RIGIDITY OF MARGINALLY OUTER TRAPPED (HYPER)SURFACES WITH NEGATIVE σ-CONSTANT

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ABSTRACT. In this paper we generalize a result of Galloway and Mendes in two different situations: in the first case for marginally outer trapped surfaces (MOTSs) of genus greater than 1 and, in the second case, for MOTSs of high dimension with negative σ-constant. In both cases we obtain a splitting result for the ambient manifold when it contains a stable closed MOTS which saturates a lower bound for the area (in dimension 2) or for the volume (in dimension ≥ 3). These results are extensions of theorems of Nunes and Moraru to general (non-time-symmetric) initial data sets.

1. INTRODUCTION

Let $M^3 = (M^3, g)$ be a Riemannian 3-manifold. It is a very interesting question to know how the topology of a minimal surface $\Sigma^2 \subset M^3$ can influence the geometry of M^3, and vice versa.

Using the Gauss–Bonnet theorem, the Gauss equation, and the stability inequality for minimal surfaces, Schoen and Yau observed that if M^3 is oriented and has positive scalar curvature, then M^3 does not admit orientable closed stable minimal surfaces of positive genus. Furthermore, they proved the following result (see [24]).

Theorem 1.1 (Schoen and Yau). *Let $M^3 = (M^3, g)$ be an oriented closed Riemannian 3-manifold with nonnegative scalar curvature. If the fundamental group of M^3 admits a subgroup abstractly isomorphic to the fundamental group of the 2-torus, then M^3 is flat.*

In order to prove this result, Schoen and Yau first proved that the hypothesis on the fundamental group of M^3 ensures the existence of a stable minimal 2-torus. After, they used the condition that if M admits a metric of nonnegative scalar curvature which is not flat, then M also admits a metric of positive scalar curvature (see [17]). Then the result follows.

As observed by Fischer-Colbrie and Schoen [10], if Σ^2 is an orientable closed stable minimal surface of genus $g(\Sigma) \geq 1$ in an oriented Riemannian 3-manifold M^3 with scalar curvature $R \geq 0$, then Σ^2 is a totally geodesic flat 2-torus in M^3, and $R = 0$ on Σ^2. In the same work (see [10, Remark 4]) Fischer-Colbrie and Schoen posed the problem of establishing a stronger (more global) rigidity statement if, say, the 2-torus is suitably area minimizing.

Partially motivated by some issues concerning the topology of back holes, Cai and Galloway [8] solved the problem posed by Fischer-Colbrie and Schoen, in a paper which has inspired a great deal of subsequent works in the subject (e.g., [1,6,19,21]). They proved the following.

Theorem 1.2 (Cai and Galloway). *Let M^3 be a Riemannian 3-manifold with nonnegative scalar curvature. If $\Sigma^2 \subset M^3$ is a two-sided embedded 2-torus which is locally area minimizing, then a neighborhood of Σ in M is isometric to the product $((-\varepsilon, \varepsilon) \times \Sigma, dt^2 + g_0)$, where g_0, the metric on Σ induced from M, is flat. Moreover, if M is complete and Σ has the least area in its isotopy class, M is globally flat.*

As observed by Cai and Galloway, in the first part of the theorem above, M need not be globally flat. Also, the second part remains true if M is a manifold with boundary $\partial M \neq \emptyset$, assuming that its boundary is mean convex. A higher-dimensional version of Theorem 1.2 was obtained in [7]; see also [12] for a simplified proof.

Two results similar to Theorem 1.2 were obtained by Bray, Brendle, and Neves [6] and by Nunes [21] under different hypotheses on the scalar curvature of M^3 and the topology of Σ^2, which we paraphrase as follows.

Theorem 1.3 (Bray, Brendle, and Neves). *Let M^3 be a Riemannian 3-manifold with scalar curvature bounded from below by $2c$ for some constant $c > 0$. If Σ^2 is an embedded 2-sphere which is locally area minimizing, then the area of Σ satisfies*

$$A(\Sigma) \leq \frac{4\pi}{c}.$$

Furthermore, if equality holds, a neighborhood of Σ in M is isometric to the product $((-\varepsilon, \varepsilon) \times \Sigma, dt^2 + g_0)$, where (Σ^2, g_0) is the round 2-sphere of Gaussian curvature $\kappa = c$. In this case, if further M is complete and Σ has the least area in its isotopy class, the universal cover of M is isometric to the product $(\mathbb{R} \times \Sigma, dt^2 + g_0)$.

Theorem 1.4 (Nunes). *Let M^3 be a Riemannian 3-manifold with scalar curvature bounded from below by $-2c$ for some constant $c > 0$. If $\Sigma^2 \subset M^3$ is a two-sided embedded closed Riemann surface of genus $g(\Sigma) \geq 2$ which is locally area minimizing, then the area of Σ satisfies*

$$A(\Sigma) \geq \frac{4\pi(g(\Sigma) - 1)}{c}.$$

Furthermore, if equality holds, a neighborhood of Σ in M is isometric to the product $((-\varepsilon, \varepsilon) \times \Sigma, dt^2 + g_0)$, where (Σ^2, g_0) has constant Gaussian curvature $\kappa = -c$. In this case, if further M is complete and Σ has the least area in its isotopy class, the universal cover of M is isometric to the product $(\mathbb{R} \times \Sigma, dt^2 + g_0)$.

The original proofs of Theorems 1.2, 1.3, and 1.4 are very different. However, Micallef and Moraru [19] presented a unified proof for them.

From the point of view of relativity, Theorems 1.2, 1.3, and 1.4 may be viewed as statements about time-symmetric (totally geodesic) initial data sets. In [13] Galloway and the author generalized Theorem 1.3 to general (non-time-symmetric) initial data sets. In that more general situation, minimal surfaces were replaced by *marginally outer trapped surfaces* (MOTSs). We proved the following (see Section 2 for definitions).

Theorem 1.5 (Galloway and Mendes). *Let $M^3 = (M^3, g, K)$ be a three-dimensional initial data set in a spacetime $\bar{M}^4 = (\bar{M}^4, \bar{g})$. Let Σ^2 be a spherical MOTS in M^3 which is weakly outermost and outer area minimizing. Suppose that $\mu - |J| \geq c$ on M_+ for some constant $c > 0$. Then the area of Σ satisfies*

$$A(\Sigma) \leq \frac{4\pi}{c}.$$

Furthermore, if equality holds, we have the following:

(1) *An outer neighborhood $U \approx [0, \varepsilon) \times \Sigma$ of Σ in M is isometric to*

$$([0, \varepsilon) \times \Sigma, dt^2 + g_0),$$

where (Σ^2, g_0) is the round 2-sphere of Gaussian curvature $\kappa = c$.
(2) *Each slice $\Sigma_t \approx \{t\} \times \Sigma$ is totally geodesic as a submanifold of spacetime. Equivalently, $\chi_+(t) = \chi_-(t) = 0$, where $\chi_\pm(t)$ are the null second fundamental forms of Σ_t in $\bar{M}^4$.*
(3) *$K(\cdot, \cdot)|_{T_x \Sigma_t} = 0$ and $K(\nu_t, \cdot)|_{T_x \Sigma_t} = 0$ for each $x \in \Sigma_t$, where ν_t is the outer unit normal to Σ_t, and $J = 0$ on U.*

Above, μ and J are defined in terms of the Einstein tensor of $\bar{M}^4$, $G = \mathrm{Ric} - \frac{\bar{R}}{2}\bar{g}$ by $\mu = G(u, u)$ and $J(\cdot) = G(u, \cdot)|_{T_p M}$, $p \in M$, where u is the future directed timelike unit normal to M.

An important generalization of Theorem 1.4 is due to Moraru [20], who extended Nunes's result to closed hypersurfaces Σ^n of dimension $n \geq 3$. He proved the following.

Theorem 1.6 (Moraru). *Let M^{n+1} be a Riemannian manifold of dimension $n+1$, $n \geq 3$, with scalar curvature bounded from below by $-2c$ for some constant $c > 0$. If $\Sigma^n \subset M^{n+1}$ is a two-sided embedded closed hypersurface with $\sigma(\Sigma) < 0$ which is locally volume minimizing, then the volume of Σ satisfies*

$$\mathrm{Vol}(\Sigma^n) \geq \left(\frac{|\sigma(\Sigma^n)|}{2c} \right)^{\frac{n}{2}}.$$

Furthermore, if equality holds, a neighborhood of Σ in M is isometric to the product $((-\varepsilon, \varepsilon) \times \Sigma, dt^2 + g_\Sigma)$, where g_Σ, the metric on Σ induced from M, is Einstein with constant scalar curvature $R_\Sigma = -2c$.

Above, $\sigma(\Sigma)$ is the topological invariant introduced by Schoen [23], called the σ-constant of Σ (see also Kobayashi [18]).

In the same spirit of Theorem 1.5, in the present paper we generalize Theorems 1.4 and 1.6 to non-time-symmetric initial data sets.

Our first result is the following.

Theorem 1.7 (Theorem 3.3). *Let $M^3 = (M^3, g, K)$ be a three-dimensional initial data set. Let Σ^2 be an orientable weakly outermost closed MOTS in M^3 of genus $g(\Sigma) \geq 2$. Suppose that $\mu - |J| \geq -c$ for some constant $c > 0$ and that K is 2-convex, both on M_+. Then the area of Σ satisfies*

$$(1.1) \qquad\qquad A(\Sigma) \geq \frac{4\pi(g(\Sigma) - 1)}{c}.$$

Furthermore, if equality holds, we have the following:

(1) *An outer neighborhood $U \approx [0, \varepsilon) \times \Sigma$ of Σ in M is isometric to*

$$([0, \varepsilon) \times \Sigma, dt^2 + g_0),$$

where (Σ, g_0) has constant Gaussian curvature $\kappa = -c$.

(2) *Each slice $\Sigma_t \approx \{t\} \times \Sigma$ is totally geodesic as a submanifold of spacetime. Equivalently, $\chi_+(t) = \chi_-(t) = 0$, where $\chi_\pm(t)$ are the null second fundamental forms of Σ_t.*

(3) *$K(\cdot, \cdot)|_{T_x \Sigma_t} = 0$ and $K(\nu_t, \cdot)|_{T_x \Sigma_t} = 0$ for each $x \in \Sigma_t$, where ν_t is the outer unit normal to Σ_t, and $J = 0$ on U.*

We point out that the area estimate (1.1) was proved by Gibbons [16] in the time-symmetric case, and by Woolgar [26] in the general case.

Our second result is the following.

Theorem 1.8 (Theorem 4.4). *Let $M^{n+1} = (M^{n+1}, g, K)$ be a $(n+1)$-dimensional initial data set, $n \geq 3$. Let Σ^n be a weakly outermost closed MOTS in M^{n+1} with σ-constant $\sigma(\Sigma) < 0$. Suppose that $\mu - |J| \geq -c$ for some constant $c > 0$, and that K is n-convex, both on M_+. Then, the volume of Σ satisfies*

$$(1.2) \qquad \qquad \mathrm{Vol}(\Sigma^n) \geq \left(\frac{|\sigma(\Sigma^n)|}{2c} \right)^{\frac{n}{2}}.$$

Furthermore, if equality holds, we have the following:

(1) *An outer neighborhood $U \approx [0, \varepsilon) \times \Sigma$ of Σ in M is isometric to*

$$([0, \varepsilon) \times \Sigma, dt^2 + g_\Sigma),$$

and (Σ, g_Σ) is Einstein with constant scalar curvature $R_\Sigma = -2c$.

(2) *Each slice $\Sigma_t \approx \{t\} \times \Sigma$ is totally geodesic as a submanifold of spacetime. Equivalently, $\chi_+(t) = \chi_-(t) = 0$, where $\chi_\pm(t)$ are the null second fundamental forms of Σ_t.*

(3) *$K(\cdot, \cdot)|_{T_x \Sigma_t} = 0$ and $K(\nu_t, \cdot)|_{T_x \Sigma_t} = 0$ for each $x \in \Sigma_t$, where ν_t is the outer unit normal to Σ_t, and $J = 0$ on U.*

The volume estimate (1.2) was obtained by Cai and Galloway [9] in the time-symmetric case, and by Galloway and Murchadha [14] in the general case.

Although we are assuming a convexity hypothesis on the initial data set M in Theorems 1.7 and 1.8, we are not assuming any minimizing hypothesis on the area (in dimension 2) or the volume (in dimension ≥ 3) of Σ.

Since in time-symmetric initial data sets $\mu - |J|$ reduces to half of the scalar curvature, the hypothesis $\mu - |J| \geq -c$ in general initial data sets is the analogue of $R \geq -2c$ in time-symmetric ones.

2. Preliminaries

Let $\bar{M}^{n+2} = (\bar{M}^{n+2}, \bar{g}, \bar{\nabla})$ be a spacetime of dimension $n + 2$, i.e., a connected, oriented and time-oriented Lorentzian $(n + 2)$-manifold, where $\bar{g}$ and $\bar{\nabla}$ denote a Lorentzian metric and its Levi-Civita connection, respectively. Let M^{n+1} be a spacelike hypersurface embedded in $\bar{M}^{n+2}$, and let u denote the future directed timelike unit normal to M. In this case, we can see M^{n+1} as an initial data set

$M^{n+1} = (M^{n+1}, g, K)$, where g is the Riemannian metric on M induced from $\bar{M}$, and K is the second fundamental form of M in $\bar{M}$. More precisely,

$$(2.1) \qquad K(V, W) = \bar{g}(\bar{\nabla}_V u, W) \qquad \text{for all } V, W \in T_p M \text{ and } p \in M.$$

Now let Σ^n be a two-sided connected closed (compact with no boundary) hypersurface embedded in M^{n+1}. Fix a unit normal to Σ in M, say, ν, that is, $\nu(x) \in T_x \Sigma^{\perp} \cap T_x M$ for all $x \in \Sigma$. By convention we say that ν is outward pointing and, in turn, $-\nu$ is inward pointing. Then we can define two future directed null normal vector fields along Σ, $l_+ = u + \nu$ and $l_- = u - \nu$, which are outward and inward pointing, respectively. The *null second fundamental forms* of Σ^n in $\bar{M}^{n+2}$, χ_+ and χ_-, are defined by

$$\chi_{\pm}(Y, Z) = K(Y, Z) \pm A(Y, Z) \qquad \text{for all } Y, Z \in T_x \Sigma \text{ and } x \in \Sigma,$$

where A is the second fundamental form of Σ in M. On the other hand, the *null mean curvatures* of Σ in $\bar{M}$, θ_+ and θ_-, are defined by taking the traces of χ_+ and χ_- with respect to Σ,

$$\theta_{\pm} = \text{tr}_{\Sigma} \chi_{\pm}.$$

Observe that

$$\theta_{\pm} = \text{div}_{\Sigma}(l_{\pm}) = \text{tr}_{\Sigma} K \pm H,$$

where $H = \text{tr} A$ is the mean curvature of Σ in M.

As introduced by Penrose, Σ is said to be a *trapped surface* if both θ_+ and θ_- are negative everywhere. Also, Σ is an *outer trapped surface* if θ_+ is negative (with no assumption on θ_-). Finally, we say that Σ is a *MOTS* if θ_+ vanishes identically (with no assumption on θ_-). For simplicity, we drop the plus sign of l_+, χ_+, and θ_+.

Remark 2.1. Observe that in the time-symmetric case, i.e., when $K \equiv 0$, χ and θ reduce to A and H, respectively. Then, in this case, a MOTS is just a minimal hypersurface.

Before presenting the notion of stability for MOTSs introduced by Andersson, Mars, and Simon [2,3], we are going to fix some notation:

- Let G be the Einstein tensor of $\bar{M} = (\bar{M}, \bar{g}, \bar{\nabla})$ given by

$$G = \bar{\text{Ric}} - \frac{\bar{R}}{2} \bar{g},$$

  where $\bar{\text{Ric}}$ and $\bar{R}$ are the Ricci tensor and the scalar curvature of $\bar{M}$, respectively.
- Denote $G(u, u)$ by μ on M and $G(u, \cdot)$ by $J(\cdot)$ on $T_p M$ for each $p \in M$.
- Define

$$Q = \frac{1}{2} R_{\Sigma} - (\mu + J(\nu)) - \frac{1}{2} |\chi|^2,$$

  where R_{Σ} is the scalar curvature of Σ with respect to the Riemannian metric $\langle\,,\,\rangle$ induced from $M = (M, g)$.
- Denote by $X \in \Gamma(T\Sigma)$ the vector field on Σ dual to $K(\nu(x), \cdot)|_{T_x \Sigma}$ for each $x \in \Sigma$.
- Denote by τ the mean curvature of M in $\bar{M}$, i.e., $\tau = \text{tr} K$.

Now consider $t \longmapsto \Sigma_t \subset M$, $t \in (-\varepsilon, \varepsilon)$, a variation of $\Sigma_0 = \Sigma$ in M with variation vector field $\frac{\partial}{\partial t}|_{t=0} = \phi\nu$, $\phi \in C^\infty(\Sigma)$. Denote by $\theta(t)$ the null mean curvature of Σ_t in $\bar{M}$ with respect to $l_t = u + \nu_t$, that is, $\theta(t) = \mathrm{div}_{\Sigma_t}(l_t)$, where ν_t is the unit normal to Σ_t in M such that $\nu_0 = \nu$. A similar calculation to the one in [3] gives

$$(2.2) \qquad \left.\frac{\partial\theta}{\partial t}\right|_{t=0} = L\phi + \left(\theta(0)\tau - \frac{1}{2}\theta(0)^2\right)\phi,$$

where $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the operator given by

$$(2.3) \qquad L\phi = -\Delta\phi + 2\langle X, \nabla\phi\rangle + (Q - |X|^2 + \mathrm{div}\, X)\phi.$$

Here, Δ and ∇ denote the Laplacian and the gradient operators on $\Sigma = (\Sigma, \langle\,,\,\rangle)$, respectively. The operator L is known as the MOTS stability operator.

Remark 2.2. Using the Gauss–Codazzi equations, we can express μ and J solely in terms of the initial data set $M = (M, g, K)$:

$$\mu = \frac{1}{2}(R + \tau^2 - |K|^2),$$
$$J = \mathrm{div}\, K - d\tau,$$

where R is the scalar curvature of M. Then, using the Gauss equation,

$$\mathrm{Ric}(\nu, \nu) + |A|^2 = \frac{1}{2}(R - R_\Sigma + |A|^2 + H^2),$$

where Ric is the Ricci tensor of M, we can see that in the time-symmetric case the operator L reduces to the classical stability operator for minimal hypersurfaces,

$$-\Delta - (\mathrm{Ric}(\nu, \nu) + |A|^2),$$

when Σ is a MOTS (in this case, a minimal hypersurface).

The operator L is not self-adjoint in general. But it has the following properties (see [3] and the references therein).

Lemma 2.3. *The following holds for the operator L.*
  (1) *There is a real eigenvalue $\lambda_1 = \lambda_1(L)$, called the* principal eigenvalue *of L, such that for any other eigenvalue μ, $\mathrm{Re}(\mu) \geq \lambda_1$. The associated eigenfunction ϕ, $L\phi = \lambda_1\phi$, is unique up to a multiplicative constant and can be chosen to be strictly positive.*
  (2) *$\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$) if and only if there exists a $\psi \in C^\infty(\Sigma)$, $\psi > 0$, such that $L\psi \geq 0$ (resp., $L\psi > 0$).*

The lemma above is true for any operator of the form $\phi \longmapsto -\Delta\phi + \langle Y, \nabla\phi\rangle + q\phi$, where $Y \in \Gamma(T\Sigma)$ and $q \in C^\infty(\Sigma)$.

A closed MOTS Σ is said to be *stable* if $\lambda_1(L) \geq 0$. It follows from the lemma above that Σ is stable if and only if there exists a $\psi \in C^\infty(\Sigma)$, $\psi > 0$, such that $L\psi \geq 0$. Observe that in the time-symmetric case, Σ is stable as a MOTS if and only if Σ is stable as a minimal hypersurface since, in this case, $\lambda_1(L)$ is the first eigenvalue of the Jacobi operator L.

Now, consider the "symmetrized" operator $L_0 : C^\infty(\Sigma) \to C^\infty(\Sigma)$,

$$L_0\phi = -\Delta\phi + Q\phi,$$

obtained formally from (2.3) by taking $X = 0$. As first observed by Galloway [11], using the Rayleigh's formula for the first eigenvalue of L_0 (which coincides with the principal eigenvalue of L_0),

$$(2.4) \qquad \lambda_1(L_0) = \inf_{f \in C^\infty(\Sigma) \setminus \{0\}} \frac{\int_\Sigma (|\nabla f|^2 + Qf^2) dA}{\int_\Sigma f^2 dA},$$

and the main argument presented in [15], we have the following result.

Lemma 2.4. $\lambda_1(L_0) \geq \lambda_1(L)$. *In particular, if* $\lambda_1(L) \geq 0$,

$$(2.5) \qquad \int_\Sigma (|\nabla f|^2 + Qf^2) dA \geq 0$$

for all $f \in C^\infty(\Sigma)$.

The next result is a very useful consequence of the proof of the main theorem proved in [11] (see also [13]).

Lemma 2.5. *Let* Σ^n *be a closed MOTS in an initial data set* $M^{n+1} = (M^{n+1}, g, K)$. *If* $\lambda_1(L) = 0$, *then there exists a variation* $t \longmapsto \Sigma_t$ *of* $\Sigma_0 = \Sigma$ *in* M, $t \in (-\varepsilon, \varepsilon)$, *such that* Σ_t *is a hypersurface with constant null mean curvature* $\theta = \theta(t)$ *for each* $t \in (-\varepsilon, \varepsilon)$. *Furthermore,* $\{\Sigma_t\}_{t \in (-\varepsilon, \varepsilon)}$ *is a foliation of a neighborhood of* Σ *in* M *with variation vector field* $\frac{\partial}{\partial t} = \phi_t \nu_t$, *where* ν_t *is the outer unit normal to* Σ_t, *and* $\phi_t : \Sigma_t \approx \{t\} \times \Sigma \to \mathbb{R}$ *is a positive function.*

Before passing to the next section, let us fix some notation and terminologies.

If Σ^n is a separating MOTS in $M^{n+1} = (M^{n+1}, g, K)$, let M_+ be the region consisting of Σ and the region outside of Σ. We say that Σ is *outermost* if there are no outer trapped or MOTSs in M_+ homologous to Σ. We say that Σ is *weakly outermost* if there are no outer trapped surfaces in M_+ homologous to Σ. Also, Σ is said to be *outer area minimizing* if its area is less than or equal to the area of any surface in M_+ homologous to Σ.

Remark 2.6. It follows from (2.2) that if Σ is a weakly outermost MOTS, then it is stable. Otherwise, taking a variation $t \longmapsto \Sigma_t$ such that $\frac{\partial}{\partial t}|_{t=0} = \phi \nu$, where $\phi > 0$ is the eigenfunction associated with $\lambda_1 = \lambda_1(L)$, we have Σ_t being an outer trapped surface for $t > 0$ small enough, which is homologous to Σ in M_+.

3. THE CASE OF HIGH GENUS

In this section we generalize the main result obtained in [13] to surfaces Σ of genus $g(\Sigma) \geq 2$.

Inequality (3.1) below is a well-known area estimate (see [16] for the time-symmetric case and [26] for the general case). But, for the sake of completeness, we will present its proof and some of the consequences when equality occurs.

Proposition 3.1. *Let* Σ^2 *be an orientable stable closed MOTS of genus* $g(\Sigma) \geq 2$ *in a three-dimensional initial data set* $M^3 = (M^3, g, K)$. *Suppose that* $\mu + J(\nu) \geq -c$ *on* Σ *for some constant* $c > 0$. *Then the area of* Σ *satisfies*

$$(3.1) \qquad A(\Sigma) \geq \frac{4\pi(g(\Sigma) - 1)}{c}.$$

Furthermore, if equality holds, Σ *has constant Gaussian curvature* $\kappa = -c$, *the null second fundamental form* $\chi = \chi_+$ *of* Σ *vanishes identically,* $\mu + J(\nu) = -c$ *on* Σ, *and* $\lambda_1(L_0) = \lambda_1(L) = 0$.

Proof. By hypothesis, $\lambda_1(L) \geq 0$, where L is the MOTS stability operator. Then, using the Gauss–Bonnet theorem and inequality (2.5) for $f = 1$, we have

$$
\begin{aligned}
0 \;\leq\; \int_\Sigma Q dA \;&=\; \int_\Sigma \left(\kappa - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2 \right) dA \\
&\leq\; \int_\Sigma \left(\kappa + c - \frac{1}{2}|\chi|^2 \right) dA \;\leq\; \int_\Sigma (\kappa + c) dA \\
&=\; 4\pi(1 - g(\Sigma)) + cA(\Sigma),
\end{aligned}
$$

which proves (3.1).

Now, if equality in (3.1) holds, all inequalities above must be equalities. Then $\chi = 0$, $\mu + J(\nu) = -c$ on Σ, and $\int_\Sigma Q dA = 0$. Observe that $Q = \kappa + c$. Using (2.5) again, for all $\alpha \in \mathbb{R}$ and $\psi \in C^\infty(\Sigma)$ we have

$$
\begin{aligned}
0 \;\leq\; & \int_\Sigma (|\nabla(\alpha + \psi)|^2 + Q(\alpha + \psi)^2) dA \\
=\; & \int_\Sigma (|\nabla\psi|^2 + Q\psi^2) dA + 2\alpha \int_\Sigma Q\psi dA.
\end{aligned}
$$

This implies that $\int_\Sigma Q\psi dA = 0$ for all $\psi \in C^\infty(\Sigma)$; hence $Q = \kappa + c = 0$. Finally, $\lambda_1(L_0) = \lambda_1(L) = 0$ follows from Lemma 2.4 and Rayleigh's formula (2.4), remembering that $\lambda_1(L) \geq 0$. $\qquad\square$

Before passing to the main result of this section, we are going to prove a calculus lemma and to present the definition of n-convexity. The proof of the following lemma is based on the techniques presented in [19].

Lemma 3.2. *Let* $f \in C^1([0,\varepsilon))$ *and* $\eta, \xi, \rho \in C^0([0,\varepsilon))$ *be functions such that* $\max\{f, \rho\} \geq 0$, $\xi \geq 0$, $\eta > 0$, $f(0) = 0$, *and*

$$
f'(t)\eta(t) \leq \int_0^t f(s)\xi(s)ds + f(t)\rho(t) \qquad \forall t \in [0,\varepsilon).
$$

Then $f \leq 0$ *everywhere. In particular, if* $f \geq 0$, *it must be identically* 0.

Proof. Define

$$
I = \{\delta \in (0,\varepsilon); f \leq 0 \text{ in } [0,\delta]\}.
$$

Claim 1. $I \neq \emptyset$.

Proof. By continuity, there exists a constant $C > 0$ satisfying

$$
\max_{t \in [0,\varepsilon/2]} \left\{ \frac{1}{\eta(t)} \int_0^t \xi(s)ds, \frac{\rho(t)}{\eta(t)} \right\} \leq C.
$$

Choose $\delta \in (0,\varepsilon/2]$ such that $0 < 1 - 2C\delta$. We claim that $\delta \in I$. Otherwise, fix $t_0 \in (0,\delta]$ with $f(t_0) > 0$, and define

$$
t_1 = \inf\{t \in [0,t_0]; f(t) \geq f(t_0)\}.
$$

By continuity, $f(t_1) = f(t_0) > 0$. Moreover, by definition, $f(t) \leq f(t_1)$ for all $t \in [0,t_1]$ (observe that $t_1 > 0$ since $f(0) = 0$). Then, by the mean value theorem,

there exists a $t^* \in (0, t_1)$ such that $f(t_1) = f'(t^*)t_1$. Therefore,

$$
\begin{aligned}
\frac{f(t_1)}{t_1} &= f'(t^*) \leq \frac{1}{\eta(t^*)} \int_0^{t^*} f(s)\xi(s)ds + f(t^*)\frac{\rho(t^*)}{\eta(t^*)} \\
&\leq \frac{f(t_1)}{\eta(t^*)} \int_0^{t^*} \xi(s)ds + f(t^*)\frac{\rho(t^*)}{\eta(t^*)} \\
&\leq Cf(t_1) + f(t^*)\frac{\rho(t^*)}{\eta(t^*)}.
\end{aligned}
$$

Now observe that $f(t^*)\rho(t^*)/\eta(t^*) \leq Cf(t_1)$. In fact, if $f(t^*)\rho(t^*) \leq 0$, we are done since $f(t_1) > 0$. If $f(t^*)\rho(t^*) > 0$, then $f(t^*) > 0$ and $\rho(t^*) > 0$ because $\max\{f(t^*), \rho(t^*)\} \geq 0$ by hypothesis, which implies that

$$
f(t^*)(\rho(t^*)/\eta(t^*)) \leq f(t^*)C \leq f(t_1)C.
$$

In any event,

$$
\frac{f(t_1)}{t_1} \leq 2Cf(t_1),
$$

and hence $1 \leq 2Ct_1 \leq 2C\delta$, which is a contradiction because $0 < 1 - 2C\delta$. This finishes the proof of Claim 1.   □

Claim 2. $\sup I = \varepsilon$.

Proof. Define $\delta_0 = \sup I \in (0, \varepsilon]$. If $\delta_0 < \varepsilon$, set $\tilde{f}(t) = f(t + \delta_0)$, $\tilde{\eta}(t) = \eta(t + \delta_0)$, $\tilde{\xi}(t) = \xi(t + \delta_0)$, and $\tilde{\rho}(t) = \rho(t + \delta_0)$ for $t \in [0, \varepsilon - \delta_0)$. Observing that, in this case, $f \leq 0$ in $[0, \delta_0]$ (i.e., $\delta_0 \in I$), we have

$$
\begin{aligned}
\tilde{f}'(t)\tilde{\eta}(t) &= f'(t + \delta_0)\eta(t + \delta_0) \\
&\leq \int_0^{t+\delta_0} f(s)\xi(s)ds + f(t + \delta_0)\rho(t + \delta_0) \\
&\leq \int_0^t \tilde{f}(r)\tilde{\xi}(r)dr + \tilde{f}(t)\tilde{\rho}(t)
\end{aligned}
$$

for all $t \in [0, \varepsilon - \delta_0)$. From the definition of δ_0, $\tilde{f}(0) = f(\delta_0) = 0$. Then, using Claim 1 for these new functions, there exists a $\delta_1 \in (0, \varepsilon - \delta_0)$ such that $\tilde{f} \leq 0$ in $[0, \delta_1]$, which implies that $f \leq 0$ in $[0, \delta_0 + \delta_1]$, contradicting the definition of δ_0. This finishes the proof of Claim 2.

The result follows from Claim 2.   □

Let $M^{n+1} = (M^{n+1}, g, K)$ be an initial data set of dimension $n+1$. We say that K is *n-convex* (on M) if $\mathrm{tr}_\pi K \geq 0$ for all $\pi \subset T_pM$ and $p \in M$, where π is a linear subspace of dimension n. Saying that K is n-convex is equivalent to say that the sum of the n smallest eigenvalues of K is nonnegative.

The main result of this section is the following.

Theorem 3.3. *Let $M^3 = (M^3, g, K)$ be a three-dimensional initial data set. Let Σ^2 be an orientable weakly outermost closed MOTS in M^3 of genus $g(\Sigma) \geq 2$.*

Suppose that $\mu - |J| \geq -c$ for some constant $c > 0$ and that K is 2-convex, both on M_+. Then if $A(\Sigma) = 4\pi(g(\Sigma) - 1)/c$, the following hold:

(1) An outer neighborhood $U \approx [0, \varepsilon) \times \Sigma$ of Σ in M is isometric to

$$([0, \varepsilon) \times \Sigma, dt^2 + g_0),$$

where (Σ, g_0) has constant Gaussian curvature $\kappa = -c$.

(2) Each slice $\Sigma_t \approx \{t\} \times \Sigma$ is totally geodesic as a submanifold of spacetime. Equivalently, $\chi_+(t) = \chi_-(t) = 0$, where $\chi_\pm(t)$ are the null second fundamental forms of Σ_t.

(3) $K(\cdot, \cdot)|_{T_x\Sigma_t} = 0$ and $K(\nu_t, \cdot)|_{T_x\Sigma_t} = 0$ for each $x \in \Sigma_t$, where ν_t is the outer unit normal to Σ_t, and $J = 0$ on U.

Proof. First, observe that Σ is stable because it is weakly outermost. Then, by Proposition 3.1, we have $\lambda_1(L) = 0$. In this case, let $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$, $\theta = \theta(t)$, and $\phi = \phi_t$ be given by Lemma 2.5. It follows from (2.2) that

$$\frac{d\theta}{dt} = -\Delta\phi + 2\langle X, \nabla\phi \rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2}\theta^2 + \theta\tau \right)\phi,$$

where $\Delta = \Delta_t$, $\langle \, , \, \rangle = \langle \, , \, \rangle_t$, $X = X_t$, $\nabla = \nabla_t$ (gradient operator), $Q = Q_t$, and $\operatorname{div} = \operatorname{div}_t$ are the respective entities associated with Σ_t for each $t \in (-\varepsilon, \varepsilon)$. Thus,

$$
\begin{aligned}
\frac{\theta'}{\phi} &= -\frac{\Delta\phi}{\phi} + 2\langle X, \frac{1}{\phi}\nabla\phi \rangle - |X|^2 + \operatorname{div} X + Q - \frac{1}{2}\theta^2 + \theta\tau \\
&= \operatorname{div} Y - |Y|^2 + Q - \frac{1}{2}\theta^2 + \theta\tau \\
&\leq \operatorname{div} Y + Q + \theta\tau,
\end{aligned}
$$

where $Y = X - \nabla \ln \phi$. Therefore, observing that $\theta'(t)$ is also constant on Σ_t and using the divergence theorem, for each $t \in [0, \varepsilon)$ we have

$$
\begin{aligned}
\theta'(t)\int_{\Sigma_t}\frac{1}{\phi}dA_t - \theta(t)\int_{\Sigma_t}\tau dA_t &\leq \int_{\Sigma_t}Q dA_t \\
&= \int_{\Sigma_t}\left(\kappa - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2 \right)dA_t \\
&\leq \int_{\Sigma_t}\left(\kappa - (\mu - |J|) - \frac{1}{2}|\chi|^2 \right)dA_t \\
&\leq \int_{\Sigma_t}(\kappa - (\mu - |J|))dA_t.
\end{aligned}
$$

Now, using the last inequality above along with the hypothesis that $\mu - |J| \geq -c$ on U, we get

$$
\begin{aligned}
\theta'(t)\int_{\Sigma_t}\frac{1}{\phi}dA_t - \theta(t)\int_{\Sigma_t}\tau dA_t &\leq \int_{\Sigma_t}(\kappa + c)dA_t \\
&= 4\pi(1 - g(\Sigma)) + cA(\Sigma_t) \\
&= -cA(\Sigma_0) + cA(\Sigma_t) \\
&= c\int_0^t\frac{d}{ds}A(\Sigma_s)ds.
\end{aligned}
$$

Above, we have used the Gauss–Bonnet theorem and the fundamental theorem of calculus. On the other side, $\theta(t) = \operatorname{tr}_{\Sigma_t} K + H_t \geq H_t$ for $t \in [0, \varepsilon)$ since K is

2-convex on U. Here, H_t is the mean curvature of Σ_t in M. Then, by the first variation of area,

$$\theta'(t) \int_{\Sigma_t} \frac{1}{\phi} dA_t - \theta(t) \int_{\Sigma_t} \tau dA_t \;\; \leq \;\; c \int_0^t \left(\int_{\Sigma_s} H_s \phi dA_s \right) ds$$

$$\leq \;\; c \int_0^t \theta(s) \left(\int_{\Sigma_s} \phi dA_s \right) ds,$$

i.e.,

$$\theta'(t) \int_{\Sigma_t} \frac{1}{\phi} dA_t \leq \int_0^t \theta(s) \left(c \int_{\Sigma_s} \phi dA_s \right) ds + \theta(t) \int_{\Sigma_t} \tau dA_t \qquad \forall t \in [0, \varepsilon).$$

It follows from Lemma 3.2 that $\theta(t) = 0$, i.e., Σ_t is a MOTS for each $t \in [0, \varepsilon)$, because $\theta(t) \geq 0$ since Σ is weakly outermost.

Since $\theta(t) = 0$ for each $t \in [0, \varepsilon)$, all inequalities above must be equalities. Then $Y = X - \nabla \ln \phi = 0$, $\chi = 0$, and $\mu + J(\nu) = \mu - |J| = -c$ on $U \approx [0, \varepsilon) \times \Sigma$. Moreover, $A(\Sigma_t) = A(\Sigma_0) = 4\pi(g(\Sigma) - 1)/c$ for all $t \in [0, \varepsilon)$. Because $\theta'(t) = 0$, it follows from (2.2) and Lemma 2.3 that Σ_t is stable. Thus, using Proposition 3.1, Σ_t has constant Gaussian curvature $\kappa_t = -c$ for each $t \in [0, \varepsilon)$.

Now, because $t \longmapsto A(\Sigma_t)$ is constant for $t \in [0, \varepsilon)$,

$$0 = \frac{d}{dt} A(\Sigma_t) = \int_{\Sigma_t} H_t \phi dA_t.$$

This implies that $H_t = 0$ for each $t \in [0, \varepsilon)$ since $0 = \theta(t) \geq H_t$ and $\phi = \phi_t > 0$. Then Σ_t is a minimal MOTS, which implies that $\mathrm{tr}_{\Sigma_t} K = 0$ for each $t \in [0, \varepsilon)$. In this case, the null mean curvature $\theta_-(t) = \mathrm{tr}_{\Sigma_t} K - H_t$ of Σ_t with respect to $l_-(t) = u - \nu_t$ also vanishes everywhere. Applying (2.2) for θ_- and $\phi_- = -\phi$ instead of $\theta = \theta_+$ and ϕ, respectively, we have

(3.2) $\quad 0 = \theta'_- = -\Delta \phi_- + 2\langle X_-, \nabla \phi_- \rangle + (Q_- - |X_-|^2 + \mathrm{div}\, X_-)\phi_-,$

where

(3.3) $\quad\quad\quad\quad\quad Q_- \;\; = \;\; \kappa - (\mu + J(-\nu)) - \frac{1}{2}|\chi_-|^2$

$$= \;\; -c - (\mu + |J|) - \frac{1}{2}|\chi_-|^2$$

$$= \;\; -2|J| - \frac{1}{2}|\chi_-|^2,$$

(3.4) $\quad\quad\quad\quad\quad X_- \;\; = \;\; -X = -\frac{1}{\phi}\nabla\phi.$

Substituting (3.3) and (3.4) into (3.2) (and remembering that $\phi_- = -\phi$), we get

$$\Delta\phi + \frac{|\nabla\phi|^2}{\phi} + \left(|J| + \frac{1}{4}|\chi_-|^2\right)\phi = 0,$$

which, after integration over Σ_t, implies that

$$|\nabla\phi| = |\chi_-| = |J| = 0 \qquad \text{on } U.$$

Equation (2.1) now implies that $K|_{T_x\Sigma_t \times T_x\Sigma_t} = 0$ and that (Σ, g_t) is totally geodesic in M for each $t \in [0, \varepsilon)$, where g_t is the Riemannian metric on $\Sigma_t \approx \{t\} \times \Sigma$ induced from (M, g). Writing $g = \phi^2 dt^2 + g_t$ on $U \approx [0, \varepsilon) \times \Sigma$ and observing that $\phi = \phi_t$ depends only on $t \in [0, \varepsilon)$, it follows that g_t does not depend on $t \in [0, \varepsilon)$.

Then, after the simple change of variable $ds = \phi(t)dt$, $\phi(t) = \phi_t$, we can see that g has the product structure $ds^2 + g_0$ on U, where (Σ, g_0) has constant Gaussian curvature $\kappa = -c$. □

4. THE CASE OF HIGH DIMENSION WITH NEGATIVE σ-CONSTANT

In this section we extend the main result of last section to the case of high-dimensional MOTSs with negative σ-constant. But before stating our result, we are going to present some terminology.

Let Σ^n be a connected closed (compact with no boundary) n-manifold, $n \geq 3$. Denote by $\mathcal{M}(\Sigma)$ the set of all Riemannian metrics on Σ. The *Einstein–Hilbert functional* $\mathcal{E} : \mathcal{M}(\Sigma) \to \mathbb{R}$ is defined by

$$\mathcal{E}(g) = \frac{\int_\Sigma R_g dv_g}{\mathrm{Vol}(\Sigma^n, g)^{\frac{n-2}{n}}},$$

where R_g is the scalar curvature of (Σ, g). Denote by $[g] = \{e^{2f}g; f \in C^\infty(\Sigma)\}$ the conformal class of $g \in \mathcal{M}(\Sigma)$. The *Yamabe invariant* of $(\Sigma, [g])$ is defined as the following conformal invariant:

$$\mathcal{Y}(\Sigma, [g]) = \inf_{\tilde{g} \in [g]} \mathcal{E}(\tilde{g}).$$

The classical solution of the Yamabe problem by Yamabe [27], Trudinger [25], Aubin [4] (se also [5]), and Schoen [22] says that every conformal class $[g]$ contains metrics $\hat{g}$, called *Yamabe metrics*, which realize the minimum:

$$\mathcal{E}(\hat{g}) = \mathcal{Y}(\Sigma, [g]).$$

Such metrics have constant scalar curvature given by

$$R_{\hat{g}} = \mathcal{Y}(\Sigma^n, [g]) \mathrm{Vol}(\Sigma^n, \hat{g})^{-\frac{2}{n}}.$$

Furthermore,

$$\mathcal{Y}(\Sigma^n, [g]) \leq \mathcal{Y}(\mathbb{S}^n, [g_{\mathrm{can}}]),$$

and equality holds if and only if (Σ^n, g) is conformally diffeomorphic to the Euclidean n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ endued with the canonical metric g_{can}.

The following topological invariant was introduced by Schoen in lectures given in 1987, which were published two years later [23] (see also Kobayashi [18]),

$$\sigma(\Sigma^n) = \sup_{g \in \mathcal{M}(\Sigma^n)} \mathcal{Y}(\Sigma^n, [g]).$$

This invariant is called the *σ-constant* of Σ.

Remark 4.1. Using the solution of the Yamabe problem, it is not difficult to prove that $\sigma(\Sigma) > 0$ if and only if Σ admits a Riemannian metric of positive scalar curvature. On the other hand, observe that if S^2 is an orientable closed surface, then its Euler–Poincaré characteristic is $\chi(S) = 2(1 - g(S)) > 0$ if and only if $g(S) = 0$, i.e., S^2 is topologically $\mathbb{S}^2$, which by Gauss–Bonnet and uniformization theorems is equivalent to saying that S^2 admits a Riemannian metric of positive Gaussian curvature. Having this in mind, the σ-constant in dimension $n \geq 3$ shares the same property with the Euler–Poincaré characteristic in dimension 2.

Now, let us present the first result of this section. The following volume estimate was obtained by Galloway and Murchadha [14] (see also [9] for the time-symmetric case). Our contribution consists in the *infinitesimal rigidity* obtained when equality occurs, which is based on [13, 20].

Proposition 4.2. *Let Σ^n be a stable closed MOTS with $\sigma(\Sigma) < 0$ in an initial data set $M^{n+1} = (M^{n+1}, g, K)$, $n \geq 3$. Suppose that $\mu + J(\nu) \geq -c$ on Σ for some constant $c > 0$. Then the volume of Σ satisfies*

$$(4.1) \qquad \mathrm{Vol}(\Sigma^n) \geq \left(\frac{|\sigma(\Sigma^n)|}{2c} \right)^{\frac{n}{2}}.$$

Furthermore, if equality holds, the metric g_Σ on Σ induced from M is Einstein and has constant scalar curvature $R_\Sigma = -2c$, the null second fundamental form $\chi = \chi_+$ of Σ vanishes identically, $\mu + J(\nu) = -c$ on Σ, and $\lambda_1(L_0) = \lambda_1(L) = 0$.

Proof. Since Σ is stable, $\lambda_1(L) \geq 0$, where L is the MOTS stability operator. Then, by (2.5),

$$(4.2) \qquad 0 \;\leq\; 2 \int_\Sigma (|\nabla u|^2 + Q u^2) dv$$

$$= \int_\Sigma (2|\nabla u|^2 + (R_\Sigma - 2(\mu + J(\nu)) - |\chi|^2) u^2) dv$$

$$(4.3) \qquad \leq \int_\Sigma (2|\nabla u|^2 + (R_\Sigma + 2c) u^2) dv$$

for all $u \in C^\infty(\Sigma)$, $u > 0$. Using $2 < \frac{4(n-1)}{n-2}$ and Hölder's inequality with (4.3), we have

$$(4.4) \quad 0 \;\leq\; \int_\Sigma \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + R_\Sigma u^2 \right) dv + 2c \int_\Sigma u^2 dv$$

$$(4.5) \qquad \leq \int_\Sigma \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + R_\Sigma u^2 \right) dv + 2c \, \mathrm{Vol}(\Sigma)^{\frac{2}{n}} \left(\int_\Sigma u^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}$$

for all $u \in C^\infty(\Sigma)$, $u > 0$. On the other hand, it is well known that for $g = u^{\frac{4}{n-2}} g_\Sigma$, $u > 0$, we have

$$(4.6) \qquad R_g u^{\frac{n+2}{n-2}} = -\frac{4(n-1)}{n-2} \Delta u + R_\Sigma u,$$

which implies that

$$(4.7) \qquad \mathcal{E}(u^{\frac{4}{n-2}} g_\Sigma) = \frac{\int_\Sigma (\frac{4(n-1)}{n-2} |\nabla u|^2 + R_\Sigma u^2) dv}{\left(\int_\Sigma u^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}}}.$$

Therefore, using (4.7) with (4.5),

$$(4.8) \qquad 0 \leq \mathcal{E}(u^{\frac{4}{n-2}} g_\Sigma) + 2c \, \mathrm{Vol}(\Sigma)^{\frac{2}{n}}$$

for all $u \in C^\infty(\Sigma)$, $u > 0$, which implies that

$$(4.9) \qquad 0 \;\leq\; \inf_{u \in C^\infty(\Sigma), \, u > 0} \mathcal{E}(u^{\frac{4}{n-2}} g_\Sigma) + 2c \, \mathrm{Vol}(\Sigma)^{\frac{2}{n}}$$

$$= \mathcal{Y}(\Sigma, [g_\Sigma]) + 2c \, \mathrm{Vol}(\Sigma)^{\frac{2}{n}}$$

$$(4.10) \qquad \leq \sigma(\Sigma) + 2c \, \mathrm{Vol}(\Sigma)^{\frac{2}{n}}.$$

This finishes the proof of (4.1).

Now suppose that equality in (4.1) holds. Then inequalities (4.9) and (4.10) must be equalities. Choose $u_0 \in C^\infty(\Sigma)$, $u_0 > 0$, such that $\hat{g} = u_0^{4/(n-2)} g_\Sigma$ is a Yamabe metric. Then inequalities (4.2), (4.3), (4.4), (4.5), and (4.8) must be equalities when $u = u_0$. Therefore, $\chi = 0$, $\mu + J(\nu) = -c$ on Σ, and $|\nabla u_0|$ vanishes identically since $2 < \frac{4(n-1)}{n-2}$. This implies that u_0 is constant (in particular, g_Σ is also a Yamabe metric) and that $\int_\Sigma Q \, dv = 0$. Proceeding analogously to the proof of Proposition 3.1, we have $R_\Sigma = -2c$ and $\lambda_1(L_0) = \lambda_1(L) = 0$. Equality in (4.10) implies that g_Σ realizes $\sigma(\Sigma)$, i.e., $\mathcal{E}(g_\Sigma) = \mathcal{Y}(\Sigma, [g_\Sigma]) = \sigma(\Sigma)$. Then, by [23, pp. 126 and 127], g_Σ is Einstein. $\qquad\square$

In the following, we are going to present the main result of this section. But before doing that, let us make an important remark.

Remark 4.3. Let Σ^n be a closed manifold of dimension $n \geq 3$. Suppose that $g \in \mathcal{M}(\Sigma)$ satisfies $\mathcal{Y}(\Sigma^n, [g]) < 0$. Then it is well known that given $c > 0$, there exists a unique metric $\tilde{g} \in [g]$ such that $R_{\tilde{g}} = -2c$. In fact, the existence follows from the solution of the Yamabe problem. To prove the uniqueness, let $g_1, g_2 \in [g]$ be such that $R_{g_1} = -2c = R_{g_2}$. Since g_1 and g_2 are conformal, we have $g_2 = u^{4/(n-2)} g_1$ for some $u \in C^\infty(\Sigma)$, $u > 0$. Then, using (4.6) for g_1 and g_2, it follows that

$$-2c u^{\frac{n+2}{n-2}} = -\frac{4(n-1)}{n-2} \Delta_{g_1} u - 2cu,$$

i.e.,

$$2c u^{\frac{n+2}{n-2}} = \frac{4(n-1)}{n-2} \Delta_{g_1} u + 2cu.$$

Then, taking $x_0 \in \Sigma$ such that $u(x_0) = \min u$,

$$2c(u(x_0))^{\frac{n+2}{n-2}} = \frac{4(n-1)}{n-2} \Delta_{g_1} u(x_0) + 2cu(x_0) \geq 2cu(x_0),$$

which implies that $u(x_0) \geq 1$. In turn, taking $x_1 \in \Sigma$ such that $u(x_1) = \max u$, we have $u(x_1) \leq 1$. Thus, $u \equiv 1$.

The main result of this section is the following. Its proof is based on [13, 20].

Theorem 4.4. *Let* $M^{n+1} = (M^{n+1}, g, K)$ *be an* $(n+1)$-*dimensional initial data set,* $n \geq 3$. *Let* Σ^n *be a weakly outermost closed MOTS in* M^{n+1} *with* σ-*constant* $\sigma(\Sigma) < 0$. *Suppose that* $\mu - |J| \geq -c$ *for some constant* $c > 0$, *and that* K *is* n-*convex, both on* M_+. *Then if* $\mathrm{Vol}(\Sigma^n) = (|\sigma(\Sigma^n)|/2c)^{n/2}$, *the following hold:*

(1) *An outer neighborhood* $U \approx [0, \varepsilon) \times \Sigma$ *of* Σ *in* M *is isometric to*

$$([0, \varepsilon) \times \Sigma, dt^2 + g_\Sigma),$$

   and (Σ, g_Σ) *is Einstein with constant scalar curvature* $R_\Sigma = -2c$.
(2) *Each slice* $\Sigma_t \approx \{t\} \times \Sigma$ *is totally geodesic as a submanifold of spacetime. Equivalently,* $\chi_+(t) = \chi_-(t) = 0$, *where* $\chi_\pm(t)$ *are the null second fundamental forms of* Σ_t.
(3) $K(\cdot, \cdot)|_{T_x \Sigma_t} = 0$ *and* $K(\nu_t, \cdot)|_{T_x \Sigma_t} = 0$ *for each* $x \in \Sigma_t$, *where* ν_t *is the outer unit normal to* Σ_t, *and* $J = 0$ *on* U.

Proof. Since weakly outermost closed MOTSs are stable (Remark 2.6), by Proposition 4.2 we have $\lambda(L) = 0$, where L is the MOTS stability operator. Then we can take $\{\Sigma_t\}_{t \in (-\varepsilon, \varepsilon)}$, $\theta = \theta(t)$ and $\phi = \phi_t$, as in Lemma 2.5. By (2.2)

(4.11)
$$
\begin{aligned}
\theta' &= -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - |X|^2 + \operatorname{div} X - \frac{1}{2}\theta^2 + \theta\tau\right)\phi \\
&= \left(-|Y|^2 + \operatorname{div} Y + Q - \frac{1}{2}\theta^2 + \theta\tau\right)\phi \\
&\leq \left(-|Y|^2 + \operatorname{div} Y + Q + \theta\tau\right)\phi,
\end{aligned}
$$

where $Y = X - \nabla\ln\phi$. Above, $\Delta = \Delta_t$, $\langle\,,\,\rangle = \langle\,,\,\rangle_t$, $X = X_t$, $\nabla = \nabla_t$ (gradient operator), $Q = Q_t$, and $\operatorname{div} = \operatorname{div}_t$ are the respective entities associated with Σ_t for each $t \in (-\varepsilon, \varepsilon)$. It follows from (4.11) that

$$
\begin{aligned}
\theta'(t)\frac{u^2}{\phi} - \theta(t)\tau u^2 \;\leq\;& -u^2|Y|^2 + u^2\operatorname{div} Y + Qu^2 \\
=\;& -u^2|Y|^2 - 2u\langle\nabla u, Y\rangle + \operatorname{div}(u^2 Y) \\
& + \left(\frac{1}{2}R_{\Sigma_t} - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2\right)u^2 \\
\leq\;& -u^2|Y|^2 + 2|u||\nabla u||Y| + \operatorname{div}(u^2 Y) \\
& + \left(\frac{1}{2}R_{\Sigma_t} - (\mu - |J|) - \frac{1}{2}|\chi|^2\right)u^2 \\
\leq\;& |\nabla u|^2 + \operatorname{div}(u^2 Y) + \left(\frac{1}{2}R_{\Sigma_t} + c\right)u^2
\end{aligned}
$$

for all $u \in C^\infty(\Sigma_t)$, $u > 0$, and $t \in [0, \varepsilon)$. Above, we have used the condition that $\mu - |J| \geq -c$ on M_+. Integrating over Σ_t, observing that $\theta'(t)$ is also constant on Σ_t, and using Hölder's inequality, we get

$$
\begin{aligned}
2\left(\theta'(t)\int_{\Sigma_t}\frac{u^2}{\phi}dv_t - \theta(t)\int_{\Sigma_t}\tau u^2 dv_t\right) \;\leq\;& \int_{\Sigma_t}(2|\nabla u|^2 + R_{\Sigma_t}u^2)dv_t + 2c\int_{\Sigma_t}u^2 dv_t \\
\leq\;& \int_{\Sigma_t}\left(\frac{4(n-1)}{n-2}|\nabla u|^2 + R_{\Sigma_t}u^2\right)dv_t \\
& + 2c\operatorname{Vol}(\Sigma_t)^{\frac{2}{n}}\left(\int_{\Sigma_t}u^{\frac{2n}{n-2}}dv_t\right)^{\frac{n-2}{n}}
\end{aligned}
$$

for all $u \in C^\infty(\Sigma_t)$, $u > 0$, and $t \in [0, \varepsilon)$. It follows from Remark 4.3 that for each $t \in [0, \varepsilon)$ there exists a unique $u_t \in C^\infty(\Sigma_t)$, $u_t > 0$, such that $\hat{g}_t = u_t^{4/(n-2)}g_{\Sigma_t}$ is a Yamabe metric with constant scalar curvature $R_{\hat{g}_t} = -2c$, given that

$$
\mathcal{Y}(\Sigma_t, [g_{\Sigma_t}]) \leq \sigma(\Sigma) < 0,
$$

where g_{Σ_t} is the Riemannian metric on Σ_t induced from M. Therefore,

$$
\frac{2\left(\theta'(t)\int_{\Sigma_t} u_t^2\phi^{-1}dv_t - \theta(t)\int_{\Sigma_t}\tau u_t^2 dv_t\right)}{\left(\int_{\Sigma_t} u_t^{\frac{2n}{n-2}}dv_t\right)^{\frac{n-2}{n}}} \leq \mathcal{E}(u_t^{\frac{4}{n-2}}g_{\Sigma_t}) + 2c\operatorname{Vol}(\Sigma_t)^{\frac{2}{n}}
$$
$$
= \mathcal{Y}(\Sigma_t,[g_{\Sigma_t}]) + 2c\operatorname{Vol}(\Sigma_t)^{\frac{2}{n}}
$$
$$
\leq \sigma(\Sigma) + 2c\operatorname{Vol}(\Sigma_t)^{\frac{2}{n}}
$$
$$
= \left(-2c\operatorname{Vol}(\Sigma)^{\frac{2}{n}} + 2c\operatorname{Vol}(\Sigma_t)^{\frac{2}{n}}\right)
$$
$$
= \frac{4c}{n}\int_0^t \operatorname{Vol}(\Sigma_s)^{\frac{2-n}{n}}\frac{d}{ds}\operatorname{Vol}(\Sigma_s)ds
$$

for each $t \in [0,\varepsilon)$. Using the condition that $\theta(t) = \operatorname{tr}_{\Sigma_t} K + H_t \geq H_t$ for $t \in [0,\varepsilon)$, since K is n-convex on M_+, together with the first variation of area (volume), we have

$$
\frac{\theta'(t)\int_{\Sigma_t} u_t^2\phi^{-1}dv_t - \theta(t)\int_{\Sigma_t}\tau u_t^2 dv_t}{\left(\int_{\Sigma_t} u_t^{\frac{2n}{n-2}}dv_t\right)^{\frac{n-2}{n}}} \leq \frac{2c}{n}\int_0^t \operatorname{Vol}(\Sigma_s)^{\frac{2-n}{n}}\left(\int_{\Sigma_s} H_s\phi dv_s\right)ds
$$
$$
\leq \frac{2c}{n}\int_0^t \theta(s)\left(\operatorname{Vol}(\Sigma_s)^{\frac{2-n}{n}}\int_{\Sigma_s}\phi dv_s\right)ds.
$$

Now, observing that by the uniqueness of $\hat{g}_t$ u_t is continuous on $t \in [0,\varepsilon)$, we can use Lemma 3.2 for

$$
f(t) = \theta(t), \qquad \eta(t) = \left(\int_{\Sigma_t} u_t^2\phi^{-1}dv_t\right)\left(\int_{\Sigma_t} u_t^{\frac{2n}{n-2}}dv_t\right)^{-\frac{n-2}{n}},
$$
$$
\rho(t) = \left(\int_{\Sigma_t}\tau u_t^2 dv_t\right)\left(\int_{\Sigma_t} u_t^{\frac{2n}{n-2}}dv_t\right)^{-\frac{n-2}{n}},
$$

and

$$
\xi(t) = \frac{2c}{n}\operatorname{Vol}(\Sigma_t)^{\frac{2-n}{n}}\int_{\Sigma_t}\phi dv_t.
$$

Thus, $\theta(t) = 0$ for all $t \in [0,\varepsilon)$. Then, continuing as in the proof of Theorem 3.3, we have the result. $\qquad\square$

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