GAP THEOREMS FOR COMPLETE SELF-SHRINKERS OF r-MEAN CURVATURE FLOWS

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Dedicated to Paolo Piccione by the occasion of his 60th birthday

Abstract. In this paper, we prove gap results for complete self-shrinkers of the r-mean curvature flow involving a modified second fundamental form. These results extend previous results for self-shrinkers of the mean curvature flow due to Cao-Li and Cheng-Peng. To prove our results we show that, under suitable curvature bounds, proper self-shrinkers are parabolic for a certain second-order differential operator which generalizes the drifted Laplacian and, even if is not proper, this differential operator satisfies an Omori-Yau type maximum principle.

1. Introdution and main results

Let $X\colon \Sigma^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a n-dimensional Riemannian manifold into the Euclidean space $\mathbb{R}^{n+1}$. Let $A\colon T\Sigma^n \to T\Sigma^n$ be its shape operator given by $A(Y) = -\overline{\nabla}_Y N$, $Y \in T\Sigma^n$, where N is a locally defined normal vector field on Σ^n and $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{R}^{n+1}$. The shape operator A is symmetric and its eigenvalues $k_1, \dots, k_n$ are the principal curvatures of the hypersurface Σ^n. The elementary symmetric functions of the principal curvatures, called the r-mean curvatures of Σ, are defined by

$$
\begin{cases}
\sigma_0 = 1, \\
\sigma_r = \displaystyle\sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \quad \text{for} \quad 1 \le r \le n, \\
\sigma_r = 0, \ \text{for } r > n.
\end{cases} \tag{1.1}
$$

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These functions appear naturally in the characteristic polynomial of A, since

$$\det(A - tI) = \sigma_n - \sigma_{n-1}t + \sigma_{n-2}t^2 - \cdots + (-1)^n t^n = \sum_{j=0}^{n} (-1)^j \sigma_{n-j} t^j.$$

Observe that

$$\sigma_1 = k_1 + k_2 + \cdots + k_n,\ \sigma_2 = \sum_{i<j} k_i k_j,\ \text{and} = \sigma_n = k_1 k_2 \cdots k_n$$

are respectively the mean curvature H, the scalar curvature, and the Gauss-Kronecker curvature K. In this article, we will assume that Σ^n has a continuous globally defined unit normal vector field N.

A family of immersions $X\colon \Sigma^n \times [0,T) \to \mathbb{R}^n$ is said to be a solution of the r-mean curvature flow if satisfies the initial value problem

$$\begin{cases} \dfrac{\partial X}{\partial t}(x,t) = \sigma_r(k_1(x,t),\ldots,k_n(x,t))N(x,t), \\ X(\cdot\,,0) = X_0. \end{cases} \tag{1.2}$$

Here, $k_1(x,t),\ldots,k_n(x,t)$ are the principal curvatures of the immersions $X_t\colon = X(\cdot,t)$, $N(\cdot,t)$ are their normal vector fields. We are adopting the convention on the normal N such that in the spheres and in every closed hypersurface the normal points inward (i.e., in the direction of the region bounded by the hypersurface). With this convention, in the n-dimensional round sphere $\mathbb{S}^n(R)$ of radius R, $X = -RN$, the principal curvatures are positive and, for spheres and cylinders $\mathbb{S}^m(R) \times \mathbb{R}^{n-m}$, $1 \le m \le n-1$, the flow contracts.

The r-mean curvature flow is a natural generalization of the well-known mean curvature flow ($r = 1$) and the Gaussian curvature flow ($r = n$) that has been widely investigated in the last four decades. Beside these cases, the r-mean curvature flow can be found in the works of several authors, as [5], [9], [10], [13], [15], [23], [25], [26], [29], [30], [34], [35] and [37].

A solution $X(\cdot,t)$ of (1.2) is said homothetic if there exists a positive $\mathcal{C}^1$-function $\phi\colon [0,T) \to \mathbb{R}$ such that $\phi(0) = 1$ and

$$X(x,t) = \phi(t)X_0(x),\ \forall x \in \Sigma^n. \tag{1.3}$$

If ϕ is a decreasing function, Σ^n shrinks homothetically under the action of the flow, then Σ^n is called a self-shrinker. It can be easily proven, after rescaling, that if Σ^n is a self-shrinker of the r-mean curvature flow, then the r^{th}-symmetric function σ_r of Σ^n satisfies the equation

$$\sigma_r = -\langle X, N\rangle, \quad 1 \leq r \leq n, \tag{1.4}$$

where X is the position vector of Σ^n in $\mathbb{R}^{n+1}$.

To state the results of this paper, we recall the definition of the Newton transformations, which can be understood as a natural generalization of the second fundamental form related to the symmetric functions σ_r. Inspired by the characteristic polynomial of A we define the r-th Newton transformation $P_r \colon T\Sigma^n \to T\Sigma^n$, $0 \leq r \leq n$, as the polynomial

$$P_r = \sigma_r I - \sigma_{r-1}A + \sigma_{r-2}A^2 - \cdots + (-1)^r A^r = \sum_{j=0}^{r}(-1)^j \sigma_{r-j}A^j, \tag{1.5}$$

where we are setting $P_0 = I$. It can be seen that the Newton transformations satisfy the recurrence relation

$$P_r = \sigma_r I - P_{r-1}A, \quad 1 \leq r \leq n, \tag{1.6}$$

and, by the Cayley-Hamilton theorem, we have that $P_n = 0$.

In the context of Differential Geometry, the Newton transformations P_r first appeared in the work of Reilly [33], in the expressions of the variational integral formulas for functions $f(\sigma_0, \ldots, \sigma_n)$ of the elementary symmetric functions σ_i's. Since we are assuming that Σ^n has a global choice of N we have that P_r globally defined.

In the following, we present some basic examples of self-shrinkers of the r-mean curvature flow.

Example 1.1. Hyperplanes passing through the origin, the round sphere $\mathbb{S}^n(\delta_n(r))$ of radius $\delta_n(r) = \binom{n}{r}^{\frac{1}{r+1}}$, and the cylinders $\mathbb{S}^m(\delta_m(r)) \times \mathbb{R}^{n-m}$ in $\mathbb{R}^{n+1}$, $r \leq m \leq n-1$, are self-shrinkers of the r-mean curvature flow. For hyperplanes passing through the origin, we have the r ^{rmth}-symmetric function $\sigma_r = 0 = -\langle X, N\rangle$. On the other hand, in $\mathbb{S}^m(\delta_m(r)) \times \mathbb{R}^{n-m}$, $m \in \{0, 1, \ldots, n\}$, the principal curvatures are $k_1 = (1/\delta_m(r)) = \binom{m}{r}^{-\frac{1}{r+1}}$ with multiplicity m and $k_2 = 0$ with multiplicity $n-m$. This gives that

$$\sigma_p = \binom{m}{p}\binom{m}{r}^{-\frac{p}{r+1}}, \quad 0 \leq p \leq n, \tag{1.7}$$

where we are using the convention that $\binom{m}{k} = 0$ if $k > m$. Therefore,

$$\sigma_r = \binom{m}{r}\binom{m}{r}^{-\frac{r}{r+1}} = \binom{m}{r}^{\frac{1}{r+1}} = -\langle X, N\rangle, \tag{1.8}$$

since $\langle X, N\rangle$ equals the negative radius in spheres and cylinders. Notice that (1.8) holds only for $r \leq m \leq n$. Indeed, if $m < r$, then $\sigma_r = 0$, and thus, the respective cylinder does not satisfy the self-shrinker equation.

Observe that for each $x \in \Sigma^n$ the linear operator $P_{r-1}(x)\colon T_x\Sigma^n \to T_x\Sigma^n$ is symmetric hence $T_x\Sigma^n$ has a basis formed with eigenvectors of $P_{r-1}(x)$ associated to eigenvalues $\lambda_1(x) \leq \lambda_2(x) \leq \ldots \leq \lambda_n(x)$. Moreover, since P_{r-1} is a polynomial in A we have that $AP_{r-1} = P_{r-1}A$ and A and $P_r - 1$ are simultaneously diagonalizable. The operator P_{r-1} is positive semidefinite if $\lambda_i(x) \geq 0$, $\forall\, x \in \Sigma^n$. The square root of P_{r-1}, as the only linear operator $\sqrt{P_{r-1}}\colon T\Sigma^n \to T\Sigma^n$ such that $(\sqrt{P_{r-1}})^2 = P_{r-1}$. Let $\{e_1, \ldots, e_n\} \subset T\Sigma^n$ be an orthonormal frame of eigenvectors of A corresponding to the eigenvalues $\{k_1, k_2, \ldots, k_n\}$. Letting $A_i\colon e_i^{\perp} \to e_i^{\perp}$ to be the restriction of A to $e_i^{\perp}$, $i = 1, \ldots, n$, we have that the eigenvalues λ_i of P_{r-1} are the symmetric functions $\lambda_i = \sigma_{r-1}(A_i) = \sigma_{r-1}(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)$ associated to A_i, see [11], p.279. This gives that $\sqrt{\sigma_{r-1}(A_i)}$, $i = 1, \ldots, n$, are the eigenvalues of $\sqrt{P_{r-1}}$.

In our main results, we will consider gap theorems involving the trace norm of the modified second fundamental form $\sqrt{P_{r-1}}A$.

$$\begin{aligned}\|\sqrt{P_{r-1}}A\|^2 &= \operatorname{trace}\left((\sqrt{P_{r-1}}A)^t \cdot \sqrt{P_{r-1}}A)\right) = \operatorname{trace}\left(P_{r-1}A^2\right)\\ &= \sum_{j=1}^{n}\langle P_{r-1}A^2(e_j), e_j\rangle = \sum_{j=1}^{n}\sigma_{r-1}(A_j)k_j^2.\end{aligned}$$

Here $(\sqrt{P_{r-1}}A)^t = (\sqrt{P_{r-1}}A)$ since the operator $\sqrt{P_{r-1}}A$ is symmetric. The quantity $\|\sqrt{P_{r-1}}A\|^2$ is quite natural in Differential Geometry in the context of σ_r. It appears in the formula of the second variation of $\int_\Sigma \sigma_r d\Sigma$, see [2], p.207, Proposition 4.4, p.284 of [11], and Theorem B, p.407 of [33]. It also appears in the definition of r-special hypersurface in [2], p.203-204, as well as in the gap theorems of Alencar, do Carmo and Santos, see [3], and Alias, Brasil and Sousa, see [6].

In the next, we calculate $\|\sqrt{P_{r-1}}A\|^2$ for the basic examples. Clearly, hyperplanes satisfy $\|\sqrt{P_{r-1}}A\|^2 = 0$. In $\mathbb{S}^m\left(\delta_m(r)\right) \times \mathbb{R}^{n-m}$, for $r \leq m \leq n$, we have, using Lemma 2.1, p.279 of [11], and (1.7),

$$\begin{aligned}\|\sqrt{P_{r-1}}A\|^2 &= \operatorname{trace}(P_{r-1}A^2)\\ &= \sigma_1\sigma_r - (r+1)\sigma_{r+1}\\ &= \binom{m}{1}\binom{m}{r}^{-\frac{1}{r+1}}\binom{m}{r}\binom{m}{r}^{-\frac{r}{r+1}} - (r+1)\binom{m}{r+1}\binom{m}{r}^{-1}\\ &= m-(m-r) = r,\end{aligned}$$

where we used that $(r+1)\binom{m}{r+1} = (m-r)\binom{m}{r}$ and the convention that $\binom{m}{r} = 0$ if $r > m$. The first result of this paper is the following gap theorem.

Theorem 1.1. *Let Σ^n be a complete, n-dimensional, properly immersed, self-shrinker of the r-mean curvature flow in $\mathbb{R}^{n+1}$, $1 \le r \le n$. Suppose the $(r-1)$-th Newton transformation P_{r-1} is positive semidefinite, bounded, and satisfies*

$$\|\sqrt{P_{r-1}}A\|^2 \le r,$$

then Σ^n is

(i) *a hyperplane in $\mathbb{R}^{n+1}$ if $\|\sqrt{P_{r-1}}A\|^2 < r$;*
(ii) *the round sphere $\mathbb{S}^n(\delta_n(r))$ or the cylinder $\Sigma^n = \mathbb{S}^m(\delta_m(r)) \times \mathbb{R}^{n-m}$ in $\mathbb{R}^{n+1}$, $r \le m \le n-1$, provided P_{r-1} is positive definite. Here $\delta_m(r) = \binom{m}{r}^{\frac{1}{r+1}}$.*

Remark 1.1. Observe that for $r = 1$, Theorem 1.1 is exactly Cao and Li's result for hypersurfaces, see Theorem 1.1 of [16], since $P_{r-1} = I$ is positive definite, bounded, $\|A\|^2 \le 1$, and, for self-shrinkers of the mean curvature flow, properness is equivalent to have polynomial volume growth, see Theorem 1.3 of [20].

Corollary 1.1 (Cao-Li for hypersurfaces, [16])**.** *If $X\colon \Sigma^n \to \mathbb{R}^{n+1}$ is a complete n-dimensional self-shrinker of the mean curvature flow, without boundary and with polynomial volume growth, and satisfies*

$$\|A\|^2 \le 1,$$

then it is one of the following:

(i) *a round sphere $\mathbb{S}^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$;*
(ii) *a cylinder $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ in $\mathbb{R}^{n+1}$, $1 \le m \le n-1$;*
(iii) *a hyperplane in $\mathbb{R}^{n+1}$.*

In particular, if $\|A\|^2 < 1$, then Σ^n is a hyperplane. Here, $\|A\|^2$ is the squared norm of the second fundamental form of Σ^n.

For $r = n$, the hypersurface Σ^n is a self-shrinker of the Gaussian curvature flow. By (1.6), it holds $KI = P_{n-1}A$, i.e.,

$$K = \langle Ke_i, e_i\rangle = \langle P_{n-1}Ae_i, e_i\rangle = k_i\sigma_{n-1}(A_i), \tag{1.9}$$

for every $i = 1, \ldots, n$.

We claim that P_{n-1} be positive semidefinite is equivalent to a choice of orientation when Σ^n is weakly convex, meaning, A is positive semidefinite. Indeed, by (1.9), if P_{n-1} positive semidefinite then $\sigma_{n-1}(A_i) \geq 0$, for all $i = 1, \ldots, n$. This gives that each k_i has the same sign of K, in particular, they have the same sign. The converse is also true. On the other hand, since $\mathrm{trace}(P_{r-1}A^2) = \sigma_1\sigma_r - (r+1)\sigma_{r+1}$ (see Lemma 2.1, p.279 of [11]), if $r = n$, then $\sigma_{n+1} = 0$ and

$$\|\sqrt{P_{n-1}}A\|^2 = \mathrm{trace}(P_{n-1}A^2) = HK \geq 0.$$

Therefore, since $\mathrm{trace}(P_{n-1}) = \sigma_{n-1}$, we have the following

Corollary 1.2. *Let Σ^n be a complete, n-dimensional, properly immersed, weakly convex, self-shrinker of the Gaussian curvature flow in $\mathbb{R}^{n+1}$. If σ_{n-1} is bounded and*

$$HK \leq n,$$

then Σ^n is one of the following:

(i) *the unitary round sphere $\mathbb{S}^n(1)$;*
(ii) *a hyperplane in $\mathbb{R}^{n+1}$.*

In particular, if $HK < n$, then Σ^n is a hyperplane in $\mathbb{R}^{n+1}$.

If we remove the properness condition of the hypotheses of Theorem 1.1 we obtain

Theorem 1.2. *Let Σ^n be a complete n-dimensional self-shrinker of the r-mean curvature flow in $\mathbb{R}^{n+1}$, for $1 \leq r \leq n$. If the $(r-1)$-th Newton transformation P_{r-1} is positive semidefinite,*

$$\sup \|A\|^2 < \infty, \quad \textit{and} \quad \sup \|\sqrt{P_{r-1}}A\|^2 < r,$$

then Σ^n is a hyperplane in $\mathbb{R}^{n+1}$.

For $r = 1$ we extend the result of Cheng and Peng for hypersurfaces, see Theorem 1.1 of [18]:

Corollary 1.3 (Cheng-Peng for hypersurfaces, [18])**.** *If Σ^n is a complete n-dimensional self-shrinker of the mean curvature flow in $\mathbb{R}^{n+1}$, then one of the following holds:*

(i) $\sup \|A\|^2 \geq 1$;

(ii) *or* $\|A\| = 0$ *and Σ^n is a hyperplane in $\mathbb{R}^{n+1}$.*

In particular, if $\sup \|A\|^2 < 1$, *then Σ^n is a hyperplane in $\mathbb{R}^{n+1}$.*

Remark 1.2. Notice that the hypothesis $\|\sqrt{P_{r-1}}A\|^2 \leq r$ in Theorem 1.1 and Theorem 1.2 does not give any natural bounds on the second fundamental form for $r > 1$, unlike Cao-Li's and Cheng-Peng's results. This drives us to impose new barriers to control the geometry and obtain the classification.

Remark 1.3. Cheng and Zhou [21], see Corollary 4 proved that complete self-shrinkers (in arbitrary codimension) of the mean curvature flow whose principal curvatures satisfy $\sup_{1\leq i\leq n} k_i^2 \leq \delta < 1$, for some constant $\delta \geq 0$, are properly immersed, have finite weighted volume, and have polynomial volume growth. Since $\sup_{1\leq i\leq n} k_i^2 \leq \|A\|^2$, if we assume that $\sup \|A\|^2 < 1$, then, taking $\delta = \sup \|A\|^2$ and using the result of Cheng and Zhou, we conclude that the self-shrinker in the hypothesis of the result of Cheng and Peng is indeed properly immersed. We also point out that the equivalence between properness and polynomial volume growth in [21] holds in a more general context, see [22].

Taking $r = n$, then we obtain the following result for self-shrinkers of the Gaussian curvature flow:

Corollary 1.4. *Let Σ^n be a n-dimensional, complete, weakly convex, self-shrinker of the Gaussian curvature flow. If*

$$\sup \|A\|^2 < \infty \quad \textit{and} \quad \sup HK < n,$$

then Σ^n is a hyperplane in $\mathbb{R}^{n+1}$.

Remark 1.4. Recently, Batista and Xavier proved in [12] results in the same direction of Theorems 1.1 and 1.2 assuming some additional hypotheses, besides assuming weak convexity, i.e., the second fundamental form is positive semidefinite. They proved that,

(i) if Σ^n is compact (without bondary), weakly convex and

$$\operatorname{trace}(P_{r-1}A^2) \leq r, \quad 1 \leq r \leq n,$$

then Σ^n is a sphere (Theorem A);

(ii) if Σ^n is complete, weakly convex, σ_1 is bounded and

$$\operatorname{trace}(P_{r-1}A^2) < r, \quad 1 \leq r \leq n,$$

then Σ^n is a hyperplane in $\mathbb{R}^{n+1}$ (Theorem B).

Notice that Theorem A is an immediate corollary of Theorem 1.1 item (i) and Theorem B is a corollary of Theorem 1.2, since $A \geq 0$ and σ_1 bounded imply that all the principal curvatures are nonnegative and bounded, which gives that P_{r-1} is positive semidefinite and bounded, but the converse is not necessarily true.

Remark 1.5. There are some conditions to deduce that P_{r-1} is positive semidefinite on a connected hypersurface. In the following, we point out some of them:

(i) if $\sigma_r = 0$, then P_{r-1} is semidefinite. If $r-1$ is odd, then we can choose an orientation such that P_{r-1} is positive semidefinite and, if $r-1$ is even and $\sigma_{r-1} \geq 0$, then P_{r-1} is positive semidefinite;

(ii) if $\sigma_r = 0$, and $\sigma_{r+1} \neq 0$, then P_{r-1} is definite. If $r-1$ is odd, then we can choose an orientation such that P_{r-1} is positive definite and, if $r-1$ is even and $\sigma_{r-1} \geq 0$, then P_{r-1} is positive definite;

(iii) if $\sigma_k > 0$ for some $1 \leq k \leq m-1$ and there exists a point where all the principal curvatures are nonnegative, then P_r is positive definite for every $1 \leq r \leq k-1$.

The proof of item (i) is a consequence of Lemma 1.1 and Equation (1.3) of [27], p.250-251, and a direct proof can be found in [4], Proposition 2.4, p.188-189. In its turn, the proof of item (ii) can be found [28], Proposition 1.5, p.873, and the proof of item (iii) can be found in [11], Proposition 3.2, p.280-281 (see also [19], Proposition 3.2, p.188).

This paper is organized as follows: in Section 2 we prove Theorem 1.1 using techniques of parabolicity for a certain second-order differential operator which generalizes the drifted Laplacian, while Section 3 is devoted to the proof of Theorem 1.2 by using an Omori-Yau type maximum principle. for the same differential operator.

2. Proof of Theorem 1.1

Let $X\colon \Sigma^n \to \mathbb{R}^{n+1}$ be a hypersurface and $f\colon \Sigma^n \to \mathbb{R}$ be a smooth function. Define the second-order differential operator

$$L_r f = \mathrm{trace}(P_r \,\mathrm{hess}\, f), \quad 0 \le r \le n-1, \tag{2.1}$$

where $\mathrm{hess}\, f(v) = \nabla_v \nabla f$ is the hessian operator and ∇f is the gradient of f on Σ^n. It can be proved that $L_r f = \mathrm{div}(P_r(\nabla f))$, see Proposition B on page 470 of [33]. We also define drifted-L_r operator by

$$\mathcal{L}_r f = L_r f - \langle X, \nabla f\rangle, \quad 0 \le r \le n-1, \tag{2.2}$$

where X is the position vector field.

Definition 2.1 (Def. 4.2, [8] p.243)**.** The operator $\mathcal{L}_r$ is strongly parabolic on Σ^n if for each nonconstant $u \in C^2(\Sigma^n)$ with $u^* = \sup_{\Sigma^n} u < +\infty$ and for each $\eta \in \mathbb{R}$ with $\eta < u^*$ we have

$$\inf_{\Omega_\eta} \mathcal{L}_r(u) < 0,$$

where $\Omega_\eta = \{x \in \Sigma^n \colon u(x) > \eta\}$.

The Khasminskii Test (Theorem 4.12 of [8]) gives sufficient conditions to guarantee strong parabolicity for the operator $\mathcal{L}_r$ on Σ^n if P_r is positive definite. However, in this article, we mostly consider positive semidefinite Newton transformations. In this case, following verbatim the proof of the Kashminskii test in [8] to $\mathcal{L}_r$ when P_r is positive semidefinite and we have the following statement.

Proposition 2.1. *Assume the existence of a function $\gamma \in C^2(\Sigma^n)$ such that*

$$\begin{cases} \gamma(x) & \to & +\infty & \text{as } x \to \infty, \\ \mathcal{L}_r \gamma & < & 0 & \text{off a compact set,} \end{cases} \tag{2.3}$$

where we are assuming that P_r is positive semidefinite. If $u \in \mathcal{C}^2(\Sigma^n)$ is not constant and satisfies $u^* = \sup_\Sigma u < \infty$, then u achieves its maximum at a point $z_0 \in \Sigma^n$ or

$$\inf_{B_\eta} \mathcal{L}_r u < 0,$$

for every $0 < \eta < u^*$*, where* $B_\eta = \{x \in \Sigma^n; u(x) > u^* - \eta\}$*. In particular, if* $\mathcal{L}_r u \geq 0$ *and* u *does not achieve its maximum, then* u *is constant. In addition, if* P_r *is positive definite, then* $\mathcal{L}_r$ *is strong parabolic* Σ^n*.*

Remark 2.1. In the proof of the Khasminskii test, the necessity to P_r to be positive definite is to show that (see Theorem 3.10 of [8]) that u can not achieve its maximum at a finite point z_0.

Proof. Assume that u^* can not be achieved in any point $z_0 \in \Sigma^n$. Let us prove that, given $u \in \mathcal{C}^2(\Sigma^n)$ with $u^* > 0$ and $0 < \eta < u^*$ fixed, but arbitrary, it holds

$$\inf_{B_\eta} \mathcal{L}_r u < 0,$$

where $B_\eta = \{x \in \Sigma^n; u(x) > u^* - \eta\}$. Suppose by contradiction that $\mathcal{L}_r u \geq 0$ on B_η. Let

$$\Omega_t = \{x \in \Sigma \colon \gamma(x) > t\} \tag{2.4}$$

and

$$\Omega_t^c = \{x \in \Sigma \colon \gamma(x) \leq t\} \tag{2.5}$$

be its complement. Notice that, since $\gamma(x) \to \infty$ when $x \to \infty$, then Ω_t^c is compact. In particular, there exists $u_t^* = \max_{\Omega_t^c} u(x)$. Notice that $\{\Omega_t^c\}_{t \in \mathbb{R}}$ is an exhaustion of Σ^n, since

$$\bigcup_{t \in \mathbb{R}} \Omega_t^c = \Sigma^n \quad \text{and} \quad \Omega_{t_1}^c \subset \Omega_{t_2}^c \quad \text{for} \quad t_1 < t_2.$$

Moreover, it holds $u_{t_1}^* \leq u_{t_2}^*$ if $t_1 < t_2$. Since u^* is not achieved, there exists a divergent sequence $t_j \to \infty$ such that $u_{t_j}^* \to u^*$. Thus, we can choose $T_1 > 0$ sufficiently large such that

$$u_{T_1}^* > u^* - \frac{\eta}{2}. \tag{2.6}$$

Now, let $\alpha \in \mathbb{R}$ such that

$$u_{T_1}^* < \alpha < u^*. \tag{2.7}$$

Since $u_{t_j}^* \to u^*$, we can find $T_2 > T_1$ such that

$$u_{T_2}^* > \alpha. \tag{2.8}$$

Select $\bar{\eta} > 0$ small enough in order to have

$$\alpha + \bar{\eta} < u_{T_2}^*. \tag{2.9}$$

For every $\delta > 0$ small, define

$$\gamma_\delta(x) = \alpha + \delta(\gamma(x) - T_1). \tag{2.10}$$

Since $\Omega_{t_1} \supset \Omega_{t_2}$ for $t_1 < t_2$, the function γ_δ satisfies the following properties:

(i) $\gamma_\delta(x) = \alpha$ for every $x \in \partial\Omega_{T_1}$;

(ii) $\mathcal{L}_r\gamma_\delta = \delta\mathcal{L}_r\gamma < 0$ on Ω_{T_1} for T_1 large enough (by hypothesis);

(iii) $\alpha < \gamma_\delta(x) \leq \alpha + \delta(T_2 - T_1)$ on $\Omega_{T_1}\backslash\Omega_{T_2}$, since $T_1 < \gamma(x) \leq T_2$ on $\Omega_{T_1}\backslash\Omega_{T_2}$.

Choosing $\delta > 0$ small enough such that $\delta(T_2 - T_1) < \bar{\eta}$ and by using (iii), we have

$$\alpha < \gamma_\sigma(x) < \alpha + \bar{\eta} \quad \text{on} \quad \Omega_{T_1}\backslash\Omega_{T_2}. \tag{2.11}$$

Since

$$\gamma_\delta(x) = \alpha > u^*_{T_1} \geq u(x) \quad \text{on} \quad \partial\Omega_{T_1},$$

we have

$$(u - \gamma_\delta)(x) \leq 0 \quad \text{on} \quad \partial\Omega_{T_1}. \tag{2.12}$$

On the other hand, since

$$\Omega_{T_1}\backslash\Omega_{T_2} = \{x \in \Sigma^n; T_1 < \gamma(x) \leq T_2\} \subset \Omega^c_{T_2}$$

and using the divergence of the sequence by taking T_1 large enough, there exists $\bar{x} \in \Omega_{T_1}\backslash\Omega_{T_2}$ such that $u(\bar{x}) = u^*_{T_2}$. This implies

$$\begin{aligned}(u - \gamma_\delta)(\bar{x}) &= u^*_{T_2} - \alpha - \delta(\gamma(x) - T_1)\\ &> u^*_{T_2} - \alpha - \delta(T_2 - T_1)\\ &> u^*_{T_2} - \alpha - \bar{\eta} > 0,\end{aligned} \tag{2.13}$$

where we used the definition of γ_δ, the fact that $\bar{x} \in \Omega_{T_1}\backslash\Omega_{T_2}$, (2.11), and (2.9). Notice that, since $u^* < \infty$ and $\gamma(x) \to \infty$ when $x \to \infty$, it holds

$$(u - \gamma_\delta)(x) < 0 \quad \text{on} \quad \Omega_{T_3} \tag{2.14}$$

for $T_3 > T_2$ sufficiently large. Thus, by (2.13) and (2.14) we conclude that there exists a positive maximum of $u - \gamma_\delta$ achieved at a point $z_0 \in \overline{\Omega}_{T_1}\backslash\Omega_3$. In particular, since P_r is positive semidefinite, it holds

$$\mathcal{L}_r(u - \gamma_\delta)(z_0) \leq 0.$$

But notice that $z_0 \in B_\eta$. Indeed, $z_0 \in \Omega_{T_1}$ and

$$\begin{aligned} u(z_0) > \gamma_\delta(z_0) &= \alpha + \delta(\gamma(z_0) - T_1) \\ &> \alpha > u^*_{T_1} > u^* - \frac{\eta}{2} > u^* - \eta. \end{aligned}$$

Therefore, since $z_0 \in B_\eta$, it holds, at z_0,

$$0 \leq \mathcal{L}_r u \leq \mathcal{L}_r \gamma_\delta = \delta \mathcal{L}_r \gamma < 0.$$

This contradiction concludes the proof. In particular, if $u \in \mathcal{C}^2(\Sigma^n)$ such that $\mathcal{L}_r u \geq 0$ with $u^* < \infty$ then then either $u(z_0) = u^*$ for some $z_0 \in \Sigma^n$ or u must be a constant function. Moreover, if P_r positive definite, then $\mathcal{L}_r$ is an elliptic operator. Thus, by the generalized Hopf maximum principle Theorem 3.10 of [8], any $\mathcal{L}_r$-subharmonic function $u \in \mathcal{C}^2(\Sigma^n)$, bounded above, can not achieves its maximum unless it is constant. Therefore, u does not achieve its maximum and the rest of the proof implies that $\mathcal{L}_r$ is strongly parabolic. □

Our next result shows that, under fairly mild geometric assumptions, Σ^n satisfies Khasminskii's conditions (2.3) for the operator $\mathcal{L}_{r-1}$.

Proposition 2.2. *Let $X\colon \Sigma^n \to \mathbb{R}^{n+1}$ be a complete properly immersed self-shrinker of the r-mean curvature flow. If there exists $0 < c < 1$, such that*

$$(n - r + 1) \limsup_{x\to\infty} \frac{\sigma_{r-1}(x)}{\|X(x)\|^2} \leq c, \tag{2.15}$$

then the function $\gamma(x) = \|X(x)\|^2$ satisfies the Khasminskii's conditions (2.3) of Proposition 2.1 for the operator $\mathcal{L}_{r-1}$. In particular, if $u \in \mathcal{C}^2(\Sigma^n)$ is bounded above and sastisfies $\mathcal{L}_{r-1}u \geq 0$, then u achieves its maximum or u is constant. Moreover, if P_{r-1} is positive definite, then $\mathcal{L}_{r-1}$ is strong parabolic Σ^n.

Proof. Since the immersion is proper, the function $\gamma(x) = \|X(x)\|^2 \to \infty$ when $x \to \infty$. On the other hand, using Lemma 1, p.208, of [1], we have that

$$\begin{aligned} \frac{1}{2} L_{r-1}\|X\|^2 &= (n - r + 1)\sigma_{r-1} + r\sigma_r\langle X, N\rangle \\ &= (n - r + 1)\sigma_{r-1} - r\langle X, N\rangle^2. \end{aligned}$$

This gives

$$\begin{aligned}\frac{1}{2}\mathcal{L}_{r-1}\|X\|^2 &= (n-r+1)\sigma_{r-1} - r\langle X, N\rangle^2 - \langle \nabla\|X\|^2, X\rangle \\ &= (n-r+1)\sigma_{r-1} - r\langle X, N\rangle^2 - \|X^\top\|^2 \\ &= (n-r+1)\sigma_{r-1} - (r-1)\langle X, N\rangle^2 - \|X\|^2 \\ &\leq (n-r+1)\sigma_{r-1} - \|X\|^2 \\ &= \left[(n-r+1)\frac{\sigma_{r-1}}{\|X\|^2} - 1\right]\|X\|^2 \\ &\leq (c-1)\|X\|^2 < 0,\end{aligned}$$

outside a suitable compact set. □

In the following lemma, we show that, for self-shrinkers of the r-mean curvature flow, σ_r satisfies a second-order partial differential equation, that is (semi-)elliptic if P_{r-1} is positive (semi)definite:

Proposition 2.3. *Let $X\colon \Sigma^n \to \mathbb{R}^{n+1}$ be a self-shrinker of the r-mean curvature flow, i.e., a hypersurface such that $\sigma_r = -\langle X, N\rangle$. Then*

$$\mathcal{L}_{r-1}\sigma_r + \left[\|\sqrt{P_{r-1}}A\|^2 - r\right]\sigma_r = 0. \tag{2.16}$$

Here, N is the unit normal vector field of the immersion X. Moreover, if P_{r-1} is positive semidefinite and $\|\sqrt{P_{r-1}}A\|^2 \leq r$, then

$$\frac{1}{2}\mathcal{L}_{r-1}\sigma_r^2 = \sigma_r^2\left[r - \|\sqrt{P_{r-1}}A\|^2\right] + \langle P_{r-1}(\nabla\sigma_r), \nabla\sigma_r\rangle \geq 0. \tag{2.17}$$

Proof. By Lemma 2, p. 209, of [1], we have, for $1 \leq r \leq n-1$,

$$L_{r-1}\langle X, N\rangle = -r\sigma_r - (\sigma_1\sigma_r - (r+1)\sigma_{r+1})\langle X, N\rangle - \langle \nabla\sigma_r, X\rangle. \tag{2.18}$$

Since Σ^n satisfies $\sigma_r = -\langle X, N\rangle$ and by Lemma 2.1, p.279, of [11],

$$\sigma_1\sigma_r - (r+1)\sigma_{r+1} = \mathrm{trace}(P_{r-1}A^2) = \|\sqrt{P_{r-1}}A\|^2,$$

we obtain

$$L_{r-1}\sigma_r = r\sigma_r - \|\sqrt{P_{r-1}}A\|^2\sigma_r + \langle \nabla\sigma_r, X\rangle,$$

i.e.,

$$\mathcal{L}_{r-1}\sigma_r = -\left[\|\sqrt{P_{r-1}}A\|^2 - r\right]\sigma_r. \tag{2.19}$$

On the other hand, since L_{r-1} satisfies

$$L_{r-1}(fg) = fL_{r-1}g + gL_{r-1}f + 2\langle P_{r-1}(\nabla f), \nabla g\rangle, \tag{2.20}$$

it holds

$$\mathcal{L}_{r-1}(fg) = f\mathcal{L}_{r-1}g + g\mathcal{L}_{r-1}f + 2\langle P_{r-1}(\nabla f), \nabla g\rangle. \tag{2.21}$$

Thus, by (2.16) and (2.21) we have

$$\begin{aligned}\frac{1}{2}\mathcal{L}_{r-1}\sigma_r^2 &= \sigma_r\mathcal{L}_{r-1}\sigma_r + \langle P_{r-1}(\nabla\sigma_r), \nabla\sigma_r\rangle \\ &= \sigma_r^2\left[r - \|\sqrt{P_{r-1}}A\|^2\right] + \langle P_{r-1}(\nabla\sigma_r), \nabla\sigma_r\rangle \geq 0.\end{aligned} \tag{2.22}$$

□

Lemma 2.1. *Let $X\colon \Sigma^n \to \mathbb{R}^{n+1}$ be a hypersurface and $f\colon \Sigma^n \to \mathbb{R}$ be a $C^2(\Sigma^n)$-function. Suppose that P_r is positive semidefinite and x_0 is a point of maximum of f. Then*

$$L_r f(x_0) = \operatorname{trace}(P_r \operatorname{hess} f(x_0) \leq 0. \tag{2.23}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{x_0}\Sigma^n$ formed with eigenvalues of $P_r(x_0)$ with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\begin{aligned}\operatorname{trace}(P_r \operatorname{hess} f)(x_0) &= \sum_{i=1}^n \langle P_r \operatorname{hess} f(e_i), e_i\rangle \\ &= \sum_{i=1}^n \langle \operatorname{hess} f(e_i), P_r e_i\rangle \\ &= \sum_{i=1}^n \lambda_i \operatorname{Hess}(e_i, e_i)(x_0) \\ &\leq 0\end{aligned}$$

Since at a point of maximum $\operatorname{Hess}(e_i, e_i)(x_0) \leq 0$. □

Proof of Theorem 1.1. Using the Cauchy-Schwarz inequality for matrices,

$$[\operatorname{trace}(BC^t)]^2 \leq \operatorname{trace}(BB^t)\operatorname{trace}(CC^t)$$

for B and C matrices, where $(\ )^t$ denotes the transpose of a matrix, we have

$$\begin{aligned}[\operatorname{trace}(P_{r-1}A)]^2 &= [\operatorname{trace}(\sqrt{P_{r-1}}(\sqrt{P_{r-1}}A))]^2 \\ &\leq \operatorname{trace}(P_{r-1})\operatorname{trace}(P_{r-1}A^2),\end{aligned}$$

since A, $\sqrt{P_{r-1}}$, and $\sqrt{P_{r-1}}A$ are symmetric matrices that commute with each other. By hypothesis, $\|\sqrt{P_{r-1}}A\|^2 = \operatorname{trace}(P_{r-1}A^2) \leq r$, P_{r-1} is bounded and by Lemma 2.1, p.279, of [11], $\operatorname{trace}(P_{r-1}A) = r\sigma_r$, we have

$$r^2\sigma_r^2 \leq \operatorname{trace}(P_{r-1})\operatorname{trace}(P_{r-1}A^2) \leq r\operatorname{trace}(P_{r-1}) < \infty, \tag{2.24}$$

i.e., σ_r^2 is a bounded function. Moreover, by Equation (2.17), $\mathcal{L}_{r-1}\sigma_r^2 \geq 0$.

Since P_{r-1} is bounded and $\mathrm{trace}(P_{r-1}) = (n-r+1)\sigma_{r-1}$, it holds that σ_{r-1} is bounded. This gives that

$$\limsup_{x\to\infty} \frac{\sigma_{r-1}(x)}{\|X(x)\|^2} = 0,$$

provided Σ^n is assumed to be properly immersed. Therefore, by Proposition 2.2, p.12, σ_r^2 achieves a maximum point $x_0 \in \Sigma^n$ or σ_r^2 is constant. If Σ^n is compact σ_r^2 has a maximum point. If σ_r^2 achieves its maximum at x_0, then $\nabla\sigma_r^2(x_0) = 0$ and, by (2.17) and (2.23),

$$0 \geq \frac{1}{2}\mathcal{L}_{r-1}(\sigma_r^2)(x_0) = \sigma_r^2(x_0)\left[r - \|\sqrt{P_{r-1}}A\|(x_0)^2\right] \geq 0. \tag{2.25}$$

Therefore, $\sigma_r^2(x_0) = 0$ or $\|\sqrt{P_{r-1}}A\|^2(x_0) = r$.

If $\|P_{r-1}A\|^2 < r$ then $\sigma_r^2 \equiv 0$ since $\sigma_r^2 \geq 0$. Thus, $\langle X, N\rangle = 0$ and Σ^n is a hyperplane. On the other hand, if P_{r-1} is positive definite, then, by Proposition 2.2, $\mathcal{L}_{r-1}$ is strongly parabolic. Therefore, since σ_r^2 is bounded and $\mathcal{L}_{r-1}\sigma_r^2 \geq 0$, we can conclude that σ_r^2 is constant. By Theorem 1 of [24], the hypersurfaces of $\mathbb{R}^{n+1}$ with constant support function $\langle X, N\rangle$ are $\Sigma^n = \mathbb{S}^m(R) \times \mathbb{R}^{n-m}$, where $0 \leq m \leq n$, for an appropriate radius R. Here we are considering that $\Sigma^n = \mathbb{R}^n$ is a hyperplane, for $m = 0$, and $\Sigma^n = \mathbb{S}^n(R)$ is the round sphere, for $m = n$. Since the principal curvatures of $\mathbb{S}^m(R) \times \mathbb{R}^{n-m}$ are $k_1 = 1/R$, with multiplicity m, and $k_2 = 0$, with multiplicity $n-m$, we have that

$$\sigma_r = \binom{m}{r}\frac{1}{R^r}, \tag{2.26}$$

where we are adopting the convention that $\binom{m}{r} = 0$ if $r > m$. Since, for $1 \leq m \leq n$, it holds $\langle X, N\rangle = -R$ in these surfaces, from the self-shrinker equation $\sigma_r = -\langle X, N\rangle$ and using (2.26), we obtain that

$$R = \binom{m}{r}^{\frac{1}{r+1}}. \tag{2.27}$$

The Example 1.1, p.3, shows us the sphere (for $m = n$) and cylinders (for $1 \leq m \leq n-1$) with radius given in (2.27) satisfy $\|\sqrt{P_{r-1}}A\|^2 = r$.

□

Remark 2.2. If $r = 1$, we have $P_{r-1} = I$ is naturally positive definite and $\mathcal{L}_{r-1} = \mathcal{L} := \Delta - \langle X, \nabla\cdot\rangle$, the so called drifted Laplacian, is parabolic. Thus, under the hypothesis, we can conclude that σ_1^2 is constant. This gives an alternative proof of Cao-Li result for hypersurfaces, see Corollary 1.1, p.5.

Proof of Corollary 1.2. In the case $r = n$, if σ_n^2 achieves a maximum at $x_0 \in \Sigma^n$, then, by (2.22), $\sigma_n(x_0)^2 = 0$ or $\|P_{n-1}A\|^2(x_0) = H(x_0)K(x_0) = n$. In the first case, we have that $\sigma_n^2 \equiv 0$, which gives that $\langle X, N\rangle = 0$ and Σ^n is a hyperplane of $\mathbb{R}^{n+1}$. In the second case, by (1.9), p.6, $\sigma_{n-1}(A_i) \neq 0$ at x_0 for every $i = 1, \ldots, n$. Thus, P_{n-1} is positive definite in a neighborhood of x_0 and, by 2.22, $\mathcal{L}_{n-1}\sigma_n^2 > 0$. Therefore, by the classical Hopf maximum principle, σ_n^2 is constant. The results comes following the conclusion of the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Let Σ^n be a n-dimensional Riemannian manifold, $f : \Sigma^n \to \mathbb{R}$ be a class $\mathcal{C}^2$ function, and $\phi : T\Sigma^n \to T\Sigma^n$ be a linear symmetric tensor. Define the second-order differential operator

$$\mathcal{L}_\phi f := \mathrm{trace}(\phi\, \mathrm{hess}\, f) - \langle V, \nabla f\rangle,$$

where V is a vector field defined on Σ^n.

The following maximum principle is a slight extension of Theorem 1, p.246, of [14] and we include a proof here for the sake of completeness.

Lemma 3.1. *Let Σ^n be an n-dimensional complete Riemannian manifold and $\phi : T\Sigma^n \to T\Sigma^n$ be a symmetric and positive semidefinite linear tensor. Let $\gamma \in \mathcal{C}^2(\Sigma^n)$ and $\psi \in \mathcal{C}^2([0,\infty))$ be positive functions. If*

(i) $\gamma(x) \to \infty$ *when* $x \to \infty$;

(ii) $\limsup_{x\to\infty} \left[\psi'(\gamma(x))\mathcal{L}_\phi\gamma(x) + \psi''(\gamma(x))\langle\phi(\nabla\gamma(x)), \nabla\gamma(x)\rangle\right] < \infty$;

(iii) $\limsup_{x\to\infty} \psi'(\gamma(x))\|\nabla\gamma(x)\| < \infty$,

then, for every function $u \in \mathcal{C}^2(\Sigma^n)$ satisfying

$$\lim_{x\to\infty} \frac{u(x)}{\psi(\gamma(x))} = 0, \tag{3.1}$$

there exists a sequence of points $x_k \in \Sigma^n$ such that

$$\|\nabla u(x_k)\| < \frac{1}{k} \quad \text{and} \quad \mathcal{L}_\phi u(x_k) < \frac{1}{k}. \tag{3.2}$$

Moreover, if instead of (3.1) *we assume that* $u^* = \sup_{\Sigma^n} u < \infty$*, then*

$$\lim_{k\to\infty} u(x_k) = u^*.$$

Proof. Let

$$f_k(x) = u(x) - \varepsilon_k \psi(\gamma(x)),$$

for each positive integer k, where $\varepsilon_k > 0$ is a sequence satisfying $\varepsilon_k \to 0$ when $k \to \infty$. Adding a positive constant to the function u, if necessary, we may assume that $f_k(x_0) > 0$ for some x_0 in Σ^n. Notice that, since

$$\frac{f_k(x_0)}{\psi(\gamma(x_0))} > 0 \quad\text{and}\quad \frac{f_k(x)}{\psi(\gamma(x))} = \frac{u(x)}{\psi(\gamma(x))} - \varepsilon_k$$

and

$$\lim_{x\to\infty} \frac{f_k(x)}{\psi(\gamma(x))} = 0,$$

then there exists a point of maximum x_k for f_k for each $k \geq 1$. Suppose that the sequence $\{x_k\}_{k\in\mathbb{N}}$ diverges, i.e., leaves any compact subset of Σ^n, otherwise we have nothing to prove. Since

$$\nabla f_k = \nabla u - \varepsilon_k \psi'(\gamma)\nabla\gamma$$

and

$$\operatorname{Hess} f_k(v,v) = \operatorname{Hess} u(v,v) - \varepsilon_k \psi''(\gamma)\langle\nabla\gamma, v\rangle^2 - \varepsilon_k \psi'(\gamma)\operatorname{Hess}\gamma(v,v),$$

we have, at x_k, that

$$\nabla u(x_k) = \varepsilon_k \psi'(\gamma(x_k))\nabla\gamma(x_k)$$

and

$$\operatorname{Hess} u(x_k)(v,v) \leq \varepsilon \left[\psi'(\gamma(x_k))\operatorname{Hess}\gamma(x_k)(v,v) + \psi''(\gamma(x_k))\langle\nabla\gamma(x_k), v\rangle^2\right].$$

First, notice that,

$$\|\nabla u(x_k)\| = \varepsilon_k |\psi'(\gamma(x_k))| \|\nabla\gamma(x_k)\| \leq \varepsilon_k C_0 < \frac{1}{k}$$

by choosing $\varepsilon_k < \frac{1}{kC_0}$.

On the other hand, letting $\{e_1, \ldots, e_n\}$ be an orthonormal frame formed with eigenvectors of $\phi\colon T\Sigma^n \to T\Sigma^n$, with nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n$, we have

$$
\begin{aligned}
\mathcal{L}_\phi u(x_k) &= \sum_{i=1}^{n} \langle \operatorname{hess} u(x_k)(e_i), \phi(e_i)\rangle - \langle V(x_k), \nabla u(x_k)\rangle \\
&= \sum_{i=1}^{n} \lambda_i \langle \operatorname{hess} u(x_k)(e_i), e_i\rangle - \langle V(x_k), \nabla u(x_k)\rangle \\
&= \sum_{i=1}^{n} \lambda_i \operatorname{Hess} u(x_k)(e_i, e_i) - \langle V(x_k), \nabla u(x_k)\rangle \\
&\leq \varepsilon_k \sum_{i=1}^{n} \lambda_i \left[\psi'(\gamma(x_k)) \operatorname{Hess} \gamma(x_k)(e_i, e_i) + \psi''(\gamma(x_k))\langle \nabla\gamma(x_k), e_i\rangle^2\right] \\
&\quad - \varepsilon_k \psi'(\gamma(x_k))\langle V(x_k), \nabla\gamma(x_k)\rangle \\
&= \varepsilon_k \left[\psi'(\gamma(x_k))\Box\gamma(x_k) + \psi''(\gamma(x_k))\langle \phi(\nabla\gamma(x_k)), \nabla\gamma(x_k)\rangle\right] \\
&\leq \varepsilon_k C_1 < \frac{1}{k},
\end{aligned}
$$

if we take $\varepsilon_k < \dfrac{1}{k \max\{C_0, C_1\}}$.

□

As an application of Lemma 3.1, we have the

Lemma 3.2. *Let Σ^n be an n-dimensional complete hypersurface of $\mathbb{R}^{n+1}$ such that $\sup_{\Sigma^n} \|A\|^2 < \infty$. If $P_{r-1} : T\Sigma^n \to T\Sigma^n$ is a positive semidefinite linear tensor, then, for every function $u \in \mathcal{C}^2(\Sigma^n)$ bounded from above, there exists a sequence of points $x_k \in \Sigma^n$ such that*

$$
\lim_{k\to\infty} u(x_k) = \sup_{\Sigma^n} u, \quad \|\nabla u(x_k)\| < \frac{1}{k} \quad \textit{and} \quad \mathcal{L}_{r-1} u(x_k) < \frac{1}{k}. \tag{3.3}
$$

Proof. Let us apply Lemma 3.1 to $\phi = P_{r-1}$, $V = X$, the position vector of Σ^n in $\mathbb{R}^{n+1,}$ $\psi(t) = \log t$, for large values of t, and $\gamma(x) = \rho(x) = \operatorname{dist}(x, x_0)$, the geodesic distance of Σ^n to a fixed point $x_0 \in \Sigma^n$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of principal directions of Σ^n and denote by $\lambda_1, \ldots, \lambda_n$, the eigenvalues of P_{r-1}. Notice that, since the extrinsic distance is less than or equal to the intrinsic distance, we have $\|X(x) - X(x_0)\| \leq \rho(x)$. This gives

$$
\frac{\|X(x)\|}{\rho(x)} \leq \frac{\|X(x) - X(x_0)\|}{\rho(x)} + \frac{\|X(x_0)\|}{\rho(x)} \leq 1 + c_0,
$$

where $c_0 = \sup_\Sigma \frac{\|X(x_0)\|}{\rho}$. Since, by the Gauss equation,

$$
K(e_i \wedge e_j) = \langle A(e_i), e_i\rangle\langle A(e_j), e_j\rangle - \langle A(e_i), e_j\rangle^2 \geq -2\|A\|^2 \geq -C,
$$

where $C := 2\sup_{\Sigma^n}\|A\|^2$, by using the hessian comparison theorem, we have, for points outside of the cut locus of x_0,

$$\begin{aligned}\psi'(\gamma(x))\mathcal{L}_{r-1}\gamma(x) &+ \psi''(\gamma(x))\langle\phi(\nabla\gamma(x)),\nabla\gamma(x)\rangle\\ &= \frac{1}{\rho(x)}\mathcal{L}_{r-1}\rho(x) - \frac{1}{(\rho(x))^2}\langle P_{r-1}(\nabla\rho(x)),\nabla\rho(x)\rangle\\ &= \frac{1}{\rho(x)}\sum_{i=1}^{n}\lambda_i \operatorname{Hess}\rho(x)(e_i,e_i) - \frac{1}{\rho(x)}\langle X(x),\nabla\rho(x)\rangle\\ &\le \frac{\sqrt{C}\coth(\sqrt{C}\rho(x))}{\rho(x)}\sum_{i=1}^{n}\lambda_i[\langle e_i,e_i\rangle - \langle\nabla\rho(x),e_i\rangle^2] + \frac{\|X(x)\|}{\rho(x)}\\ &\le \frac{2\sqrt{C}\operatorname{trace}(P_{r-1}(x))}{\rho(x)} + 1 + c_0 < \infty,\end{aligned}$$

where we used that P_{r-1} is positive semidefinite and bounded and that $\coth(\sqrt{C}\rho) < 2$ for $\rho \gg 1$. For points in the cut locus of x_0 we use the Calabi trick as it was done by Cheng and Yau in [17], p.341-342. The result then follows from Lemma 3.1, since the other inequalities are immediate. □

We conclude the paper with the proof of Theorem 1.2:

Proof of Theorem 1.2. If $\sup(\|\sqrt{P_{r-1}}A\|^2) \ge r$ there is nothing to prove. If $\sup\|\sqrt{P_{r-1}}A\|^2 < r$, then, by (2.24), p.14,

$$\begin{aligned}r^2\sigma_r^2 &\le [\operatorname{trace}(P_{r-1}A)]^2\\ &\le \operatorname{trace}(P_{r-1}A^2)\operatorname{trace}(P_{r-1})\\ &< r\operatorname{trace}(P_{r-1}) < \infty,\end{aligned}$$

since $\|A\|$ is bounded P_{r-1} is bounded. Using Lemma 3.2 in (2.17), p.13, we have

$$\begin{aligned}0 &\ge \limsup \mathcal{L}_{r-1}\sigma_r^2\\ &= \sup\sigma_r^2 \sup[r - \|\sqrt{P_{r-1}}A\|^2]\\ &\ge \sup\sigma_r^2[r - \sup\|\sqrt{P_{r-1}}A\|^2]\\ &\ge 0.\end{aligned}$$

This gives

$$\sup\sigma_r^2 = 0,$$

i.e., $-\langle X,N\rangle = \sigma_r = 0$ and, thus Σ^n is a hyperplane. □

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