Mini-course in Maceio on embedded constant mean curvature surfaces in $\mathbb{R}^3$

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Based on joint work with Giuseppe Tinaglia.

Some preliminary papers on the lecture material can be found on Tinaglia's web page at Kings College London.

Outline of 3 lectures

1. Lecture 1: Background material, statements of the main results.
2. Lecture 2: Proof of extrinsic curvature estimates for **H**-disks.
3. Lecture 3: Applications:
    1. Intrinsic curvature and radius estimates for **H**-disks.
    2. Chord-arc results and **1**-sided curvature estimates for **H**-disks.
    3. Curvature estimates for **H**-annuli.
    4. Classification of **0** and **1**-connected **H**-surfaces, $\mathbf{H} > 0$.

Theorem (Intrinsic Curvature Estimates for $\mathbf{1}$-Disks, Meeks-Tinaglia)

Fix $\varepsilon > 0$ and $\mathbf{H} = \mathbf{1}$. $\exists\ \mathbf{C} = \mathbf{C}(\mathbf{1}, \varepsilon) \geq \pi$ such that for every embedded $\mathbf{1}$-disk $\mathbf{D} \subset \mathbf{R}^3$ and every $p \in \mathbf{D}$ with $\mathbf{dist_D}(p, \partial\mathbf{D}) \geq \varepsilon$,

$$|\mathbf{A_D}|(p) \leq \mathbf{C}.$$

Brief idea/ingredients of the proof.

- **One-sided curvature estimates** for $\mathbf{H}$-disks.
- **Deep weak-chord arc type theorem** reduces the proof to the failure of an **extrinsic curvature estimate**:

  Curvature estimate fails for $\mathbf{D}$ = disk with $\partial\mathbf{D} \subset \partial\mathbb{B}(\delta)$ and $\vec{0} \in \mathbf{D}$ is a point of large second fundamental form.
- **Rescaling arguments** imply **helicoid-type surfaces occur** near $\vec{0}$.
- **Pair of highly-sheeted multigraphs around $\vec{0}$ extends** to pair of highly-sheeted multigraphs for a fixed distance proportional to δ, impossible for $\mathbf{H} = 1$. □

Theorem (Radius Estimates for H-Disks, Meeks-Tinaglia 2013)

$\exists\ \mathbf{R_0} \geq \pi$ such that every embedded **H**-disk in $\mathbf{R}^3$ has radius $< \frac{\mathbf{R_0}}{\mathbf{H}}$.

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- Rescale any surfaces in the proof so that $\mathbf{H} = \mathbf{1}$.

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- Arguing by contradiction, suppose that there exists a sequence $\mathbf{D}_n$ of $\mathbf{1}$-disks and points $p_n \in \mathbf{D}_n$ with $\mathbf{d}_{\mathbf{D}_n}(p_n, \partial \mathbf{D}_n) > n$.

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- But the **CMC** flux of $\mathbf{D}_n$ is 0, which is a contradiction!
- **Alternate proof** that a Delaunay surface cannot occur is by way of the Alexandrov reflection principle (argument on the blackboard). □

Application of the Radius Estimate to get the Curvature Estimate!

Theorem (Intrinsic Curvature Estimates for H-Disks, Meeks-Tinaglia)

Let $\mathbf{C}(\mathbf{1}, \delta)$ be the curvature estimate for embedded $\mathbf{1}$-disks at points of distance $\geq \delta$ from their boundaries and let $\mathbf{R}_0$ be their radius estimate. Fix $\varepsilon, \mathcal{H} > 0$. Then $\forall$ embedded $\mathbf{H}$-disks $\mathbf{D} \subset \mathbf{R}^3$ with $\mathbf{H} \geq \mathcal{H}$ and $\forall p \in \mathbf{D}$ with $\mathbf{dist}_{\mathbf{D}}(p, \partial \mathbf{D}) \geq \varepsilon$,

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- Rescale the $\mathbf{D}$ by $\mathbf{H}$ to obtain the new surfaces $\mathbf{H} \cdot \mathbf{D}$ with mean curvature $\mathbf{H} = \mathbf{1}$; note the radius of $\mathbf{H} \cdot \mathbf{D}$ is at least $\mathbf{H} \cdot \varepsilon$.

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- By radius estimates for $\mathbf{1}$-disks, $\mathbf{H} \in [\mathcal{H}, \frac{\mathbf{R}_0}{\varepsilon}]$.
- Then $\mathbf{C}(\mathcal{H}, \varepsilon) = \mathbf{C}(\mathbf{1}, \varepsilon\mathcal{H})\frac{\mathbf{R}_0}{\varepsilon}$ works by scaling.

□

Theorem (One-sided curvature estimate for **H**-disks, Meeks, Tinaglia)

There exist $\varepsilon \in (0, \frac{1}{2})$ and $C > 0$ such that for any $R > 0$, the following holds. Let $\mathbf{\Sigma}$ be an $\mathbf{H}$-disk such that

$$\mathbf{\Sigma} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \emptyset \quad \text{and} \quad \partial\mathbf{\Sigma} \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \emptyset.$$

Then:

$$\sup_{x \in \mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}} |\mathbf{A}_{\mathbf{\Sigma}}|(x) \leq \frac{C}{R}. \qquad (1)$$

In particular, if $\mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\} \neq \emptyset$, then $\mathbf{H} \leq \frac{C}{R}$.

Sketch of the Proof.

- After scaling, assume $R = 1$.
- It suffices to prove that for some $\varepsilon > 0$ the tangent planes to $\mathbf{\Sigma} \cap \mathbb{B}(\varepsilon)$ are not vertical.
- Suppose $\exists$ a sequence of $E(n)$ of $\mathbf{H}_n$-disks satisfying the conditions of $\mathbf{\Sigma}$ with points q_n with vertical tangent planes and $q_n \to \vec{0}$.
- By extrinsic curvature estimates for $\mathbf{H}$-disks with $\mathbf{H} > 0$, $\mathbf{H}_n \to 0$.
- $\mathcal{B}_{E(n)}(q_n, 2x_3(q_n))$ cannot be a graph of gradient less than or equal to 1 over its orthogonal projection to $T_{q_n}E(n)$. □

Continuation of Proof.

- Let $r(n) \in (0, 2x_3(q_n))$ be the largest number such that $\mathcal{B}_{E(n)}(q_n, r(n))$ is a graph of gradient at most 1 over its projection to $T_{q_n}E(n)$; by the previous discussion, $\lim_{n\to\infty} r(n) = 0$.

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- Next show that a sub-sequence of translated and scaled surfaces

$$\boldsymbol{\Sigma}(n) = \frac{1}{r(n)}(E(n) - q_n).$$

  converges to a vertical helicoid; this proof is **technical**.

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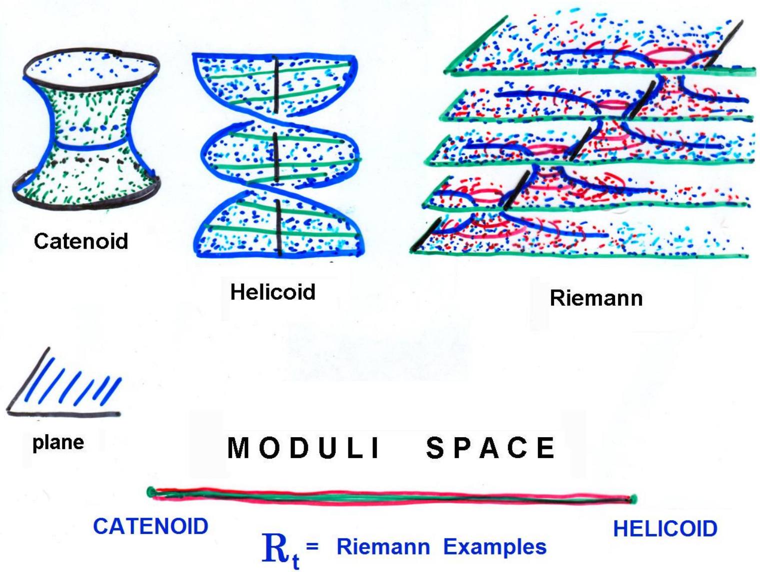
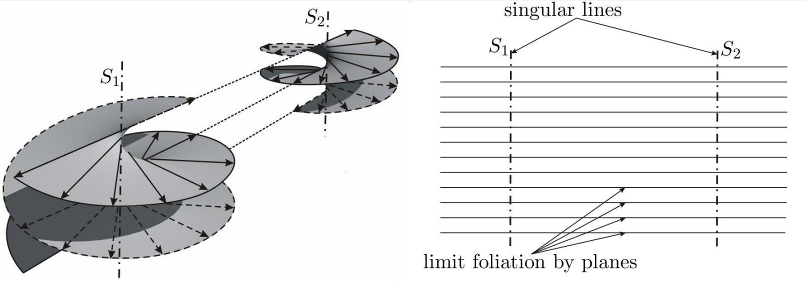
- See blackboard for arguments to obtain a contradiction. □

Moduli space of genus-0 minimal examples - Meeks, Pérez & Ros



Riemann minimal examples near helicoid limits

x



- By appropriately scaling, the Riemann examples $\mathcal{R}_t$ converge as $t \to \infty$ to a foliation $\mathcal{F}$ of $\mathbf{R}^3$ by horizontal planes.
- The set of non-smooth convergence $\mathbf{S}(\mathcal{F})$ to $\mathcal{F}$ consists of **2** vertical lines $\mathbf{S_1}, \mathbf{S_2}$ perpendicular to the planes in $\mathcal{F}$.

Theorem (Chord-Arc Theorem, Meeks-Tinaglia)

There exists a positive constant C such that if $\mathbf{\Sigma} \subset \mathbf{R}^3$ is an $\mathbf{H}$-disk, $B_{\mathbf{\Sigma}}(\vec{0}, CR) \subset \mathbf{\Sigma} - \partial\mathbf{\Sigma}$ and $\sup_{\mathbb{B}_{\mathbf{\Sigma}}(\vec{0},r_0)} |\mathbf{A}_{\mathbf{\Sigma}}| \geq \frac{1}{r_0}$ where $R > r_0$, then for $x \in B_{\mathbf{\Sigma}}(\vec{0}, R)$,

$$\frac{1}{6}\,\mathrm{dist}_{\mathbf{\Sigma}}(x, \vec{0}) < |x| + r_0. \tag{2}$$

Proof.

- Clever application of Limit Lamination Theorem for $\mathbf{H}$-planar domains with positive injectivity radius function $\geq \delta > 0$ away their boundaries, which generalizes the main theorem by Colding-Minicozzi for minimal planar domains in the final paper $\#\mathbf{5}$ in their Annals series.
- Given $\mathbf{H}_n$-planar domains $\mathbf{M}_n$, $\partial\mathbf{M}_n \to \infty$, $|\mathbf{A}_{\mathbf{M}_n}|(\vec{0}) \geq \varepsilon > 0$, then a subsequence converges to planes, catenoids, helicoids, Riemann minimal examples, to a foliation $\mathcal{F}$ of $\mathbf{R}^3$ by parallel planes with singular set $\mathbf{S}(\mathcal{F})$ of convergence consisting of one or two lines orthogonal to $\mathcal{F}$ or to a properly "embedded" genus-$\mathbf{0}$ ($\mathbf{H} > 0$)-planar domain.
- If $\mathrm{Inj}_{\mathbf{M}_n}(\vec{0}) \leq 1$, then $\exists\ \eta > 0$ depending on the limit and $\mathbf{1}$-cycles α_n on $\mathbf{M}_n$ with flux vector of length $F \in [\eta, 2\eta]$. □

Definition (Scalar flux of an H-annulus)

For an **H**-annulus **E** with generator $[\alpha]$ of $\mathbf{H_1}(\mathbf{E})$, the **scalar flux** of **E**, denoted by $F(\mathbf{E})$, is the length of the flux vector of α.

Proposition (Curvature Estimates for **H**-annuli)

Given $\rho > 0$ and $\delta \in (0,1)$ there exists a positive constant $\mathbf{I}_0 := \mathbf{I}(\rho,\delta)$ (or $\mathbf{I}_0 := \mathbf{I}(\delta)$) such that if $\mathbf{E}$ is a compact $\mathbf{1}$-annulus with flux $F(\mathbf{E}) \geq \rho$ (or $F(\mathbf{E}) = 0$), then

$$\inf_{\{p\in\mathbf{E} \mid d_{\mathbf{E}}(p,\partial\mathbf{E})\geq\delta\}} \mathbf{I}_{\mathbf{E}}(p) \geq \mathbf{I}_0,$$

where $\mathbf{I}_{\mathbf{E}}\colon \mathbf{E} \to [0,\infty)$ is the injectivity radius function of $\mathbf{E}$.

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Proof.

- Suppose $\exists$ a sequence $\mathbf{E}(n)$ of $\mathbf{1}$-annuli with $F(\mathbf{E}(n)) \geq \rho > 0$ (or $F(\mathbf{E}(n)) = 0$), with $\mathbf{I}_n \colon \mathbf{E}(n) \to [0,\infty)$ and points $p(n)$ in $\{q \in \mathbf{E}(n) \mid d_{\mathbf{E}(n)}(q,\partial\mathbf{E}(n)) \geq \delta\}$ with $\mathbf{I}_n(p(n)) \leq \dfrac{1}{n}$.

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$$\inf_{\{p\in\mathbf{E} \mid d_{\mathbf{E}}(p,\partial\mathbf{E})\geq\delta\}} \mathbf{I}_{\mathbf{E}}(p) \geq \mathbf{I}_0,$$

where $\mathbf{I}_{\mathbf{E}} \colon \mathbf{E} \to [0,\infty)$ is the injectivity radius function of $\mathbf{E}$.

Proof.

- Suppose $\exists$ a sequence $\mathbf{E}(n)$ of $\mathbf{1}$-annuli with $F(\mathbf{E}(n)) \geq \rho > 0$ (or $F(\mathbf{E}(n)) = 0$), with $\mathbf{I}_n \colon \mathbf{E}(n) \to [0,\infty)$ and points $p(n)$ in $\{q \in \mathbf{E}(n) \mid d_{\mathbf{E}(n)}(q, \partial\mathbf{E}(n)) \geq \delta\}$ with $\mathbf{I}_n(p(n)) \leq \dfrac{1}{n}$.
- For each $p(n)$ consider a point $q(n) \in B_{\mathbf{E}(n)}(p(n),\delta)$ where the following function obtains its maximum value:

$$f(x) = \frac{d_{\mathbf{E}(n)}(x, \partial B_{\mathbf{E}(n)}(p(n),\delta))}{\mathbf{I}_n(x)}.$$

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- But $F(\mathbf{M}_n) \notin [\eta, 2\eta]$ for any $\eta > 0$, which gives a contradiction. □

Theorem (Curvature Estimates for H-Disks, Meeks-Tinaglia)

For $\delta, \varepsilon > 0$, $\exists$ $\mathbf{C} \geq 1$ such that for any $\mathbf{H} \geq \varepsilon$ and any complete Riemannian $\mathbf{3}$-manifold $\mathbf{N}$ with absolute sectional curvature at most $\mathbf{1}$, the following hold:

- An embedded $\mathbf{H}$-surface $\mathbf{M}$ with $\mathbf{I_M} \geq \delta$ satisfies

$$|\mathbf{A_M}| \leq \mathbf{C}.$$

- If $\mathbf{N}$ is locally homogeneous and $\mathbf{D} \subset \mathbf{N}$ is an embedded $\mathbf{H}$-disk, then for $p \in \mathbf{D}$ with $\mathbf{dist_D}(p, \partial\mathbf{D}) \geq \varepsilon$,

$$|\mathbf{A_D}|(p) \leq \mathbf{C}.$$

Conjecture (Meeks-Tinaglia)

For $\mathbf{H} > 0$, a complete embedded $\mathbf{H}$-surface $\mathbf{M}$ of finite topology in a complete locally homogeneous three-manifold $\mathbf{X}$ has bounded second fundamental form. (Already proved true for many homogeneous geometries including $\mathbb{H}^3$.)

Conjecture (Meeks-Tinaglia)

Suppose that $\mathbf{X}$ is a non-compact simply connected homogeneous $\mathbf{3}$-manifold with Cheeger constant $\mathbf{Ch}(\mathbf{X})$. Given $\varepsilon > 0$, there exists radius estimates $\mathbf{R}(\varepsilon)$ for embedded $\mathbf{H}$-disks whenever $\mathbf{H} \geq \frac{1}{2}\mathbf{Ch}(\mathbf{X}) + \varepsilon$.

Conjecture (Embedded Calabi-Yau Problem for finite genus H-surfaces, Meeks-Perez-Ros-Tinaglia)

Complete embedded **H** finite genus surfaces $\mathbf{M} \subset \mathbf{R}^3$ are properly embedded and when $\mathbf{H} > 0$, then such an **M** has cubical volume growth.

Conjecture (Meeks-Tinaglia)

- Suppose that **X** is a homology **3**-manifold with a Riemannian metric.
- Given $n_0 \in \mathbb{N}$ and positive numbers $a < b$, there exists a constant $\mathbf{A}_{a,b}$ such that every compact embedded genus-n_0 **H**-surface $\mathbf{M} \subset \mathbf{X}$ with $\mathbf{H} \in [a, b]$ satisfies:

$$\mathbf{Area}(\mathbf{M}) \leq \mathbf{A}_{a,b}.$$

- Furthermore there is a natural compactification of the moduli space of examples with fixed $\mathbf{H} > 0$ and genus at most n_0.