# Mini-course in Maceio on embedded constant mean curvature surfaces in $\mathbb{R}^3$

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**Based on joint work with Giuseppe Tinaglia.**

Some preliminary papers on the lecture material can be found on Tinaglia's web page at Kings College London.

## Outline of 3 lectures

1. Lecture 1: Background material, statements of the main results.
2. Lecture 2: Proof of extrinsic curvature estimates for **H**-disks.
3. Lecture 3: Applications:
   1. Intrinsic curvature and radius estimates for **H**-disks.
   2. Chord-arc results and **1**-sided curvature estimates for **H**-disks.
   3. Curvature estimates for **H**-annuli.
   4. Classification of **0** and **1**-connected **H**-surfaces, $H > 0$.

## Definition

- An continuous map $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ between topological spaces is **proper**, if for each compact set $\mathbf{\Delta} \subset \mathbf{Y}$, $\mathbf{f}^{-1}(\mathbf{\Delta})$ is compact in $\mathbf{X}$.
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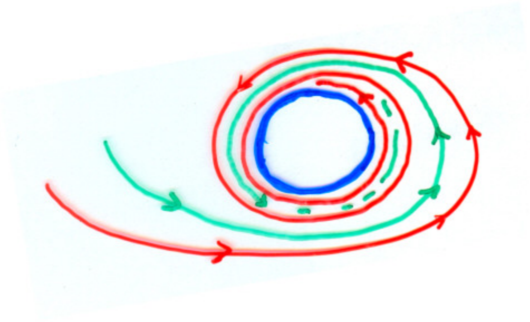
- An embedded surface $\mathbf{M} \subset \mathbf{R^3}$ is **proper** if its inclusion map is proper.

### Remark

A smooth embedded noncompact surface $\mathbf{M} \subset \mathbf{R^3}$ that is **not** proper must have accumulation points, i.e., $\exists$ a sequence of points $\mathbf{p_n} \in \mathbf{M}$ such that $\lim_{n\to\infty} \mathbf{p_n} = \mathbf{p} \in \mathbf{R^3}$, but this sequence fails to converge in the intrinsic Riemannian metric space structure on $\mathbf{M}$.

## Example (Non-proper curves in $\mathbf{R^2}$)

Below is a picture of the union $\mathcal{L}$ of **2** infinite **non-proper** **green** and **red** spirals in $\mathbf{R^2}$ with the blue circle $\mathbf{S^1}$ as its accumulation or limit set.



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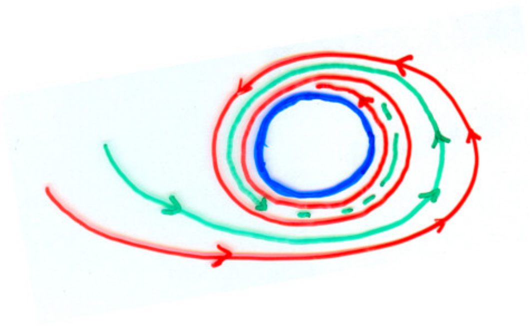
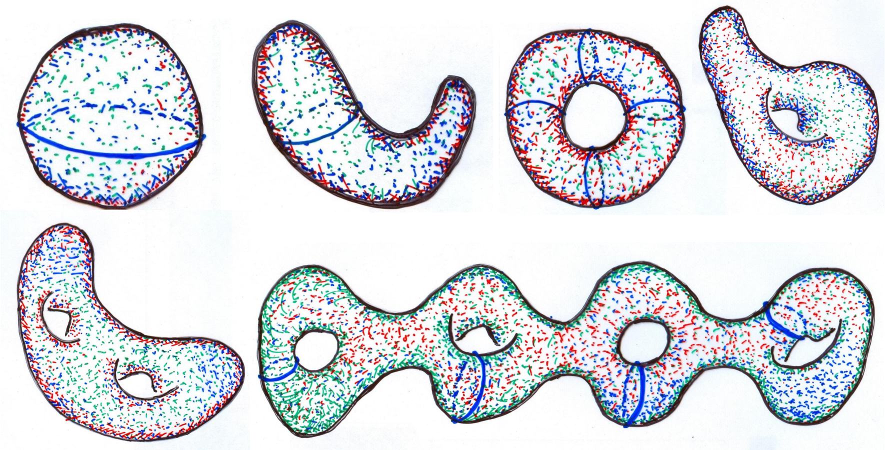


#### Example (Non-proper embedded surfaces in $\mathbf{R}^3$)

The cross product $(\mathcal{L} \times \mathbb{R}) \subset \mathbf{R}^2 \times \mathbb{R} = \mathbf{R}^3$ corresponds to 2 non-proper embedded topological planes that spiral into the cylinder $(\mathbf{S}^1 \times \mathbb{R}) \subset \mathbf{R}^3$.

## Theorem (Classification of compact surfaces in $\mathbf{R}^3$)

An embedded compact surface $\mathbf{M}$ in $\mathbf{R}^3$ is topologically equivalent to a sphere $\mathbf{S}$ with $\mathbf{g}$-handles attached. The integer $\mathbf{g} = \mathbf{genus}(\mathbf{M}) = \max \#$ of pairwise disjoint simple closed curves which do not separate $\mathbf{M}$.



## Definition

- The **genus** $\mathbf{g}$ of a surface $\mathbf{M}$ is the maximum number of pairwise disjoint simple closed curves which do not separate the surface; note that if $\mathbf{M}$ is a sphere with $\mathbf{g}$-handles attached, then it has genus $\mathbf{g}$.
- A surface $\mathbf{M}$ has **finite topology** if is topologically equivalent to a compact surface with a finite subset of points $\mathbf{E} = \{p_1, p_2, \ldots, p_n\}$ removed; $\mathbf{E}$ is called the set of **ends** of $\mathbf{M}$.
- A surface $\mathbf{M} \subset \mathbf{R}^3$ is a **planar domain** if it is topologically equivalent to a connected open set of the plane $\mathbf{R}^2$.
- An embedded surface $\mathbf{M} \subset \mathbf{R}^3$ is **complete** if with respect to its Riemannian distance function, it is a complete metric space.

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## Remark (Properness versus Completeness)

Every properly immersed surface $\mathbf{M} \looparrowright \mathbf{R}^3$ is complete, since metric spaces where every closed ball $\overline{B}_{\mathbf{M}}(\mathbf{p}, 1) \subset \mathbf{M}$ of radius $\mathbf{1}$ is compact are always complete.

## Theorem (Classification of noncompact genus $g = 0$ surfaces)

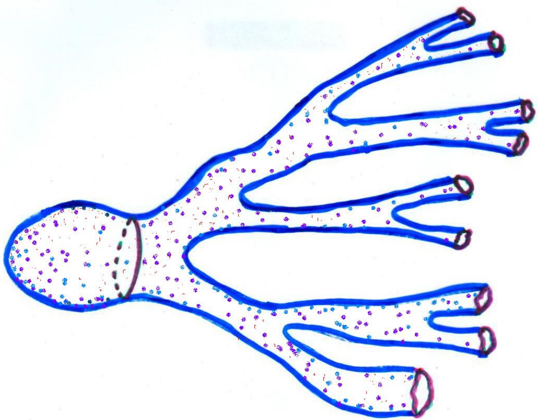
View the sphere $\mathbf{S}^2$ as $\mathbf{R}^2 \cup \{\infty\}$.

- A connected noncompact surface $\mathbf{M}_\mathbf{E}$ of **genus** $= \mathbf{0}$ can be parameterized by $\mathbf{S}^2 - \mathbf{E}$, where $\mathbf{E} \subset \mathbf{S}^1 \subset \mathbf{S}^2$ is a totally disconnected compact set called the **space of ends** of $\mathbf{M}_\mathbf{E}$. Hence:

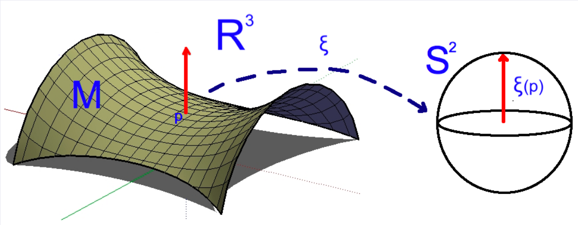
  **Noncompact genus 0 surfaces are planar domains.**

- Two planar domains $\mathbf{M}_{\mathbf{E}(1)}$, $\mathbf{M}_{\mathbf{E}(2)}$ are homeomorphic $\Leftrightarrow$ their spaces of ends $\mathbf{E}(1)$, $\mathbf{E}(2)$ are homeomorphic.

# A proper $g = 0$ surface $M_E$ with $E$ = a Cantor set.



$S^2$ - Cantor set




Let **M** be an oriented surface in $\mathbf{R}^3$, let $\xi$ be the unit vector field normal to **M**:

$$\mathbf{A_p} = -\mathbf{d}\xi \colon T_{\mathbf{p}}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

is the **shape operator** of **M**. $\mathbf{A_p}$ is symmetric linear transformation.

**Definition**

- The eigenvalues $k_1, k_2$ of $\mathbf{A_p}$ are the **principal curvatures** of $\mathbf{M}$ at $\mathbf{p}$.
- $\mathbf{K} = \det(\mathbf{A}) = k_1 k_2$ is the **Gauss curvature** function.
- $\mathbf{H} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) = \frac{k_1+k_2}{2}$ is the **mean curvature** function.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the shape operator.**

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## Gauss equation

$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K} \quad (\mathbf{K} = \text{Gaussian curvature})$$

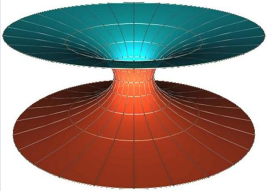
In particular:

1. When $\mathbf{H}(\mathbf{p}) = 0$, then $\mathbf{K}(\mathbf{p}) \leq 0$.
2. When $\mathbf{H}(\mathbf{p}) = \mathbf{1}$, then $\mathbf{K}(\mathbf{p}) = 2 - \frac{1}{2}|\mathbf{A}|(\mathbf{p})$, and so estimates for $|\mathbf{A}|$ give estimates on the Gaussian curvature when $\mathbf{M}$ has constant mean curvature $\mathbf{1}$.

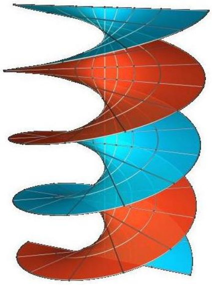
# Part 2: Introduction to the theory of H-surfaces.

**Definition**

An **H**-surface **M** is a **minimal surface** $\iff$ **H** $\equiv 0$ $\iff$ **M** is a critical point for the area functional under compactly supported variations.



- Catenoid

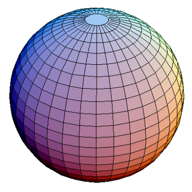


- Helicoid

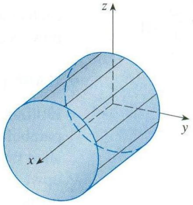
# Introduction to the theory of H-surfaces.

**Definition**

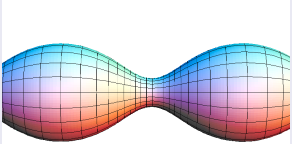
**M** is a **H-surface** $\Longleftrightarrow$ **M** is a critical point for the area functional under compactly supported variations **preserving the volume**.



- Sphere



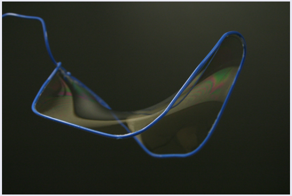

- Cylinder



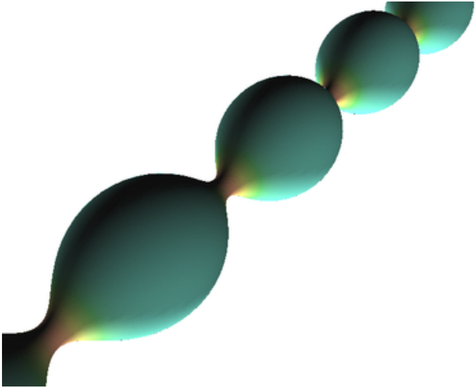
- Delaunay surfaces

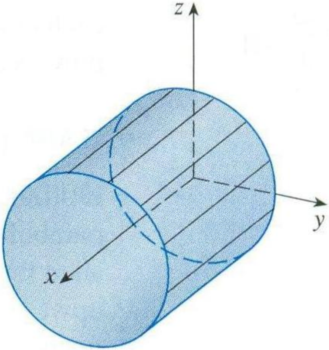
# H-surfaces in nature.

Soap films are minimal surfaces.

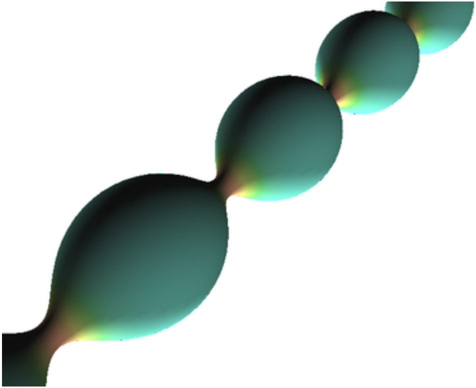


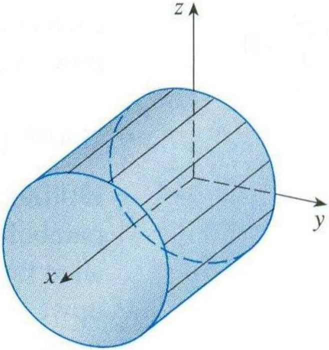
Soap bubbles are nonzero H-surfaces.






- In 1845, **Delaunay** discovered and classified the surfaces of revolution with constant mean curvature $\mathbf{H} = 1$.



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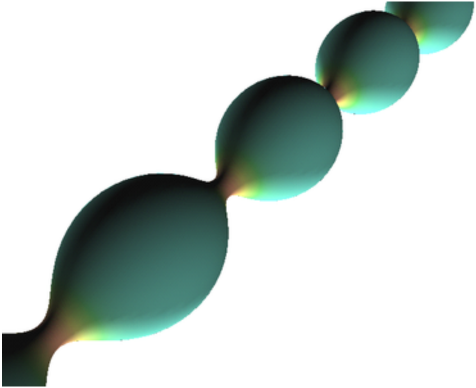
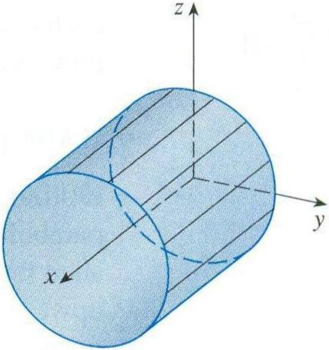
- The Sphere **S** of radius 1 and the Cylinder **C** of radius $\frac{1}{2}$ were already known.



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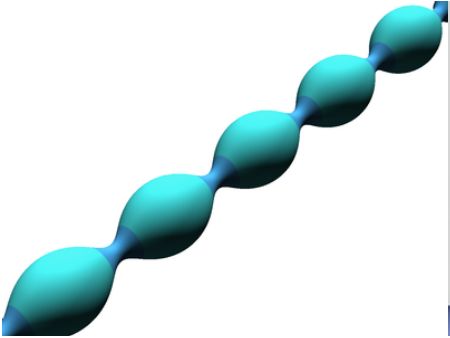
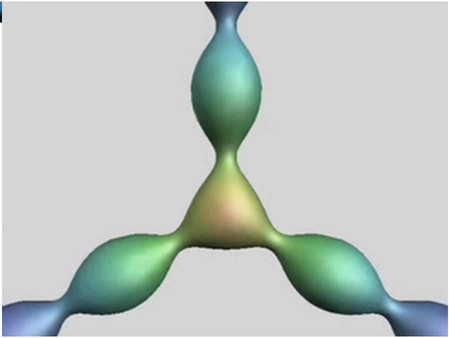
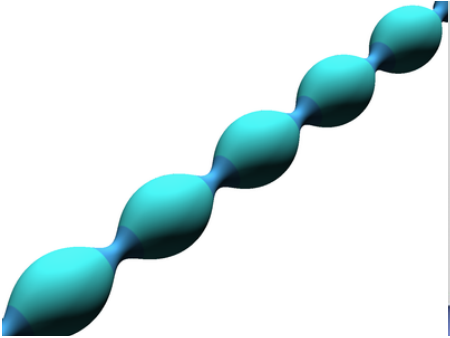
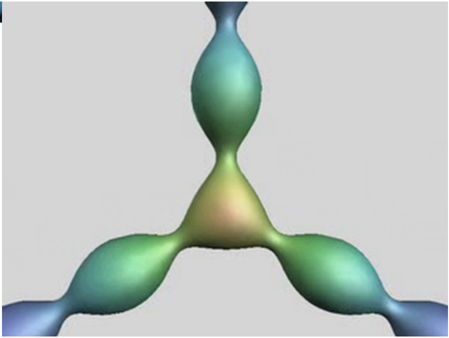
- The Sphere **S** of radius 1 and the Cylinder **C** of radius $\frac{1}{2}$ were already known.
- He wrote down a 1-parameter family $\mathcal{D}_t$, called **unduloids** or **Delaunay surfaces**, where

$$\lim_{t \to 0} \mathcal{D}_t = \mathbf{S} \qquad \lim_{t \to \infty} \mathcal{D}_t = \mathbf{C}.$$





- A different Delaunay surface and an **H**-surface called a **CMC** Trinoid or just **Trinoid**.

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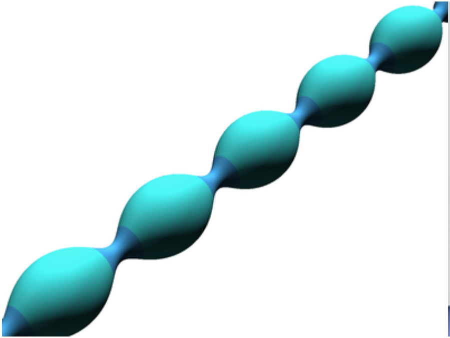

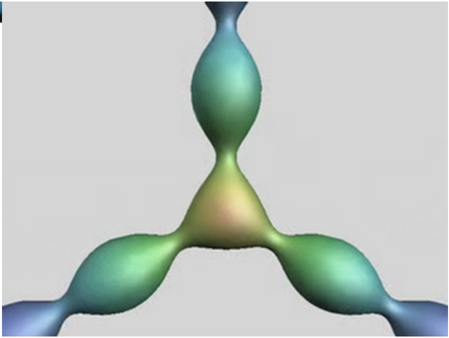

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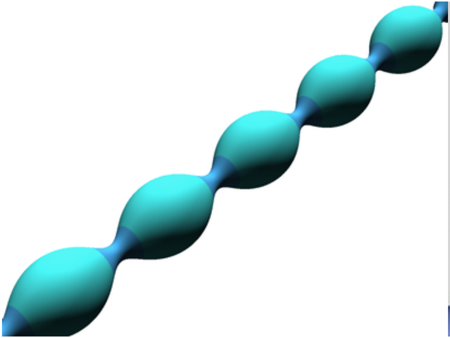

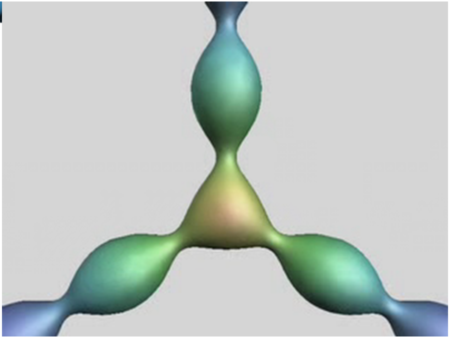

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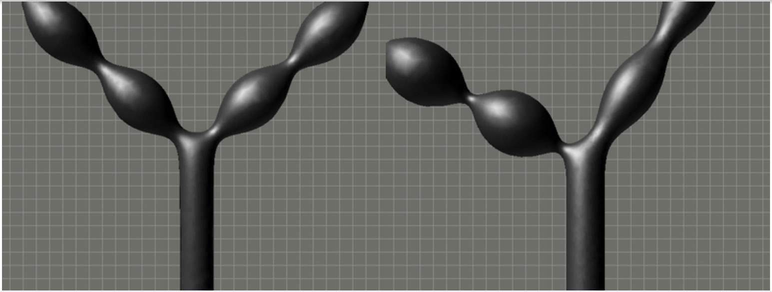
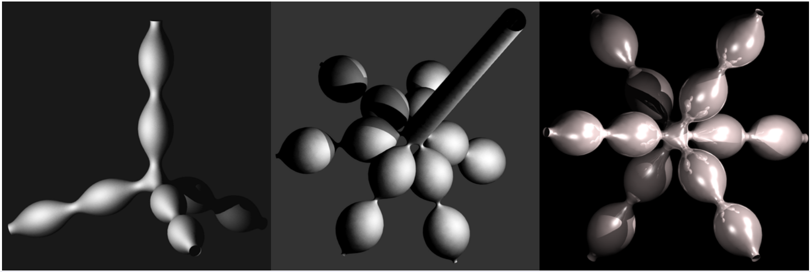
- A Trinoid is topologically a planar domain with three ends, each end is topologically an annulus, **asymptotic** to the end of a cylinder or to the end of some Delaunay surface.

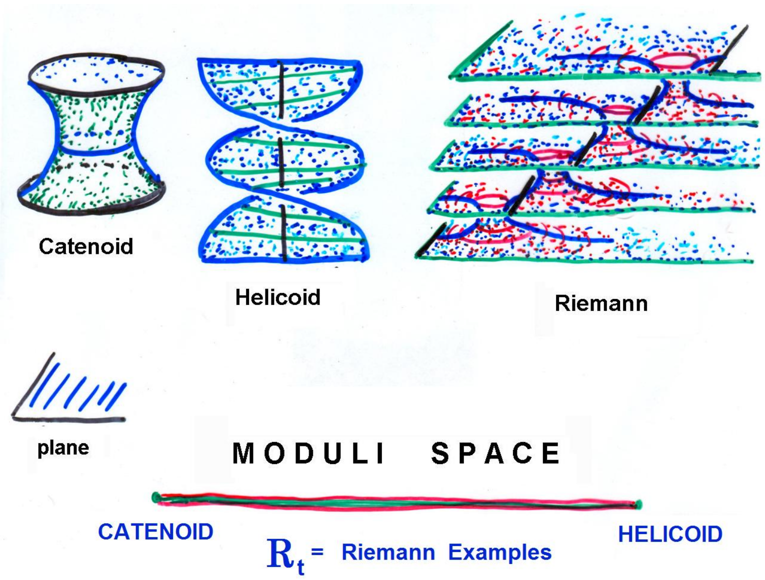


**2** Trinoids each with 1 cylindrical and 2 Delaunay-type ends.



**n**-noids with the middle one having 1 cylindrical-type end.

# Moduli space of genus-0 minimal examples - Meeks, Pérez & Ros



## Definition (Smyth and Tinaglia)

- Let $\gamma$ be a piecewise-smooth 1-cycle in an **H**-surface **M**.
- The **flux** of $\gamma$ is $\int_\gamma (\mathbf{H}\gamma + \xi) \times \dot{\gamma}$, where $\xi$ is the unit normal to **M** along $\gamma$.
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### Remark (Application of flux)

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If $\mathbf{M}_n \subset \mathbf{R}^3$ is a sequence of $\mathbf{H}_n$-disks that converge smoothly to a non-flat properly embedded minimal surface $\mathbf{M}_\infty$, then $\mathbf{M}_\infty$ is a helicoid!

## Proof.

- By curve liftings, the limit surface surface must have genus $\mathbf{0}$.
- By the classification of properly embedded minimal planar domains, $\mathbf{M}_\infty$ is a helicoid, a catenoid or a Riemann minimal example.
- The fluxes of the $\mathbf{M}_n$ are $\mathbf{0}$, so the flux of $\mathbf{M}_\infty$ is $\mathbf{0}$.
- But flux of circles on catenoids or Riemann examples $\neq \mathbf{0}$.

□

# Motivating question for the Main Results.

Do there exist complete, embedded $\mathbf{M}$ in $\mathbf{R}^3$ having constant mean curvature $\mathbf{H} \neq \mathbf{0}$ which are topologically the plane $\mathbf{R}^2$ ?

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- More generally, the work of **Meeks, Meeks-Rosenberg**, **Colding-Minicozzi**, **Collin, Lopez-Ros, Korevaar, Kusner, Solomon**, the next theorem by **Meeks-Tinaglia** completes the **classification** of complete, embedded **H**-surfaces with genus **0** and **0**, **1** or **2** ends.

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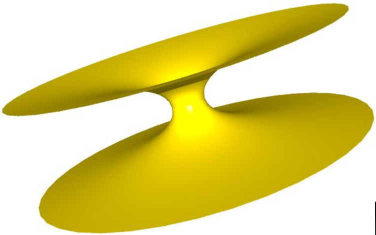
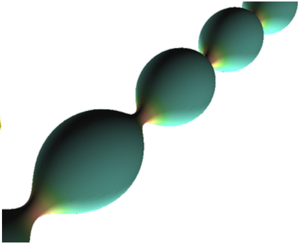
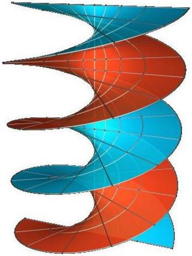
- They are:
  **Planes, spheres, catenoids, unduloids, helicoids**.







**Theorem (Radius Estimates for H-Disks, Meeks-Tinaglia 2014)**

$\exists\ \mathbf{R_0} \geq \pi$ such that every embedded **1**-disk in $\mathbf{R^3}$ has radius $< \mathbf{R_0}$.

# Part 3: Summary of the Main Results

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**Corollary (Meeks-Tinaglia 2014)**

A complete simply connected **H**-surface embedded in $\mathbf{R^3}$ with $\mathbf{H} > 0$ is a round sphere.

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**Theorem (Curvature Estimates for H-Disks, Meeks-Tinaglia 2014)**

Fix $\varepsilon > 0$ and $\mathbf{H} = 1$. $\exists\ \mathbf{C} \geq 1$ such that for every embedded **1**-disk $\mathbf{D} \subset \mathbf{R^3}$ and every $p \in \mathbf{D}$ with $\mathbf{dist_D}(p, \partial \mathbf{D}) \geq \varepsilon$,

$$|\mathbf{A_D}|(p) \leq \mathbf{C}.$$

**Theorem (One-sided curvature estimate for H-disks, Meeks-Tinaglia)**

There exist $\varepsilon \in (0, \frac{1}{2})$ and $C \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds. Let $\mathbf{\Sigma}$ be an **H**-disk such that

$$\mathbf{\Sigma} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \emptyset \quad \text{and} \quad \partial \mathbf{\Sigma} \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \emptyset.$$

Then:

$$\sup_{x \in \mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}} |\mathbf{A}_{\mathbf{\Sigma}}|(x) \leq \frac{C}{R}. \tag{1}$$

In particular, if $\mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\} \neq \emptyset$, then $\mathbf{H} \leq \frac{C}{R}$.

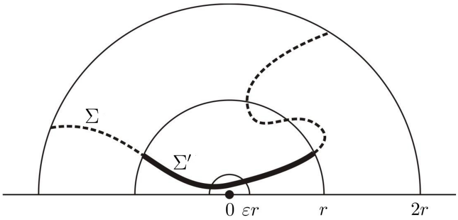


Figure: The one-sided curvature estimate.

**Theorem (One-side Curvature Estimate, Colding-Minicozzi)**

- There exists an $\varepsilon > 0$ such that the following holds.
- Given $r > 0$ and an embedded minimal disk $\mathbf{\Sigma} \subset \mathbb{B}(2r) \cap \{x_3 > 0\}$ with $\partial \mathbf{\Sigma} \subset \partial \mathbb{B}(2r)$, then for any component $\mathbf{\Sigma}'$ of $\mathbf{\Sigma} \cap \mathbb{B}(r)$ which intersects $\mathbb{B}(\varepsilon r)$,

$$r^2 \sup_{\mathbf{\Sigma}'} |\mathbf{K}_{\mathbf{\Sigma}}| \leq 1. \tag{2}$$

**Theorem (Chord-Arc Theorem, Meeks-Tinaglia)**

There exists a positive constant $C$ such that if $\boldsymbol{\Sigma} \subset \mathbf{R}^3$ is an $\mathbf{H}$-disk, $B_{\boldsymbol{\Sigma}}(\vec{0}, CR) \subset \boldsymbol{\Sigma} - \partial\boldsymbol{\Sigma}$ and $\sup_{\mathbb{B}_{\boldsymbol{\Sigma}}(\vec{0}, r_0)} |\mathbf{A}_{\boldsymbol{\Sigma}}| \geq \frac{1}{r_0}$ where $R > r_0$, then for $x \in B_{\boldsymbol{\Sigma}}(\vec{0}, R)$,

$$\frac{1}{6}\,\mathrm{dist}_{\boldsymbol{\Sigma}}(x, \vec{0}) < |x| + r_0. \tag{3}$$

- This theorem generalizes a similar result by Colding-Minicozzi for $\mathbf{H} = 0$.
- The above theorem implies that a complete simply connected $\mathbf{H}$-surface embedded in $\mathbf{R}^3$ is proper!!

Let $\mathbf{M}$ be an $\mathbf{H}$-surface properly embedded in $\mathbf{R}^3$, $\mathbf{H} > 0$.

# Theoretical results on complete embedded H-surfaces

Let **M** be an **H**-surface properly embedded in $\mathbf{R}^3$, $\mathbf{H} > 0$.

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- Recently **Meeks** and **Tinaglia** proved that if $\mathbf{\Sigma} \subset \mathbf{R^3}$ is a complete, embedded **H**-surface with finite topology, then $\mathbf{\Sigma}$ is properly embedded. **(***Proved for* $\mathbf{H} = 0$ *by* **Colding-Minicozzi**, *2008***)**

## Definition (Injectivity Radius)

- Given a Riemannian surface $\mathbf{M}$, the injectivity radius function $\mathbf{I_M}\colon \mathbf{M} \to (0, \infty]$ is defined by: $\mathbf{I_M}(\mathbf{p}) = \sup\{R > 0 \mid \exp_{\mathbf{p}}\colon B(R) \subset \mathbf{T_p M} \to \mathbf{M} \text{ is a diffeomorphism.}\}$
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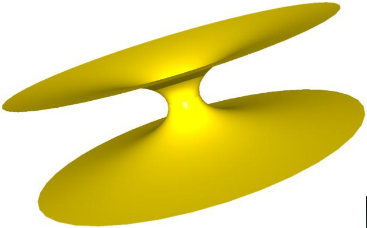
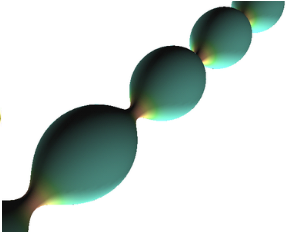
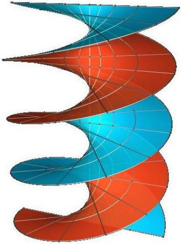
## Theorem (Meeks-Tinaglia, based on previous work of Colding-Minicozzi & Meeks-Rosenberg)

- Complete embedded $\mathbf{H}$-surfaces $\mathbf{M} \subset \mathbf{R^3}$ with finite topology have positive injectivity radius.
- Let $\mathbf{M} \subset \mathbf{R^3}$ be a complete, connected embedded $\mathbf{H}$-surface with $\mathbf{H} > 0$ and positive injectivity radius. Then $\mathbf{M}$ has bounded second fundamental form and it is properly embedded in $\mathbf{R^3}$.

- This theorem by **Meeks-Tinaglia** and work of **Meeks-Rosenberg**, **Colding-Minicozzi**, **Collin**, **Lopez-Ros** when $\mathbf{H} = 0$, and **Meeks** and **Korevaar-Kusner-Solomon** when $\mathbf{H} \neq 0$, completes the **classification** of complete, embedded **H**-surfaces of genus **0** with **0**, **1** or **2** ends.
- They are **planes, spheres, catenoids, unduloids, helicoids**.







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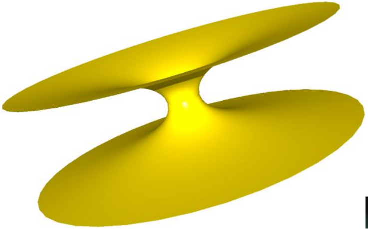
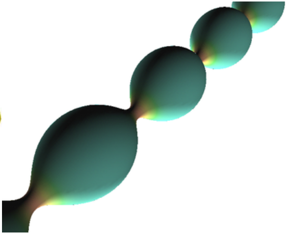
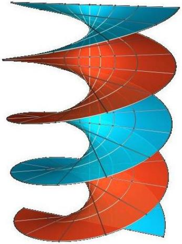

**Remark**

One **Main Objective** of this course is to present the theory behind this classification for the special case where $\mathbf{H} > 0$.