γ_t) is Ricci flat, $P_t = 0$, and u_t is constant. In particular, $\mathcal{L}_t u_t = 0$, which is a contradiction because $\mathcal{L}_t u_t = \lambda_1(\mathcal{L}_t) u_t > 0$. Here we have used that $\Sigma_t \cong \{t\} \times \Sigma$ does not admit a metric of positive scalar curvature with minimal boundary. This finishes the proof of Claim 1.

Claim 1 gives that $e^{-ct}\theta^+(t) \leq \theta^+(0) = 0$ for $t \in [0, \epsilon)$. Then $\theta^+(t) = 0$ for every $t \in [0, \epsilon)$, since Σ is weakly outermost. Using this information into (10), we obtain $L_t\varphi = 0$ for each $t \in [0, \epsilon)$. In particular, each such Σ_t is a stable MOTS. Theorem 1.4 then follows from Proposition 3.3 applied to Σ_t for each $t \in [0, \epsilon)$.

5. A splitting result

The class of initial data sets (M, g, K) covered by [Theorem 1.4](#) is expected to be fairly wide (see Example 4.2 in [\[9\]](#)). Therefore, if we want to prove a stronger rigidity result, it is natural to assume some extra conditions. In what follows, we show (among other things) that, under natural *convexity* and *volume-minimizing* conditions, (M, g) splits in an outer neighborhood of Σ .

We say that K is *n-convex* with respect to g if, at every point of M , the sum of the smallest n eigenvalues of K with respect to g is nonnegative (see [\[9,15\]](#)). In particular, if K is *n-convex*, then $\text{tr}_S K \geq 0$ for every hypersurface $S \subset M$. We say that Σ is *outer volume-minimizing* if $\text{vol}(\Sigma) \leq \text{vol}(S)$ for every hypersurface S outside of, and homologous to, Σ .

Theorem 5.1. *Under the assumptions of [Theorem 1.4](#), if K is n-convex and Σ is outer volume-minimizing, then there exists an outer neighborhood U of Σ in M such that the following conditions hold:*

- (1) (Σ, γ_0) is Ricci flat with totally geodesic boundary, where γ_0 is the metric on Σ induced from (M, g) .
- (2) $(U, g|_U)$ is isometric to $([0, \epsilon) \times \Sigma, dt^2 + \gamma_0)$.
- (3) $K = a dt^2$ on U , where a depends only on $t \in [0, \epsilon)$.
- (4) $\mu = 0$ and $J = 0$ on U .
- (5) $\Pi^{\partial M} = 0$ and $(\iota_\rho \pi)^\top = 0$ on $U \cap \partial M$.

Proof. Let φ and $\Sigma_t, t \in [0, \epsilon)$, be as in the proof of [Theorem 1.4](#). Remember that $L_t \varphi = 0, \chi_t^+ = 0, (\Sigma_t, \gamma_t)$ is Ricci flat with totally geodesic boundary, and $\mu = |J| = -J(N_t)$ on Σ_t (see the proof of [Proposition 3.3](#)). In particular,

$$Q = \frac{1}{2} R^{\Sigma_t} - (\mu + J(N_t)) - \frac{1}{2} |\chi_t^+|^2 = 0.$$

From Eq. [\(11\)](#), we have

$$\begin{aligned} \int_{\Sigma_t} |X - \nabla \ln \varphi|^2 dv_t &= \int_{\Sigma_t} \text{div}(X - \nabla \ln \varphi) dv_t \\ &= \int_{\partial \Sigma_t} \left(\langle X, \nu_t \rangle - \frac{1}{\varphi} \frac{\partial \varphi}{\partial \nu_t} \right) ds_t \\ &= \int_{\partial \Sigma_t} \left(\langle X, \nu_t \rangle - \Pi^{\partial M}(N_t, N_t) \right) ds_t, \end{aligned}$$

where above we have used item [\(3\)](#) of [Lemma 3.6](#). On the other hand, it follows from the proof of [Proposition 3.3](#) that

$$\Pi^{\partial M}(N_t, N_t) = H^{\partial M} = |(\iota_\rho \pi)^\top| = (\iota_\rho \pi)^\top(N_t) = \langle X, \nu_t \rangle \text{ along } \partial \Sigma_t.$$

Therefore

$$\int_{\Sigma_t} |X - \nabla \ln \varphi|^2 dv_t = 0,$$

i.e.

$$X = \nabla \ln \varphi \text{ on } \Sigma_t.$$

Now, observing that $\text{tr}_{\Sigma_t} K \geq 0$, since K is *n-convex*, we obtain

$$H(t) \leq \text{tr}_{\Sigma_t} K + H(t) = \theta^+(t) = 0,$$

where $H(t) = \text{div}_{\Sigma_t} N_t$ is the mean curvature of Σ_t in (M, g) . Then the first variation formula of volume (see [\[3, Proposition 14\]](#)) gives

$$\frac{d}{dt} \text{vol}(\Sigma_t) = \int_{\Sigma_t} \varphi H(t) dv_t + \int_{\partial \Sigma_t} \varphi \langle N_t, \nu_t \rangle ds_t = \int_{\Sigma_t} \varphi H(t) dv_t \leq 0.$$

Therefore $t \mapsto \text{vol}(\Sigma_t)$, $t \in [0, \epsilon)$, is a nonincreasing function. In particular, $\text{vol}(\Sigma_t) \leq \text{vol}(\Sigma_0)$ for each $t \in [0, \epsilon)$. Thus, because we are assuming that $\Sigma = \Sigma_0$ is outer volume-minimizing, it follows that $\text{vol}(\Sigma_t) = \text{vol}(\Sigma_0)$ and, in particular, $H(t) = \text{tr}_{\Sigma_t} K = 0$, for each $t \in [0, \epsilon)$.

Remember that

$$g|_U = \varphi^2 dt^2 + \gamma_t,$$

where $U \cong [0, \epsilon) \times \Sigma$. We claim that each leaf $\Sigma_t \cong \{t\} \times \Sigma$ is totally geodesic in (M, g) and φ is constant on Σ_t , i.e. $\varphi = \varphi(t)$ depends only on $t \in [0, \epsilon)$. Provided that is the case, it is not difficult to see that γ_t does not depend on t and, after the simple change of variable $ds = \varphi(t)dt$, the metric g has the product structure $ds^2 + \gamma_0$ on U , where (Σ, γ_0) is Ricci flat with totally geodesic boundary, which guarantees items (1) and (2). In order to prove the claim, we consider the first variation of the inward null mean curvature $\theta^-(t)$ of Σ_t in (M, g, K) .

The first variation of $\theta^-(t) = \text{tr}_{\Sigma_t} K - H(t) = 0$ is given by

$$-\Delta\varphi^- + 2\langle X^-, \nabla\varphi^- \rangle + (Q^- - |X^-|^2 + \text{div} X^-)\varphi^- = \frac{d\theta^-}{dt} = 0, \tag{12}$$

where $\varphi^- = -\varphi$,

$$Q^- = \frac{1}{2}R^{\Sigma_t} - (\mu + J(-N_t)) - \frac{1}{2}|\chi_t^-|^2 = -2|J| - \frac{1}{2}|\chi_t^-|^2, \tag{13}$$

and X^- is the tangent vector field of Σ_t that is dual to the 1-form $K(-N_t, \cdot)|_{\Sigma_t} = -K(N_t, \cdot)|_{\Sigma_t}$, i.e.

$$X^- = -X = -\nabla \ln \varphi \text{ on } \Sigma_t. \tag{14}$$

Substituting (13) and (14) into (12), with $\varphi^- = -\varphi$, we get

$$\Delta\varphi + \frac{|\nabla\varphi|^2}{\varphi} + \left(|J| + \frac{1}{4}|\chi_t^-|^2\right)\varphi = 0.$$

Therefore, integrating over Σ_t and using that

$$\frac{\partial\varphi}{\partial\nu_t} = \text{II}^{\partial M}(N_t, N_t)\varphi = |(\iota_{\varrho}\pi)^\top|\varphi \geq 0,$$

we obtain

$$0 \leq \int_{\partial\Sigma_t} \frac{\partial\varphi}{\partial\nu_t} ds_t = \int_{\Sigma_t} \Delta\varphi dv_t = - \int_{\Sigma_t} \left(\frac{|\nabla\varphi|^2}{\varphi} + \left(|J| + \frac{1}{4}|\chi_t^-|^2\right)\varphi \right) dv_t \leq 0,$$

that is,

$$|\nabla\varphi| = |J| = |\chi_t^-| = 0 \text{ on } \Sigma_t. \tag{15}$$

In particular, φ is constant on Σ_t . On the other hand, $\chi_t^+ = \chi_t^- = 0$ gives that $K|_{\Sigma_t} = 0$ and Σ_t is totally geodesic in (M, g) .

Observe that $K(N_t, \cdot)|_{\Sigma_t} = 0$, since $X = \nabla \ln \varphi = 0$. Therefore, because $K|_{\Sigma_t} = 0$ for each $t \in [0, \epsilon)$, we have $K = a dt^2$ on $U \cong [0, \epsilon) \times \Sigma$. Then we can use that $d \text{tr} K = \text{div} K$ (since $J = 0$) to see that a is constant on each leaf Σ_t . This proves item (3).

Item (4) follows directly from Eq. (15) and the fact that $\mu = |J|$ on each leaf Σ_t .

Finally, since $|(\iota_{\varrho}\pi)^\top| = \text{II}^{\partial M}(N_t, N_t) = \langle X, \nu_t \rangle = 0$ along $\partial\Sigma_t$, $\partial\Sigma_t$ is totally geodesic in (Σ_t, γ_t) , and Σ_t is free boundary in (M, g) , we obtain that $(\iota_{\varrho}\pi)^\top = 0$ and $\text{II}^{\partial M} = 0$ along $\partial\Sigma_t$ for each $t \in [0, \epsilon)$. This finishes the proof of [Theorem 5.1](#) \square

A similar splitting result to [Theorem 5.1](#) was established by M. Eichmair, G. J. Galloway, and the author [9] for closed MOTS Σ in $n = 2, \dots, 6$ dimensions. It is important to mention that they *did not* suppose Σ is weakly outermost. Instead, under a suitable barrier condition, they *proved* that Σ is weakly

outermost and, in particular, stable. (Example 5.3 in [9] shows that the convexity and the volume-minimizing conditions are in fact necessary for the splitting result.)

Alaee, Lesourd, and Yau [1] established an analogous result to [Theorem 5.1](#) in $n = 2$ dimensions, provided $\mu - |J|$ is bounded from below by a positive constant. As far as we know, they were the pioneers in considering this kind of problem for free boundary MOTS.

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