HOPF TYPE THEOREMS FOR SURFACES IN THE DE SITTER-SCHWARZSCHILD AND REISSNER-NORDSTROM MANIFOLDS

HILÁRIO ALENCAR AND GREGÓRIO SILVA NETO

Dedicated to Renato Tribuzy by the occasion of his 75th birthday

Abstract. In 1951, H. Hopf proved that the only surfaces, homeomorphic to the sphere, with constant mean curvature in Euclidean space are the round (geometrical) spheres. These results were generalized by S. S. Chern and then by Eschenburg and Tribuzy for surfaces homeomorphic to the sphere in Riemannian manifolds with constant sectional curvature whose mean curvature function satisfies some bound on its differential. In this paper, we extend these results for surfaces in a wide class of warped product manifolds, which includes, besides the classical space forms of constant sectional curvature, the de Sitter-Schwarzschild manifolds and the Reissner-Nordstrom manifolds, which are time slices of solutions of the Einstein field equations of general relativity.

1. Introduction

In 1951, H. Hopf, see [24] and [25], proved that the only surfaces with constant mean curvature in $\mathbb{R}^3$, homeomorphic to the sphere, are the round spheres. After 32 years, the result of Hopf was extended to three-dimensional Riemannian manifolds of constant sectional curvature in 1983 by S.-S. Chern, see [16], proving that the only surfaces with constant mean curvature in these spaces, homeomorphic to the sphere, are the geodesic spheres. Later, in 1991, J. Eschenburg and R. Tribuzy (see Theorem 3, p. 151 of [19]) observed that, to obtain a Hopf-type result, it is not necessary for the immersion to have constant mean curvature, but just that the differential of the mean curvature function satisfies some upper bound, namely

Theorem 1.1 (Eschenburg-Tribuzy)**.** *Let Q_c^3 be a three-dimensional Riemannian manifold with constant sectional curvature $c \in \mathbb{R}$. Let $X : \Sigma \to Q_c^3$ be an immersed surface with mean curvature function H. Assume that Σ is homeomorphic to the sphere. If there exists a local L^p, $p > 2$, function $f : \Sigma \to \mathbb{R}$ such that*

$$|dH| \leq f\sqrt{H^2 - K + c}, \tag{1.1}$$

where K is the Gaussian curvature of Σ, then $X(\Sigma)$ is totally umbilical.

Hilário Alencar and Gregório Silva Neto were partially supported by the National Council for Scientific and Technological Development - CNPq of Brazil.

Date: September 08, 2023.

2020 *Mathematics Subject Classification.* Primary 53C42, 53C21; Secondary 30F30, 30F10, 30A10, 58J05.

Key words and phrases. Hopf differential, mean curvature, de Sitter-Schwarzschild, Reissner-Nordstrom, isometric immersion, conformally flat, warped product.

In this paper, we generalize the Eschenburg-Tribuzy theorem for the more general class of three-dimensional Riemannian manifolds $M^3 = I \times \mathbb{S}^2$, where $I = (0, b)$ or $I = (0, \infty)$, with the metric

$$\langle \cdot, \cdot \rangle = dt^2 + h(t)^2 d\omega^2, \tag{1.2}$$

where $h : I \to \mathbb{R}$ is a smooth function, called warping function, and $d\omega^2$ denotes the canonical metric of the 2-dimensional round sphere $\mathbb{S}^2$. With the metric (1.2), the product $M^3 = I \times \mathbb{S}^2$ is called a warped product manifold and generalizes the space forms with constant sectional curvature. In fact, the metrics of the space forms of constant sectional curvature $c \in \mathbb{R}$ can be written in polar coordinates as in (1.2), where

$$h(t) = t \text{ for } \mathbb{R}^3,\ h(t) = \frac{1}{\sqrt{c}} \sin(\sqrt{c}t) \text{ for } \mathbb{S}^3(c),\ h(t) = \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}t) \text{ for } \mathbb{H}^3(c).$$

The warped product manifold M^3 has two different sectional curvatures depending only on the parameter t: one tangent to the slices $\{t\} \times \mathbb{S}^2$, denoted by $K_{\tan}(t)$, and another relative to the planes which contain the radial direction ∇t, denoted by $K_{\text{rad}}(t)$. In terms of the warping function, we can write

$$K_{\tan}(t) = \frac{1 - h'(t)^2}{h(t)^2} \text{ and } K_{\text{rad}}(t) = -\frac{h''(t)}{h(t)}, \tag{1.3}$$

where $X, Y \in TM^3,\ X \perp \nabla t,\ Y \perp \nabla t$.

These manifolds were first introduced by Bishop and O' Neill in 1969, see [12], and are gaining increasing importance due to their applications as model spaces in general relativity. Part of these applications comes from the metrics that are solutions of the Einstein equations, such as the de Sitter-Schwarzschild metric and the Reissner-Nordstrom metric, which we introduce later.

In recent years, immersions in warped product manifolds have been extensively studied, with many interesting papers in this subject; for instance, see [29], [30], [11], [33], [10], [17], [3], [4], [5], [7], [9], [37], [21], [20], [34], [38], [1], [23], [2], [22], and [35]. We can also cite the book of Petersen, see [32], for a modern presentation of warped product manifolds, and the book of Besse [8] for an introduction to general relativity and the deduction of Schwarzschild space-time from the Einstein equations.

The main result of this paper is the following generalization of Theorem 1.1 for a class of warped product manifolds that contain the de Sitter-Schwarzschild and the Reissner-Nordstrom manifolds:

Theorem 1.2. *Let Σ be a surface, homeomorphic to the sphere, immersed in a warped product manifold $M^3 = I \times \mathbb{S}^2$, with mean curvature function H. If there exists a non-negative L^p, $p > 2$, function $f : \Sigma \to \mathbb{R}$ such that*

$$|dH + (K_{\tan}(t) - K_{\text{rad}}(t))\nu dt| \leq f \sqrt{H^2 - K + K_{\tan}(t) - (1 - \nu^2)(K_{\tan}(t) - K_{\text{rad}}(t))}, \tag{1.4}$$

then Σ is totally umbilical.

Moreover, if $K_{\tan}(t) \neq K_{\text{rad}}(t)$, except possibly for a discrete set of values $t \in I$, and Σ has constant mean curvature, then Σ is a slice.

Remark 1.1. Actually, some additional hypothesis, such as (1.4), is needed in order to classify the slices as the only constant mean curvature spheres. In fact, it was observed by Brendle (see [13], Theorem 1.5,

p. 250) that a result of Pacard and Xu (see [31], Theorem 1.1, p. 276) implies that in some warped product manifolds there are small spheres with constant mean curvature that are not umbilical.

Remark 1.2. To obtain the slice in the second part of Theorem 1.2, the assumption over M^3 that $K_{\tan}(t) \neq K_{\mathrm{rad}}(t)$, except possibly for a discrete set of values $t \in I$, is necessary. In fact, if $K_{\tan}(t) = K_{\mathrm{rad}}(t)$ for some interval $(t_0, t_1) \subset I$, then all the sectional curvatures of M^3 will depend only on t. This will imply, by the classical Schur's Theorem, that $\widetilde{M}^3 := (t_0, t_1) \times \mathbb{S}^2$ has constant sectional curvature. In this case, there exist spheres, other than the slices, with constant mean curvature (in fact, the geodesic spheres centered at some point of $\widetilde{M}^3$).

Remark 1.3. Again, to obtain the slice in the second part of Theorem 1.2, the assumption that H is constant is necessary. In fact, defining $\varphi : (r_0, r_1) \subset \mathbb{R} \to \mathbb{R}_+$ by the equations

$$\frac{dr}{\varphi(r)} = dt \quad \text{and} \quad \frac{r}{\varphi(r)} = h(t),$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$, we can interpret the warped product $M^3 = I \times \mathbb{S}^2$ as the Euclidean space $\mathbb{R}^3$ (actually, a ring $0 \leq r_0^2 \leq x_1^2 + x_2^2 + x_3^2 \leq r_1^2 \leq \infty$) with the conformal metric

$$\langle \cdot, \cdot \rangle_\varphi = \frac{1}{\varphi(r)^2}(dr^2 + r^2 d\omega^2).$$

Since the spheres $\mathbb{S}^2(C_0, R)$ of any center $C_0 \in \mathbb{R}^3$ and any radius $R > 0$ are umbilical surfaces of $\mathbb{R}^3$ with umbilicity factor $1/R$, $\mathbb{S}^2(C_0, R)$ remains umbilical for the conformal metric $\langle \cdot, \cdot \rangle_\varphi$, but with umbilicity factor (see [14], p.183)

$$\lambda = \frac{\varphi(r)}{R} - \varphi'(r) N(r),$$

where $N(r)$ denotes the derivative of r in the direction of the unitary (in the Euclidean metric) inner normal vector field N of $\mathbb{S}^2(C_0, R)$. Notice that, if the sphere is not centered at the origin (i.e., $r =$ constant, which is equivalent to being a slice in the warped product manifold), the umbilicity factor λ is not constant and the sphere does not have constant mean curvature. Therefore, if we drop the condition that H is constant, we obtain other umbilical spheres that are not slices.

Two of the most famous examples of warped product manifolds are the de Sitter-Schwarzschild manifolds and the Reissner-Nordstrom manifolds, which we describe below.

Definition 1.1 (The de Sitter-Schwarzschild manifolds)**.** Let $m > 0$, $c \in \mathbb{R}$, and

$$(s_0, s_1) = \{r > 0; 1 - mr^{-1} - cr^2 > 0\}.$$

If $c \leq 0$, then $s_1 = \infty$. If $c > 0$, assume that $cm^2 < \frac{4}{27}$. The de Sitter-Schwarzschild manifold is defined by $M^3(c) = (s_0, s_1) \times \mathbb{S}^2$ endowed with the metric

$$\langle \cdot, \cdot \rangle = \frac{1}{1 - mr^{-1} - cr^2} dr^2 + r^2 d\omega^2.$$

In order to write the metric in the form (1.2), define $F : [s_0, s_1) \to \mathbb{R}$ by

$$F'(r) = \frac{1}{\sqrt{1 - mr^{-1} - cr^2}}, \ F(s_0) = 0.$$

Taking $t = F(r)$, we can write $\langle\cdot,\cdot\rangle = dt^2 + h(t)^2 d\omega^2$, where $h : [0, F(s_1)) \to [s_0, s_1)$ denotes the inverse function of F. The function h clearly satisfies

$$h'(t) = \sqrt{1 - mh(t)^{-1} - ch(t)^2},\ h(0) = s_0,\ \text{and } h'(0) = 0. \tag{1.5}$$

For the de Sitter-Schwarzschild manifolds, we have

$$K_{\tan}(t) = c + \frac{m}{h(t)^3} \quad \text{and} \quad K_{\mathrm{rad}}(t) = c - \frac{m}{2h(t)^3}.$$

Replacing these facts in Theorem 1.2 and writing f in the place of $\frac{2}{3m}h^3 f$ in order to clean the presentation (since the function f in Theorem 1.2 is an arbitrary L^p function, $p > 2$) we obtain

Corollary 1.1 (The de Sitter-Schwarzschild manifolds). *Let Σ be a surface, homeomorphic to the sphere, immersed in the de Sitter-Schwarzschild manifold with constant mean curvature. If there exists a non-negative L^p, $p > 2$, function $f : \Sigma \to \mathbb{R}$ such that*

$$|\nu dt| \le f\sqrt{H^2 - K + c + \frac{m(3\nu^2 - 1)}{2h(t)^3}},$$

then Σ is a slice.

Here, K is the Gaussian curvature of Σ, $\nu = \langle \nabla t, N\rangle$ is the angle function, and N is the unitary normal vector field of Σ in the de Sitter-Schwarzschild manifold.

Definition 1.2 (The Reissner-Nordstrom manifolds). The Reissner-Nordstrom manifold is defined by $M^3 = (s_0, \infty) \times \mathbb{S}^2$, with the metric

$$\langle\cdot,\cdot\rangle = \frac{1}{1 - mr^{-1} + q^2 r^{-2}} dr^2 + r^2 d\omega^2,$$

where $m > 2q > 0$ and $s_0 = \frac{2q^2}{m - \sqrt{m^2 - 4q^2}}$ is the larger of the two solutions of $1 - mr^{-1} + q^2 r^{-2} = 0$. In order to write the metric in the form (1.2), define $F : [s_0, \infty) \to \mathbb{R}$ by

$$F'(r) = \frac{1}{\sqrt{1 - mr^{-1} + q^2 r^{-2}}},\ F(s_0) = 0.$$

Taking $t = F(r)$, we can write $\langle\cdot,\cdot\rangle = dt^2 + h(t)^2 d\omega^2$, where $h : [0, \infty) \to [s_0, \infty)$ denotes the inverse function of F. The function h clearly satisfies

$$h'(t) = \sqrt{1 - mh(t)^{-1} + q^2 h(t)^{-2}},\ h(0) = s_0,\ \text{and } h'(0) = 0. \tag{1.6}$$

For the Reissner-Nordstrom manifolds, we have

$$K_{\tan}(t) = \frac{m}{h(t)^3} - \frac{q^2}{h(t)^4} \quad \text{and} \quad K_{\mathrm{rad}}(t) = -\frac{m}{2h(t)^3} + \frac{q^2}{h(t)^4}.$$

Moreover,

$$K_{\tan}(t) - K_{\mathrm{rad}(t)} = \frac{3m}{2h(t)^3} - \frac{2q^2}{h(t)^4} = \frac{3mh(t) - 4q^2}{2h(t)^4} > 0,$$

for every $t \in (s_0, \infty)$, since $4q^2/3m < s_0$.

Replacing these facts in Theorem 1.2 and writing f in the place of $\frac{2h^4 f}{3mh - 4q^2}$ in order to clean the presentation (since the function f in Theorem 1.2 is an arbitrary L^p function, $p > 2$) we obtain

Corollary 1.2 (The Reissner-Nordstrom manifolds). *Let Σ be a surface, homeomorphic to the sphere, immersed in the Reissner-Nordstrom manifold with constant mean curvature. If there exists a non-negative L^p, $p > 2$, function $f : \Sigma \to \mathbb{R}$ such that*

$$|\nu dt| \leq f\sqrt{H^2 - K + \frac{m(3\nu^2-1)}{2h(t)^3} + \frac{q^2(1-2\nu^2)}{h(t)^4}},$$

then Σ is a slice.

Here, K is the Gaussian curvature of Σ, $\nu = \langle \nabla t, N\rangle$ is the angle function, and N is the unitary normal vector field of Σ in the Reissner-Nordstrom manifold.

Remark 1.4. Since the warped product manifold is smooth at $t = 0$ if and only if $h(0) = 0,\ h'(0) = 1$, and all the even order derivatives are zero at $t = 0$, i.e., $h^{(2k)}(0) = 0,\ k > 0$, see [32], Proposition 1, p. 13, we can see the de Sitter-Schwarzschild manifolds and the Reissner-Nordstrom manifolds are singular at $t = 0$.

Acknowledgements. We would like to thank the referee for the careful reading of the early version of this paper and for giving valuable comments and constructive suggestions.

2. Preliminaries

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth and positive function. We will denote by

$$M_F^3 = (\mathbb{R}^3, \langle\cdot,\cdot\rangle_F),\ \text{where } \langle\cdot,\cdot\rangle_F = \frac{1}{F(x_1,x_2,x_3)^2}(dx_1^2 + dx_2^2 + dx_3^2) \tag{2.1}$$

be the conformally flat three dimensional manifold. Denote by $f = \log F$ and by $f_i = \dfrac{\partial f}{\partial x_i}$, $i = 1,2,3$. If Γ_{ij}^k, $i,j,k = 1,2,3$, are the Christoffel symbols of M^3, then

$$\begin{aligned}
&\Gamma_{11}^1 = -f_1,\ \Gamma_{11}^2 = f_2,\ \Gamma_{11}^3 = f_3,\ \Gamma_{12}^1 = \Gamma_{21}^1 = -f_2,\ \Gamma_{12}^2 = \Gamma_{21}^2 = -f_1,\ \Gamma_{12}^3 = \Gamma_{21}^3 = 0,\\
&\Gamma_{13}^1 = \Gamma_{31}^1 = -f_3,\ \Gamma_{13}^2 = \Gamma_{31}^2 = 0,\ \Gamma_{13}^3 = \Gamma_{31}^3 = -f_1,\ \Gamma_{22}^1 = f_1,\ \Gamma_{22}^2 = -f_2,\ \Gamma_{22}^3 = f_3,\\
&\Gamma_{23}^1 = \Gamma_{32}^1 = 0,\ \Gamma_{23}^2 = \Gamma_{32}^2 = -f_3,\ \Gamma_{23}^3 = \Gamma_{32}^3 = -f_2,\ \Gamma_{33}^1 = f_1,\ \Gamma_{33}^2 = f_2,\ \Gamma_{33}^3 = -f_3.
\end{aligned} \tag{2.2}$$

Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$ with the canonical metric. The canonical orthonormal frame of M^3 is

$$E_1(x_1,x_2,x_3) = F(x_1,x_2,x_3)e_1,$$

$$E_2(x_1,x_2,x_3) = F(x_1,x_2,x_3)e_2,$$

$$E_3(x_1,x_2,x_3) = F(x_1,x_2,x_3)e_3.$$

Lemma 2.1. *Let us denote by ∇ the connection $\mathbb{R}^3$ with the metric $\langle\cdot,\cdot\rangle_F$. We have*

$$\nabla_{E_1}E_1 = F_2E_2 + F_3E_3,\ \nabla_{E_1}E_2 = -F_2E_1,\ \nabla_{E_1}E_3 = -F_3E_1,$$

$$\nabla_{E_2}E_1 = -F_1E_2,\ \nabla_{E_2}E_2 = F_1E_1 + F_3E_3,\ \nabla_{E_2}E_3 = -F_3E_2,$$

$$\nabla_{E_3}E_1 = -F_1E_3,\ \nabla_{E_3}E_2 = -F_2E_3,\ \nabla_{E_3}E_3 = F_1E_1 + F_2E_2,$$

where $F_i = \dfrac{\partial F}{\partial x_i}$, $i = 1,2,3$.

Proof. We have

$$\begin{aligned}\nabla_{E_i}E_j &= F^2\nabla_{e_i}e_j + FF_ie_j\\ &= F^2(\Gamma_{ij}^1e_1+\Gamma_{ij}^2e_2+\Gamma_{ij}^3e_3)+F^2f_ie_j.\end{aligned}$$

The result then follows by replacing the values of Γ_{ij}^k given by (2.2) and noticing that $f_i = F_i/F$, $i=1,2,3$. □

Let Σ be a smooth two-dimensional Riemannian surface whose metric, in a local coordinate system $\Phi : D\subset\mathbb{R}^2\to\Sigma$, is given by

$$ds^2 = E(u,v)du^2+2F(u,v)dudv+G(u,v)dv^2,\ (u,v)\in D.$$

A local coordinate system is called a system of isothermal parameters if $E=G$ and $F=0$. By the results of Korn [26] and Lichtenstein [28] (for instance, see the work of Chern [15] for an elementary proof), if the functions $E,F,G : D\subset\mathbb{R}^2\to\mathbb{R}$ are Hölder continuous of order $0<\lambda<1$, then every point of D has a neighborhood whose local coordinates are isothermal parameters (remember that a function $f : D\subset\mathbb{R}^2\to\mathbb{R}$ is Hölder continuous of order $\lambda>0$ if $|f(u_2,v_2)-f(u_1,v_1)|\leq Cr^\lambda$, where $r=\sqrt{(v_2-v_1)^2+(u_2-u_1)^2}$).

Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by taking $z=u+iv$ and $\bar{z}=u-iv$, we have $dz=du+idv$ and $d\bar{z}=du-idv$, which gives the rules of differentiation

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\right)\quad\text{and}\quad\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right).$$

By using this complexification, we can write

$$ds^2=\alpha|dz+\mu d\bar{z}|^2, \tag{2.3}$$

where

$$\alpha=\frac{1}{4}\left(E+G+2\sqrt{EG-F^2}\right)\quad\text{and}\quad\mu=\frac{1}{4\alpha}(E-G+2iF).$$

Here, $\alpha>0$ and $|\mu|<1$. If (x,y) are isothermal coordinates for Σ, then we can write

$$ds^2=E(u,v)(du^2+dv^2)=\alpha(z)|dz|^2.$$

Remark 2.1. The existence of isothermal coordinates can also be proved by applying known existence theorems to the Beltrami equation. In fact, since the change of coordinates satisfies

$$ds^2=\alpha|dz|^2=\alpha|z_w|^2\left|dw^2+\frac{z_{\bar{w}}}{z_w}d\bar{w}\right|^2, \tag{2.4}$$

comparing (2.3) and (2.4), there exist isothermal parameters in a neighborhood of Σ is and only if there exists a solution of the Beltrami differential equation

$$\frac{\partial z}{\partial \bar{w}}=\mu\frac{\partial z}{\partial w}.$$

By using L^p estimates for singular integral operators of Calderón and Zygmund, it can be proved that the solution exists in any neighborhood where $\|\mu\|_\infty<1$ (see, for instance [36], p.20-21 and p.97).

In this section, we will consider a smooth Riemannian surface Σ and an isometric immersion $X:\Sigma\to M^3_F$. Taking isothermal parameters u and v in a neighborhood of Σ and complexifying the parameters by taking $z=u+iv$, we can identify this neighborhood of Σ with a subset of $\mathbb{C}$ and obtain

$$\langle X_z, X_{\bar z}\rangle_F=\frac{\alpha(z)}{2},$$

where $X_z=\frac{\partial X}{\partial z}$, $X_{\bar z}=\frac{\partial X}{\partial \bar z}$, and $\alpha(z)$ is the conformal factor of Σ, i.e., $ds^2=\alpha(z)|dz|^2$ is the metric of Σ. In this case, the second fundamental form becomes

$$II=Pdz^2+H\alpha|dz|^2+\bar P d\bar z^2,$$

where

$$Pdz^2=\langle\nabla_{X_z}X_z,N_F\rangle_F dz^2$$

is the Hopf differential of X, i.e., the $(2,0)$-part of the complexified second fundamental form. For more details about the complexification and the Hopf differential, we refer to chapter VI of the classical book of Hopf, see [25].

Remark 2.2. Here and after, the bar over a quantity will mean the complex conjugate of the quantity.

Since $X_z=\frac{1}{2}(X_u-iX_v)$ and $X_{\bar z}=\frac{1}{2}(X_u+iX_v)$, where $X_u=\frac{\partial X}{\partial u}$ and $X_v=\frac{\partial X}{\partial v}$, we have $X_u=X_z+X_{\bar z}$ and $X_v=i(X_z-X_{\bar z})$. This implies

$$X_u\times X_v=i(X_z+X_{\bar z})\times(X_z-X_{\bar z})=2iX_{\bar z}\times X_z,$$

where $\times$ means the usual vector product of $\mathbb{R}^3$. On the other hand,

$$\|X_u\times X_v\|_F=\sqrt{\|X_u\|_F^2\|X_v\|_F^2-\langle X_u,X_v\rangle_F^2}=2\langle X_z,X_{\bar z}\rangle_F=\alpha,$$

where $\|Y\|_F^2=\langle Y,Y\rangle_F,\ Y\in\mathbb{R}^3$. Therefore, the unitary normal vector field of the immersion, with the canonical orientation, is given by

$$N_F=\frac{X_u\times X_v}{\|X_u\times X_v\|_F}=\frac{2i}{\alpha}X_{\bar z}\times X_z. \tag{2.5}$$

We also have the following fundamental equations

$$\begin{cases}\nabla_{X_z}X_z=\dfrac{\alpha_z}{\alpha}X_z+PN_F\\ \nabla_{X_{\bar z}}X_z=\dfrac{\alpha H}{2}N_F\\ \nabla_{X_{\bar z}}X_{\bar z}=\dfrac{\alpha_{\bar z}}{\alpha}X_{\bar z}+\bar P N_F\end{cases}\qquad \begin{cases}\nabla_{X_z}N=-HX_z-\dfrac{2P}{\alpha}X_{\bar z}\\ \nabla_{X_{\bar z}}N=-\dfrac{2\bar P}{\alpha}X_z-HX_{\bar z}.\end{cases} \tag{2.6}$$

Since

$$\begin{aligned}P&=\langle\nabla_{X_z}X_z,N_F\rangle_F=\frac{1}{4}\langle\nabla_{X_u-iX_v}X_u-iX_v,N_F\rangle_F\\ &=\frac{1}{4}[\langle\nabla_{X_u}X_u,N_F\rangle_F-\langle\nabla_{X_v}X_v,N_F\rangle_F-i(\langle\nabla_{X_u}X_v,N_F\rangle_F+\langle\nabla_{X_v}X_u,N_F\rangle_F)]\\ &=\frac{1}{4}[II(X_u,X_u)-II(X_v,X_v)-2iII(X_u,X_v)],\end{aligned} \tag{2.7}$$

where II is the second fundamental form of Σ in M^3_F, we have $P = 0$ if and only if II is umbilical. Moreover,

$$\begin{aligned}|P|^2 &= \frac{1}{16}[(II(X_u,X_u) - II(X_v,X_v))^2 + 4II(X_u,X_v)^2]\\ &= \frac{1}{16}[(II(X_u,X_u) + II(X_v,X_v))^2 - 4(II(X_u,X_u)II(X_v,X_v) - II(X_u,X_v)^2)]\\ &= \frac{1}{16}[(\operatorname{trace} II)^2 - 4(\det II)].\end{aligned}$$

Since $H = \frac{1}{2}\operatorname{trace} II$ is the mean curvature and, by the Gauss equation $\det II = K - \overline{K}(T\Sigma)$, we have

$$|P|^2 = \frac{1}{4}(H^2 - K + \overline{K}(T\Sigma)). \tag{2.8}$$

Here K is the Gaussian curvature of Σ and $\overline{K}(T\Sigma)$ is the sectional curvature of M^3_F relative to the two dimensional subspace $T\Sigma$.

In order to prove our main theorem, we will need some computational lemmas.

Lemma 2.2. *If $Pdz^2 = \langle\nabla_{X_z}X_z, N_F\rangle_F dz^2$ is the Hopf differential of an isometric immersion $X : \Sigma \to M^3_F$, then*

$$P_{\bar z} = \frac{\alpha}{2}H_z + \langle\overline{R}(X_z, X_{\bar z})X_z, N_F\rangle_F, \tag{2.9}$$

where $\overline{R}$ is the curvature tensor of M^3_F and H is the mean curvature of X.

Proof. We have

$$\begin{aligned}P_{\bar z} &= \frac{\partial}{\partial \bar z}\langle\nabla_{X_z}X_z, N_F\rangle_F\\ &= \langle\nabla_{X_{\bar z}}\nabla_{X_z}X_z, N_F\rangle_F + \langle\nabla_{X_z}X_z, \nabla_{X_{\bar z}}N_F\rangle_F\\ &= \langle\bar R(X_z, X_{\bar z})X_z, N_F\rangle_F + \langle\nabla_{X_z}\nabla_{X_{\bar z}}X_z, N_F\rangle_F + \langle\nabla_{X_z}X_z, \nabla_{X_{\bar z}}N_F\rangle_F\\ &= \langle\bar R(X_z, X_{\bar z})X_z, N_F\rangle_F + \frac{\partial}{\partial z}\left(\langle\nabla_{X_{\bar z}}X_z, N_F\rangle_F\right) - \langle\nabla_{X_{\bar z}}X_z, \nabla_{X_z}N_F\rangle_F + \langle\nabla_{X_z}X_z, \nabla_{X_{\bar z}}N_F\rangle_F\\ &= \langle\bar R(X_z, X_{\bar z})X_z, N_F\rangle_F + \frac{\partial}{\partial z}\left(\frac{\alpha H}{2}\right) - \left\langle\frac{\alpha H}{2}N_F, -HX_z - \frac{2P}{\alpha}X_{\bar z}\right\rangle_F\\ &\quad + \left\langle\frac{\alpha_z}{\alpha}X_z + PN_F, -\frac{2\overline{P}}{\alpha}X_z - HX_{\bar z}\right\rangle_F\\ &= \langle\bar R(X_z, X_{\bar z})X_z, N_F\rangle_F + \frac{\alpha H_z}{2}.\end{aligned}$$

□

In order to calculate the expression for $\langle\bar R(X_z, X_{\bar z})X_z, N_F\rangle_F$, we will use the following result, whose proof can be found in [18], p. 98. Here we use the expression as it is written in the classical work of Kulkarni [27], Proposition 2.2, p. 318.

Lemma 2.3. *Let (M, g) be a Riemannian manifold and $\bar{g} = e^{2\phi}g$ be a conformal metric. If R and $\bar{R}$ denote the curvature tensors of M with metrics g and $\bar{g}$, respectively, then*

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ [\operatorname{Hess}\phi(Y,Z) - (Y\phi)(Z\phi) + g(Y,Z)\|\nabla\phi\|^2]X \\ &- [\operatorname{Hess}\phi(X,Z) - (X\phi)(Z\phi) + g(X,Z)\|\nabla\phi\|^2]Y \\ &+ g(Y,Z)[\nabla_X\nabla\phi - (X\phi)\nabla\phi] - g(X,Z)[\nabla_Y\nabla\phi - (Y\phi)\nabla\phi].\end{aligned}$$

Here, $\operatorname{Hess}\phi$, *and* ∇ *are, respectively, the Hessian and the connection (and the gradient) relative to the metric* g.

Lemma 2.4. *If $X : (\Sigma, \alpha(z)|dz|^2) \to M^3_F$ be an isometric immersion with normal vector N_F, then*

$$\langle \bar{R}(X_z, X_{\bar{z}})X_z, N_F\rangle_F = -\frac{\alpha}{2F}\operatorname{Hess}F(X_z, N_F), \tag{2.10}$$

where $\operatorname{Hess}F$ *is the Euclidean hessian of* F.

Proof. First, notice that

$$\begin{aligned}\langle \bar{R}(X_z, X_{\bar{z}})X_z, N_F\rangle_F &= \frac{1}{8}\langle \bar{R}(X_u - iX_v, X_u + iX_v)(X_u - iX_v), N_F\rangle_F \\ &= \frac{i}{4}\langle \bar{R}(X_u, X_v)X_u, N_F\rangle_F + \frac{1}{4}\langle \bar{R}(X_u, X_v)X_v, N_F\rangle_F.\end{aligned}\tag{2.11}$$

Denoting by $U \cdot V$ the Euclidean inner product of the vectors $U \in \mathbb{R}^3$ and $V \in \mathbb{R}^3$, we have

$$\langle U, V\rangle_F = \frac{1}{F^2}(U \cdot V). \tag{2.12}$$

By using Lemma 2.3 for $\phi = -\log F$, we have

$$\begin{aligned}\langle \bar{R}(X_u, X_v)X_u, N_F\rangle_F &= (X_v \cdot X_u)[\langle \nabla_{X_u}\nabla\phi, N_F\rangle_F - (X_u\phi)\langle\nabla\phi, N_F\rangle_F] \\ &\quad - (X_v \cdot X_u)[\langle \nabla_{X_v}\nabla\phi, N_F\rangle_F - (X_v\phi)\langle\nabla\phi, N_F\rangle_F] \\ &= -\langle X_u, X_u\rangle_F[((\nabla_{X_v}\nabla\phi)\cdot N_F) - (X_v\phi)(\nabla\phi\cdot N_F)] \\ &= -\langle X_u, X_u\rangle_F[\operatorname{Hess}\phi(X_v, N_F) - (\nabla\phi\cdot X_v)(\nabla\phi\cdot N_F)] \\ &= -\alpha[\operatorname{Hess}\phi(X_v, N_F) - (\nabla\phi\cdot X_v)(\nabla\phi\cdot N_F)].\end{aligned}\tag{2.13}$$

Analogously,

$$\langle \bar{R}(X_u, X_v)X_u, N_F\rangle_F = \alpha[\operatorname{Hess}\phi(X_u, N_F) - (\nabla\phi\cdot X_u)(\nabla\phi\cdot N_F)]. \tag{2.14}$$

Replacing (2.13) and (2.14) in (2.11), gives

$$\langle \bar{R}(X_z, X_{\bar{z}})X_z, N_F\rangle_F = \frac{\alpha}{2}[\operatorname{Hess}\phi(X_z, N_F) - (\nabla\phi\cdot X_z)(\nabla\phi\cdot N_F)].$$

On the other hand, since

$$\nabla\phi = -\frac{\nabla F}{F} \text{ and } \operatorname{Hess}\phi(U,V) = -\frac{1}{F}\operatorname{Hess}F(U,V) + \frac{1}{F^2}(\nabla F\cdot U)(\nabla F\cdot V),$$

we obtain

$$\langle \bar{R}(X_z, X_{\bar{z}})X_z, N_F\rangle_F = -\frac{\alpha}{2F}\operatorname{Hess}F(X_z, N_F).$$

□

Lemma 2.5. *If $X:(\Sigma,\alpha(z)|dz|^2)\to M_F^3$ be an isometric immersion, then*

$$\frac{4}{\alpha(F(X))^2}\|X\|_z\|X\|_{\bar z}+\left(\frac{X}{\|X\|}\cdot N\right)^2=1, \tag{2.15}$$

where $\|X\|$ denotes the Euclidean norm of X, $N=FN_F$, and $U\cdot V$ denotes the Euclidean inner product of the vectors U and V.

Proof. Identifying $T_p\Sigma$ with the complex plane $\mathbb{C}$, for each point $p\in\Sigma$, and since $N=FN_F$ is normal to $T\Sigma$, we can consider the adapted frame $\{X_z,X_{\bar z},N\}$ in $T_p\Sigma\times\mathbb{R}\equiv\mathbb{C}\times\mathbb{R}\equiv\mathbb{R}^3\equiv M_F^3$, where the $\equiv$ sign means linear isomorfism. Thus, we can write

$$X=aX_z+bX_{\bar z}+cN,$$

for smooth functions $a,b,c:\Sigma\to\mathbb{C}$. Since $X_z\cdot X_z=X_{\bar z}\cdot X_{\bar z}=0$ implies

$$\begin{aligned}X\cdot X_z&=a(X_z\cdot X_z)+b(X_{\bar z}\cdot X_z)+c(N\cdot X_z)\\&=b(F(X))^2\langle X_{\bar z},X_z\rangle_F=b(F(X))^2\frac{\alpha}{2},\end{aligned}$$

and analogously for $X\cdot X_{\bar z}$ and $X\cdot N$, we have

$$X=\frac{2}{\alpha(F(X))^2}(X\cdot X_{\bar z})X_z+\frac{2}{\alpha(F(X))^2}(X\cdot X_z)X_{\bar z}+(X\cdot N)N.$$

This implies

$$\begin{aligned}\|X\|^2=X\cdot X&=\frac{8}{\alpha^2(F(X))^4}(X\cdot X_z)(X\cdot X_{\bar z})(X_{\bar z}\cdot X_z)+(X\cdot N)^2\\&=\frac{8}{\alpha^2(F(X))^4}(X\cdot X_z)(X\cdot X_{\bar z})\frac{\alpha(F(X))^2}{2}+(X\cdot N)^2\\&=\frac{4}{\alpha^2(F(X))^2}(X\cdot X_z)(X\cdot X_{\bar z})+(X\cdot N)^2.\end{aligned}$$

By using $X\cdot X_z=\|X\|\|X\|_z$ and $X\cdot X_{\bar z}=\|X\|\|X\|_{\bar z}$, we obtain the result by dividing the expression above by $\|X\|^2$. □

Lemma 2.6. *Let $X:(\Sigma,\alpha(z)|dz|^2)\to M_F^3$ be an isometric immersion. If $\overline{K}(T\Sigma)$ denotes the sectional curvature of M_F^3 relative to the plane $dX(T\Sigma)$, then*

$$\overline{K}(T\Sigma)=-\|\nabla F\|^2+\frac{4}{\alpha(z)F(X(z))}\operatorname{Hess}F(X_z,X_{\bar z}),$$

where $\|\nabla F\|^2=\left(\frac{\partial F}{\partial x_1}(X(z))\right)^2+\left(\frac{\partial F}{\partial x_2}(X(z))\right)^2+\left(\frac{\partial F}{\partial x_3}(X(z))\right)^2$ and $\operatorname{Hess}F$ denotes the Euclidian hessian of F.

Proof. Considering the frame $\{X_z,X_{\bar z}\}$ in $T\Sigma$, we have

$$\overline{K}(T\Sigma)=\frac{\langle\overline{R}(X_z,X_{\bar z})X_z,X_{\bar z}\rangle_F}{\langle X_z,X_z\rangle_F\langle X_{\bar z},X_{\bar z}\rangle_F-\langle X_{\bar z},X_z\rangle_F^2}=-\frac{\langle\overline{R}(X_z,X_{\bar z})X_z,X_{\bar z}\rangle_F}{\langle X_{\bar z},X_z\rangle_F^2},$$

i.e.,

$$\overline{K}(T\Sigma)=-\frac{4}{\alpha(z)^2}\langle\overline{R}(X_z,X_{\bar z})X_z,X_{\bar z}\rangle_F. \tag{2.16}$$

Since

$$\begin{aligned}\langle\overline{R}(X_z, X_{\bar{z}})X_z, X_{\bar{z}}\rangle_F &= \frac{1}{16}\langle\overline{R}(X_u - iX_v, X_u + iX_v)(X_u - iX_v), X_u + iX_v\rangle_F \\ &= -\frac{1}{4}\langle\overline{R}(X_u, X_v)X_u, X_v\rangle_F,\end{aligned} \tag{2.17}$$

using Lemma 2.3 for $\phi = -\log F$, we obtain

$$\begin{aligned}\langle\overline{R}(X_u, X_v)X_u, X_v\rangle_F &= -[\operatorname{Hess}\phi(X_u, X_u) - (X_u \cdot \nabla\phi)^2 + (X_u \cdot X_u)\|\nabla\phi\|^2]\langle X_v, X_v\rangle_F \\ &\quad - (X_u \cdot X_u)[\langle\nabla_{X_v}\nabla\phi, X_v\rangle_F - (X_v \cdot \nabla\phi)\langle\nabla\phi, X_v\rangle_F] \\ &= -\alpha[\operatorname{Hess}\phi(X_u, X_u) - (X_u \cdot \nabla\phi)^2 + (X_u \cdot X_u)\|\nabla\phi\|^2] \\ &\quad - \alpha[\operatorname{Hess}\phi(X_v, X_v) - (X_v \cdot \nabla\phi)^2],\end{aligned}$$

where $U \cdot V$ denotes the Euclidean inner product of the vectors $U, V \in \mathbb{R}^3$. Since

$$\nabla\phi = \frac{1}{F}\nabla F, \ \operatorname{Hess}\phi(U, V) = -\frac{1}{F}\operatorname{Hess}F(U, V) + \frac{1}{F^2}(\nabla F \cdot U)(\nabla F \cdot V)$$

and $X_u \cdot X_u = F^2\langle X_u, X_u\rangle_F = F^2\alpha$, we have

$$\langle\overline{R}(X_u, X_v)X_u, X_v\rangle_F = -\alpha\left[-\frac{1}{F}\left(\operatorname{Hess}F(X_u, X_u) + \operatorname{Hess}F(X_v, X_v)\right) + \alpha\|\nabla F\|^2\right].$$

On the other hand,

$$\operatorname{Hess}F(X_z, X_{\bar{z}}) = \frac{1}{4}\operatorname{Hess}F(X_u - iX_v, X_u + iX_v) = \frac{1}{4}[\operatorname{Hess}F(X_u, X_u) + \operatorname{Hess}F(X_v, X_v)],$$

which gives

$$\langle\overline{R}(X_u, X_v)X_u, X_v\rangle_F = -\alpha^2\|\nabla F\|^2 + \frac{4\alpha}{F}\operatorname{Hess}F(X_z, X_{\bar{z}}). \tag{2.18}$$

The result then comes by replacing (2.18) in (2.17) and then in (2.16). □

3. Proof of the main result

Warped product manifolds can be seen as conformally flat Riemannian manifolds with radial weight, as follows. By taking the spherical coordinates in $\mathbb{R}^3$, we obtain

$$dx_1^2 + dx_2^2 + dx_3^2 = dr^2 + r^2 d\omega^2,$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $d\omega^2$ is the canonical metric of the round sphere $\mathbb{S}^2$. Let

$$A(r_0, r_1) = \{(x_1, x_2, x_3) \in \mathbb{R}^3; r_0^2 \le x_1^2 + x_2^2 + x_3^2 \le r_1^2\}.$$

If $F : A(r_0, r_1) \subset \mathbb{R}^3 \to \mathbb{R}$ is a radial function, then there exists a positive real function $\varphi : (r_0, r_1) \subset \mathbb{R} \to \mathbb{R}$ such that $F(x_1, x_2, x_3) = \varphi(r)$. In this case, we have

$$\langle\cdot,\cdot\rangle_F = \frac{1}{\varphi(r)^2}(dr^2 + r^2 d\omega^2).$$

Define $G : (r_0, r_1) \to \mathbb{R}$ by $G'(r) = 1/\varphi(r)$. Since $G'(r) > 0$, we have that the function G is invertible. Let $G^{-1} : I \subset \mathbb{R} \to (r_0, r_1)$ be the inverse function of G, where $I = G((r_0, r_1))$, and let $t = G(r)$. Defining

$$h(t) = \frac{G^{-1}(t)}{\varphi(G^{-1}(t))},$$

we have

$$\frac{dr}{\varphi(r)} = dt \quad \text{and} \quad \frac{r}{\varphi(r)} = h(t). \tag{3.1}$$

The metric $\langle\cdot,\cdot\rangle_F$ thus becomes the warped metric

$$\langle\cdot,\cdot\rangle = dt^2 + h(t)^2 d\omega^2, \tag{3.2}$$

where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a smooth function, called the warping function, and M^3_F can be seen as the product $M^3 = I \times \mathbb{S}^2$ with the metric (3.2), where $I \subset \mathbb{R}$ is an interval.

To prove our main theorem, we will need the following Lemma, which is an adaptation of a result due to Eschenburg and Tribuzy, see [19] (see also the main Lemma of [6]):

Lemma 3.1 (Eschenburg-Tribuzy, [19])**.** *Let $Q : U \subset \mathbb{C} \to \mathbb{C}$ be a complex function defined in an open set U of the complex plane. Assume that*

$$|Q_{\bar z}| \le f(z)|Q(z)|$$

where $f \in L^p$, $p > 2$, is a continuous, non-negative real function. Assume further that $z = z_0 \in U$ is a zero of Q. Then either $Q \equiv 0$ in a neighborhood $V \subset U$ of z_0 or

$$Q(z) = (z - z_0)^k Q_k(z),\ \ z \in V,\ \ k \ge 1,$$

where $Q_k(z)$ is a continuous function with $Q_k(z_0) \neq 0$.

This lemma has the following consequence for surfaces homeomorphic to the sphere. The argument is contained in the proof of the main theorem of [6], and we include a proof here for the sake of completeness.

Lemma 3.2. *Let Σ be a Riemann surface homeomorphic to the sphere. Let Qdz^2 denote a complex quadratic differential on Σ. Assume that*

$$|Q_{\bar z}| \le f_0|Q|, \tag{3.3}$$

where $f_0 : \Sigma \to \mathbb{R}$ is a non-negative and L^p function, $p > 2$, and z is a local conformal parameter. Then $Q \equiv 0$ in Σ.

Proof. Let $U \subset \Sigma$ be an open set covered by isothermal coordinates. Assume that the set of zeros of Q in U is not empty and let $z_0 \in U$ be a zero of Q. By the Lemma 3.1, either Q is identically zero in a neighborhood V of z_0 or this zero is isolated and the index of a direction field determined by $Im[Qdz^2] = 0$ is $-k/2$ (hence negative). If, for some coordinate neighborhood V of zero, $Q \equiv 0$, this will be so for the whole Σ, otherwise, the zeroes on the boundary of V will contradict Lemma 3.1. So if Q is not identically zero, all zeroes are isolated and have negative indices. Since Σ has genus zero, the sum of the indices of the singularities of any field of directions is 2 (hence positive) by the Poincaré Index Theorem. This contradiction shows that Q is identically zero. Notice also that Q must have a zero by the Poincaré Index theorem, since the sum of the index is 2 (hence nonzero). $\square$

Proof of Theorem 1.2. Differentiating the second equation of (3.1) relative to r, and using the first one, we have

$$r = \varphi(r)h(t) \Rightarrow 1 = \varphi'(r)h(t) + \varphi(r)h'(t)\frac{dt}{dr} \Rightarrow 1 = \varphi'(r)h(t) + h'(t)$$

which implies

$$\varphi'(r) = \frac{1 - h'(t)}{h(t)}. \tag{3.4}$$

Differentiating (3.4) relative to r and using the first equation of (3.1), we obtain

$$\varphi''(r) = \frac{d}{dt}\left(\frac{1 - h'(t)}{h(t)}\right)\frac{dt}{dr} = \left(\frac{-h''(t)h(t) - (1 - h'(t))h'(t)}{h(t)^2}\right)\frac{1}{\varphi(r)},$$

i.e.,

$$\varphi''(r)\varphi(r) = -\frac{h''(t)}{h(t)} - \frac{(1 - h'(t))h'(t)}{h(t)^2}. \tag{3.5}$$

On the other hand, for $F(x_1, x_2, x_3) = \varphi(r)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, we have

$$\frac{\partial F}{\partial x_i} = x_i\frac{\varphi'(r)}{r} \Rightarrow \frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\varphi'(r)}{r}\delta_{ij} + \frac{x_i x_j}{r^2}\left(\varphi''(r) - \frac{\varphi'(r)}{r}\right).$$

This implies, for $V, W \in \mathbb{R}^3$,

$$\operatorname{Hess} F(V, W) = \frac{\varphi'(r)}{r}(V \cdot W) + \frac{1}{r^2}\left(\varphi''(r) - \frac{\varphi'(r)}{r}\right)(X \cdot V)(X \cdot W).$$

We observe that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial r}\frac{dr}{dt} = \varphi(r)\frac{\partial}{\partial r} \tag{3.6}$$

and $N_F = \varphi(r)N$ gives

$$\begin{aligned}\nu = \left\langle \frac{\partial}{\partial t}, N_F \right\rangle &= \left\langle \varphi(r)\frac{\partial}{\partial r}, \varphi(r)N \right\rangle_F = \varphi(r)^2\left[\frac{1}{\varphi(r)^2}\left(\frac{\partial}{\partial r} \cdot N\right)\right] \\ &= \left(\frac{X}{\|X\|} \cdot N\right),\end{aligned}$$

provided $\partial/\partial r = X/\|X\|$. Since

$$X \cdot X_z = \|X\|(\|X\|)_z = r r_z, \; X \cdot X_{\bar{z}} = \|X\|(\|X\|)_{\bar{z}} = r r_{\bar{z}},$$

we can rewrite the equation (2.15) of Lemma 2.5 as

$$\frac{4}{\alpha\varphi(r)^2} r_z r_{\bar{z}} + \nu^2 = 1. \tag{3.7}$$

This implies,

$$
\begin{aligned}
\frac{4}{\alpha F}\operatorname{Hess} F(X_z, X_{\bar z}) &= \frac{4}{\alpha\varphi(r)}\frac{\varphi'(r)}{r}(X_z\cdot X_{\bar z}) + \frac{4}{\alpha\varphi(r)r^2}\left(\varphi''(r)-\frac{\varphi'(r)}{r}\right)(X\cdot X_z)(X\cdot X_{\bar z})\\
&= \frac{4\varphi'(r)\varphi(r)}{r}\frac{\langle X_z, X_{\bar z}\rangle_F}{\alpha} + \frac{4}{\alpha\varphi(r)r^2}\left(\varphi''(r)-\frac{\varphi'(r)}{r}\right)(rr_z)(rr_{\bar z})\\
&= \frac{2\varphi'(r)\varphi(r)}{r} + \left(\varphi''(r)\varphi(r)-\frac{\varphi(r)\varphi'(r)}{r}\right)\frac{4}{\alpha\varphi(r)^2}r_z r_{\bar z}\\
&= \frac{2\varphi'(r)\varphi(r)}{r} + \left(\varphi''(r)\varphi(r)-\frac{\varphi(r)\varphi'(r)}{r}\right)(1-\nu^2)\\
&= \frac{2(1-h'(t))}{h(t)^2} - \left(\frac{h''(t)}{h(t)} + \frac{(1-h'(t))h'(t)}{h(t)^2} + \frac{1-h'(t)}{h(t)^2}\right)(1-\nu^2)\\
&= \frac{2(1-h'(t))}{h(t)^2} - \left(\frac{h''(t)}{h(t)} + \frac{1-h'(t)^2}{h(t)^2}\right)(1-\nu^2).
\end{aligned}
$$

Since, using (3.4),

$$
\|\nabla F\|^2 = \varphi'(r)^2 = \frac{(1-h'(t))^2}{h(t)^2},
$$

we have

$$
\begin{aligned}
-\|\nabla F\|^2 + \frac{4}{\alpha F}\operatorname{Hess} F(X_z, X_{\bar z}) &= -\frac{(1-h'(t))^2}{h(t)^2} + \frac{2(1-h'(t))}{h(t)^2} - \left(\frac{h''(t)}{h(t)} + \frac{1-h'(t)^2}{h(t)^2}\right)(1-\nu^2)\\
&= \frac{1-h'(t)^2}{h(t)^2} - \left(\frac{h''(t)}{h(t)} + \frac{1-h'(t)^2}{h(t)^2}\right)(1-\nu^2)\\
&= K_{\tan}(t) - (K_{\tan}(t) - K_{\mathrm{rad}}(t))(1-\nu^2).
\end{aligned}
$$

Thus, using Lemma 2.6, p. 10, and the the equation (2.8), p.8, we have

$$
\begin{aligned}
|P| &= \frac{1}{2}\sqrt{H^2 - K + \overline{K}(T\Sigma)}\\
&= \frac{1}{2}\sqrt{H^2 - K - \|\nabla F\|^2 + \frac{4}{\alpha F}\operatorname{Hess} F(X_z, X_{\bar z})}\\
&= \frac{1}{2}\sqrt{H^2 - K + K_{\tan}(t) - (K_{\tan}(t) - K_{\mathrm{rad}}(t))(1-\nu^2)}.
\end{aligned}
\tag{3.8}
$$

On the other hand, since

$$
\frac{\partial r}{\partial z} = \frac{dr}{dt}\frac{\partial t}{\partial z} = \varphi(r)\frac{\partial t}{\partial z},
$$

we have

$$
\begin{aligned}
\operatorname{Hess} F(X_z, N) &= \frac{1}{r^2}\left(\varphi''(r)-\frac{\varphi'(r)}{r}\right)(X\cdot X_z)(X\cdot N)\\
&= \left(\varphi''(r)\varphi(r)-\frac{\varphi'(r)\varphi(r)}{r}\right)\frac{\nu}{\varphi(r)}r_z\\
&= -\left(\frac{h''(t)}{h(t)} + \frac{1-h'(t)^2}{h(t)^2}\right)\frac{\nu}{\varphi(r)}r_z\\
&= -(K_{\tan}(t) - K_{\mathrm{rad}}(t))\frac{\nu}{\varphi(r)}r_z\\
&= -(K_{\tan}(t) - K_{\mathrm{rad}}(t))\nu\frac{\partial t}{\partial z}.
\end{aligned}
\tag{3.9}
$$

Replacing (3.9) in (2.10) and then in (2.9), we obtain

$$P_{\bar{z}} = \frac{\alpha}{2}H_z - \frac{\alpha}{2}\operatorname{Hess} F(X_z, N) = \frac{\alpha}{2}[H_z + (K_{\tan}(t) - K_{\mathrm{rad}}(t))\nu t_z].$$

Since

$$\begin{aligned}
|P_{\bar{z}}| &= \frac{\alpha}{2}|H_z + (K_{\tan}(t) - K_{\mathrm{rad}}(t))\nu t_z| \\
&= \frac{\alpha}{2}|dH(X_z) + (K_{\tan}(t) - K_{\mathrm{rad}}(t))\nu dt(X_z)| \\
&\leq \frac{\alpha}{2}|dH + (K_{\tan}(t) - K_{\mathrm{rad}}(t))\nu dt|\|X_z\| \\
&= \left(\frac{\alpha}{2}\right)^{3/2}|dH + (K_{\tan}(t) - K_{\mathrm{rad}}(t))\nu dt|,
\end{aligned}$$

the hypothesis (1.4) and (3.8) imply

$$|P_{\bar{z}}| \leq 2\left(\frac{\alpha}{2}\right)^{3/2} f|P|.$$

This implies, using Lemma 3.2, p. 12, for $f_0 = 2\left(\frac{\alpha}{2}\right)^{3/2}f$, that $P = 0$ everywhere in Σ. Therefore, by (2.7), Σ is umbilic.

If H is constant, then $|P| \equiv 0$ gives

$$|K_{\tan}(t) - K_{\mathrm{rad}}(t)||\nu||dt| \equiv 0.$$

This and the hypothesis that $K_{\tan}(t) \neq K_{\mathrm{rad}}(t)$, except possibly by a discrete set of values $t \in I$, gives $|K_{\tan}(t) - K_{\mathrm{rad}}(t)| = 0$ only for discrete set, which implies, by continuity, that

$$|\nu||dt| \equiv 0.$$

Let $D_1 = \{z \in \Sigma; \nu = 0\}$ and $D_2 = \{z \in \Sigma; dt = 0\}$. We have $D_1 \cup D_2 = \Sigma$. Observe that, using

$$r_z = \frac{\partial r}{\partial z} = \frac{dr}{dt}\frac{\partial t}{\partial z} = \varphi(r)t_z$$

and analogous expression for $r_{\bar{z}}$, in (3.7), we have

$$\frac{4}{\alpha}t_z t_{\bar{z}} + \nu^2 = 1.$$

This gives that $\nu^2 = 1$ in D_2. Thus, by continuity of ν, we have $D_1 \cap D_2 = \emptyset$, and $D_1 = \emptyset$ or $D_1 = \Sigma$. But, since Σ is compact, there exist $t_0, t_1 \in I$ such that $\Sigma \subset [t_0, t_1] \times \mathbb{S}^2$. Taking the least value of t_1 with this property, we have that Σ is tangent to the slice $\{t_1\} \times \mathbb{S}^2$ and, at this point of tangency, we have $\nu^2 = 1$, i.e., $D_2 \neq 0$. Thus, $D_1 = \emptyset$ and $D_2 = \Sigma$, which implies $dt = 0$ everywhere, and this gives that t is constant, i.e., $X(\Sigma)$ is a slice.

□

References

[1] Juan A. Aledo and Rafael M. Rubio, *Stable minimal surfaces in Riemannian warped products*, J. Geom. Anal. **27** (2017), no. 1, 65–78, DOI 10.1007/s12220-015-9673-8. MR3606544

[2] Hilário Alencar and Gregório Silva Neto, *Isoperimetric inequalities and monotonicity formulas for submanifolds in warped products manifolds*, Rev. Mat. Iberoam. **34** (2018), no. 4, 1821–1852, DOI 10.4171/rmi/1045. MR3896251

[3] Luis J. Alías and Marcos Dajczer, *Uniqueness of constant mean curvature surfaces properly immersed in a slab*, Comment. Math. Helv. **81** (2006), no. 3, 653–663, DOI 10.4171/CMH/68. MR2250858

[4] ______, *Constant mean curvature hypersurfaces in warped product spaces*, Proc. Edinb. Math. Soc. (2) **50** (2007), no. 3, 511–526, DOI 10.1017/S0013091505001069. MR2360513

[5] Luis J. Alías, Debora Impera, and Marco Rigoli, *Hypersurfaces of constant higher order mean curvature in warped products*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 591–621, DOI 10.1090/S0002-9947-2012-05774-6. MR2995367

[6] Hilario Alencar, Manfredo do Carmo, and Renato Tribuzy, *A theorem of Hopf and the Cauchy-Riemann inequality*, Comm. Anal. Geom. **15** (2007), no. 2, 283–298. MR2344324

[7] G. P. Bessa, S. C. García-Martínez, L. Mari, and H. F. Ramirez-Ospina, *Eigenvalue estimates for submanifolds of warped product spaces*, Math. Proc. Cambridge Philos. Soc. **156** (2014), no. 1, 25–42, DOI 10.1017/S0305004113000443. MR3144209

[8] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR867684

[9] K. S. Bezerra, A. Caminha, and B. P. Lima, *On the stability of minimal cones in warped products*, Bull. Braz. Math. Soc. (N.S.) **45** (2014), no. 3, 485–503, DOI 10.1007/s00574-014-0059-5. MR3264802

[10] Hubert Bray and Frank Morgan, *An isoperimetric comparison theorem for Schwarzschild space and other manifolds*, Proc. Amer. Math. Soc. **130** (2002), no. 5, 1467–1472, DOI 10.1090/S0002-9939-01-06186-X. MR1879971

[11] Hubert L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267. MR1908823

[12] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49, DOI 10.2307/1995057. MR0251664

[13] Simon Brendle, *Constant mean curvature surfaces in warped product manifolds*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 247–269, DOI 10.1007/s10240-012-0047-5. MR3090261

[14] Manfredo Perdigão do Carmo, *Riemannian geometry*, Portuguese edition, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. MR1138207

[15] Shiing-shen Chern, *An elementary proof of the existence of isothermal parameters on a surface*, Proc. Amer. Math. Soc. **6** (1955), 771–782, DOI 10.2307/2032933. MR74856

[16] Shiing Shen Chern, *On surfaces of constant mean curvature in a three-dimensional space of constant curvature*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 104–108, DOI 10.1007/BFb0061413. MR730266

[17] Marcos Dajczer and Jaime Ripoll, *An extension of a theorem of Serrin to graphs in warped products*, J. Geom. Anal. **15** (2005), no. 2, 193–205, DOI 10.1007/BF02922192. MR2152479

[18] Luther Pfahler Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton, N. J., 1949. 2d printing. MR0035081

[19] J.-H. Eschenburg and R. Tribuzy, *Conformal mappings of surfaces and Cauchy-Riemann inequalities*, Differential geometry, Pitman Monogr. Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 149–170. MR1173039

[20] Sandra C. García-Martínez, Debora Impera, and Marco Rigoli, *A sharp height estimate for compact hypersurfaces with constant k-mean curvature in warped product spaces*, Proc. Edinb. Math. Soc. (2) **58** (2015), no. 2, 403–419, DOI 10.1017/S0013091514000157. MR3341446

[21] Vicent Gimeno, *Isoperimetric inequalities for submanifolds. Jellett-Minkowski's formula revisited*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 3, 593–614, DOI 10.1112/plms/pdu053. MR3342099

[22] Pengfei Guan, Junfang Li, and Mu-Tao Wang, *A volume preserving flow and the isoperimetric problem in warped product spaces*, Trans. Amer. Math. Soc. **372** (2019), no. 4, 2777–2798, DOI 10.1090/tran/7661. MR3988593

[23] Pengfei Guan and Siyuan Lu, *Curvature estimates for immersed hypersurfaces in Riemannian manifolds*, Invent. Math. **208** (2017), no. 1, 191–215, DOI 10.1007/s00222-016-0688-y. MR3621834

[24] Heinz Hopf, *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr. **4** (1951), 232–249, DOI 10.1002/mana.3210040122 (German). MR0040042

[25] ———, *Differential geometry in the large*, Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, Berlin, 1983. Notes taken by Peter Lax and John Gray; With a preface by S. S. Chern. MR707850

[26] Arthur Korn, *Zwei Anwendungen der Methode der sukzessiven Annäherungen*, Schwarz-Festschr. (1914), 215–229 (German). Zbl 45.0568.01

[27] Ravindra Shripad Kulkarni, *Curvature and metric*, Ann. of Math. (2) **91** (1970), 311–331, DOI 10.2307/1970580. MR257932

[28] L. Lichtenstein, *Zur Theorie der konformen Abbildung nichtanalytischer, singularitätenfreier Flächenstücke auf ebene Gebiete*, Krak. Anz. (1916), 192–217 (German). Zbl 46.0547.01

[29] Sebastián Montiel, *Stable constant mean curvature hypersurfaces in some Riemannian manifolds*, Comment. Math. Helv. **73** (1998), no. 4, 584–602, DOI 10.1007/s000140050070. MR1639892

[30] Sebastián Montiel, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), no. 2, 711–748, DOI 10.1512/iumj.1999.48.1562. MR1722814 (2001f:53131)

[31] F. Pacard and X. Xu, *Constant mean curvature spheres in Riemannian manifolds*, Manuscripta Math. **128** (2009), no. 3, 275–295, DOI 10.1007/s00229-008-0230-7. MR2481045

[32] Peter Petersen, *Riemannian geometry*, 2nd ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR2243772

[33] Manuel Ritoré, *Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces*, Comm. Anal. Geom. **9** (2001), no. 5, 1093–1138, DOI 10.4310/CAG.2001.v9.n5.a5. MR1883725

[34] Juan J. Salamanca and Isabel M. C. Salavessa, *Uniqueness of ϕ-minimal hypersurfaces in warped product manifolds*, J. Math. Anal. Appl. **422** (2015), no. 2, 1376–1389, DOI 10.1016/j.jmaa.2014.09.028. MR3269517

[35] Gregório Silva Neto, *Stability of constant mean curvature surfaces in three-dimensional warped product manifolds*, Ann. Global Anal. Geom. **56** (2019), no. 1, 57–86, DOI 10.1007/s10455-019-09656-x. MR3962026

[36] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors. MR1215481

[37] Jie Wu and Chao Xia, *On rigidity of hypersurfaces with constant curvature functions in warped product manifolds*, Ann. Global Anal. Geom. **46** (2014), no. 1, 1–22, DOI 10.1007/s10455-013-9405-x. MR3205799

[38] ———, *Hypersurfaces with constant curvature quotients in warped product manifolds*, Pacific J. Math. **274** (2015), no. 2, 355–371, DOI 10.2140/pjm.2015.274.355. MR3332908

Instituto de Matemática, Universidade Federal de Alagoas, Maceió, AL, 57072-900, Brazil,
Email address: `hilario@mat.ufal.br`

Instituto de Matemática, Universidade Federal de Alagoas, Maceió, AL, 57072-900, Brazil,
Email address: `gregorio@im.ufal.br`