Hypersurfaces With Constant Mean Curvature in Space Forms*

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Invited Paper

ABSTRACT

It has been found recently that a number of results on minimal submanifolds that involve the second fundamental form can be naturally extended to the case of constant mean curvature if one replaces the second fundamental form by a related tensor. This paper describes some of these results and raises further questions.

Key words: constant mean curvature, spheres, stable, hyperbolic space, Morse index.

1. Surfaces with constant mean curvature behave in some questions in a manner strikingly different from minimal surfaces. For instance, the Gauss map of a complete minimal surface in R^3 fill the whole sphere except for four points, whereas the Gauss map of the complete circular cylinder is merely a circle in the sphere, that is, it is as thin as it can be. Also, the Morse index of a complete minimal surface M in R^3 is finite if and if the total curvature of M is finite, whereas all complete noncompact surfaces with nonzero constant mean curvature have infinite index.

On the other hand, in some other questions they behave in such a similar way that theorems on minimal surfaces have natural extensions to surfaces with nonzero constant mean curvature. They both satisfy elliptic partial differential equations which are the Euler equations of similar variational problem, and their Plateau's problems follow patterns reasonably alike. Actually, the first investigators in the theory of surfaces of constant mean curvature explored quite well these similarities and only recently it became clear that the differences are an important part of the whole picture.

In this paper, we want to mention some recent results in the theory of hypersurfaces with constant mean curvature in space forms that are natural extensions of theorems for minimal submanifolds.

2. We will begin with a well known result for minimal submanifolds. First, let us fix some notation. Let $f\colon M^n \to \overline{M}^{n+p}$ be an immersion of an n-manifold M^n into a Riemannian $(n+p)$-manifold $\overline{M}^{n+p}$. Fix $p \in M$ and choose a local orthonormal frame of normal fields $e_{n+1}, \dots, e_{p+n}$ around p. For each α, $\alpha = n+1, \dots, p+n$, define a linear map $A_\alpha\colon T_pM \to T_pM$ by

$$\langle A_\alpha X, Y\rangle = \langle \overline{\nabla}_X Y, e_\alpha\rangle,$$

where X, Y are tangent vector fields and $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$. The map A_α can be diagonalized, i.e., for each α, there exists a tangent basis $\{e_1^\alpha, \dots, e_n^\alpha\}$ such that $A_\alpha e_i^\alpha = k_i^\alpha\, e_i^\alpha$,

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$i = 1, \dots, n$. We then define the mean curvature normal vector:

$$H = \sum_{\alpha} \left(\frac{1}{n} \sum_{i} k_i^{\alpha} \right) e_{\alpha},$$

and the square of the normal of the second fundamental form:

$$|A|^2 = \sum_{i\alpha} (k_i^{\alpha})^2 .$$

It is known that, up to orientations, the above objects do not depend on the choices made.

Let us now specialize to the case where M^n is compact, $\overline{M}^{n+p}$ is the sphere $S^{n+p}(c) \subset R^{n+p+1}$ with curvature $c > 0$ in the euclidean space R^{n+p+1}, and f is a minimal immersion ($H \equiv 0$).

Theorem A. (Simons, 1968, Chern-do-Carmo-Kobayashi, 1970, Lawson, 1969). *Assume that* $|A|^2 \leq nc\left(2 - \frac{1}{p}\right)^{-1}$. *Then:*

(i) *Either* $|A|^2 \equiv 0$ *(and* M^n *is totally geodesic in* $S^{n+p}(c)$*) or* $|A|^2 \equiv nc\left(2 - \frac{1}{p}\right)^{-1}$.

(ii) $|A|^2 \equiv nc\left(2 - \frac{1}{p}\right)^{-1}$ *if and only if:*

a) $p = 1$, *and* M^n *is locally a Clifford torus, i.e., a product of spheres of appropriate radii, in* $S^{n+1}(c)$.

b) $p = 2$, *and* $M^n = M^2$ *is locally a Veronese surface in* $S^4(c)$.

We want to extend the above theorem to the case of constant mean curvature. For that, it is convenient to modify slightly the second fundamental form and to introduce a new linear map $\phi_{\alpha} : T_pM \to T_pM$ by

$$\langle \phi_{\alpha} X, Y \rangle = -\langle A_{\alpha} X, Y \rangle + \langle H, e_{\alpha} \rangle \langle X, Y \rangle.$$

We define a tensor $\phi \colon T_pM \to T_pM^{\perp}$ by $\phi(X , Y) = \sum_{\alpha} \langle \phi_{\alpha} X, Y \rangle e_{\alpha}$. The map ϕ_{α} can also be diagonalized:

$$\phi_{\alpha}(e_i^{\alpha}) = \mu_i^{\alpha} e_{\alpha}.$$

It can be readily checked that trace $\phi_{\alpha} = 0$, and that

$$|\phi|^2 \stackrel{\text{def}}{=} \sum_{i\alpha} (\mu_i^{\alpha})^2 = \frac{1}{2n} \sum_{ij\alpha} (k_i^{\alpha} - k_j^{\alpha})^2.$$

Notice that while $|A|^2$ measures how far is $M^n \subset \overline{M}^{n+p}$ from being totally geodesic, $|\phi|^2$ measures how far it is from being totally umbilic.

It turns out that ϕ is precisely what is needed to extend the above theorem to constant mean curvature.

Consider first the case of codimension $p = 1$ and let us restrict ourselves, for convenience, to the unit sphere $S^{n+1}(1)$. Thus M^n is compact and $f \colon M^n \to S^{n+1}$ is a hypersurface with constant mean curvature H. Without loss of generality, we can assume that $H \geq 0$. For each H, set

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n}(n-1)} Hx - n(H^2 + 1),$$

and let B_H be the square of the positive root of $P_H(x) = 0$. Notice that $B_0 = n$.

Before the statement of the next theorem, we need a definition. An $H(r)$-*torus* in $S^{n+1}(1)$ is obtained by taking the product immersion $S_r^{n-1} \times S^1_{\sqrt{1-r^2}} \subset R^n \times R^2$, where, for instance, S_r^{n-1} is a sphere of radius r in R^n. It is easily checked that an $H(r)$-*torus* is actually contained in $S^{n+1}(1)$ and has constant mean curvature.

THEOREM 1. (Alencar & do Carmo, 1994). *Assume that* $|\phi|^2 \leq B_H$. *Then:*

(i) *Either* $|\phi|^2 \equiv 0$ *(and* M *is totally umbilic) or* $|\phi|^2 \equiv B_H$.

(ii) $|\phi|^2 \equiv B_H$ *if and only if:*

a) $H = 0$ *and* M^n *is locally a Clifford torus.*

b) $H \neq 0$, $n \geq 3$, *and* M^n *is locally an* $H(r)$*-torus with* $r^2 < \frac{n-1}{n}$.

c) $H \neq 0$, $n = 2$, *and* M^n *is locally an* $H(r)$*-torus, for any* $r \neq \frac{n-1}{n}$, $0 < r < 1$.

REMARK 1. A curious fact is that not all $H(r)$*-tori* appear in the equality case (b), but only those for which $r^2 < \frac{n-1}{n}$. It can be easily checked that those $H(r)$-tori for which $r^2 > \frac{n-1}{n}$ have $|\phi|^2 > B_H$. We will come back to this later.

REMARK 2. An upper bound for $|\phi|^2$ which implies that below this bound M^n is totally umbilic has been obtained by (Okumura, 1974). However, the bound obtained there is not sharp.

We now consider the case of codimension $p > 1$. In this situation, the natural definition of constant mean curvature is that the mean curvature vector H is parallel in the normal connection. It is curious that, apparently, there is no variational interpretation of this condition for $p > 1$. At any rate, the problem of extending Theorem 1 to this situation makes sense and was solved by Walcy Santos in her Doctor's thesis at IMPA. The results are as follows.

Let W be compact and let $f \colon M^n \to S^{n+p}(1)$ be an immersion with parallel mean curvature vector H. When $H \equiv 0$, we assume that there exists a normal parallel direction. Set $\phi_H = \langle \phi, H \rangle$.

We will denote by S^n_r an n-sphere of radius r and by $S^n(k)$ an n-sphere of curvature k. For simplicity, we will write $S^n(k) \hookrightarrow_u S^{n+1}(1)$ to mean that $S^n(k)$ is an umbilic hypersurface of $S^{n+1}(1)$.

THEOREM 2. (Santos, 1992). *Assume that* $p > 1$ *and that*

$$|\phi|^2 \leq \left(\frac{p-1}{2p-3}\right)\left(-\frac{n(n-2)}{\sqrt{n(n-1)}}|\phi_H| + n(H^2+1)\right). \tag{1}$$

Then:

(i) $|\phi| = const.$, *and either* $|\phi|^2 = 0$ *or equality holds in* (1).

(ii) *Equality holds in (1) if and only if one of the following cases occur:*

a) M *is a minimal Clifford torus*

$$S^m_{r_1} \times S^{n-m}_{r_2} \subset S^{n+1}(1+H^2) \hookrightarrow_u S^{n+2}(1),$$

where

$$r_1 = \left(\frac{m}{n(1+H^2)}\right)^{1/2}, \qquad r^2 = \left(\frac{n+m}{n(1+H^2)}\right)^{1/2},$$

or $M = M^2$ *is a Veronese surface*

$$M^2 \subset S^4(1+H^2) \hookrightarrow_u S^5(1).$$

b) *For all* H_0, $0 \leq H_0 < H$, M *is an* $H_1(r)$*-torus*

$$S^{n-1}_r \text{ x } S^1_{r_1} \subset S^{n+1}(1+H_0^2) \hookrightarrow_u S^{n+2}(1),$$

where

$$H_1^2 + H_0^2 = H^2, \;\; r^2 + r_1^2 = (1+H_0^2)^{-1}.$$

If $n \geq 3$, we have only those $H_1(r)$-tori with $r^2 < \frac{n-1}{n}(1+H_0^2)^{-1}$; if $n = 2$, the only condition is $r^2 \neq \frac{1}{2}(1+H_0^2)^{-1}$.

A further question raised by Theorem 1 can be described as follows. Let us restrict ourselves to $p = 1$, although the problems makes sense for any p. Consider the set of compact hypersurfaces M^n of $S^{n+1}(1)$ with constant mean curvature H and with $|\phi|^2 = \text{const}$. The question asks whether the set of values of $|\phi|^2$ in this situation is discrete.

For $H \equiv 0$, this question was posed in (Chern, do Carmo & Kobayashi, 1970) about 20 years ago and, even in this simpler case, little progress has been made. The most important contribution is due to (Peng & Terng, 1983) who proved that if $n = 3$, $H \equiv 0$, $|A|^2 = \text{const.}$, and $3 < |A|^2 \leq 6$, then $|A|^2 \equiv 6$; in this latter case, M^3 is an isoparametric hypersurface of $S^4(1)$ with three distinct principal curvatures.

Let us recall that a hypersurface of a space form is called *isoparametric* if all principal curvatures are constant. In the case of hypersurfaces $M^3 \subset S^4(1)$, they are well known and are: umbilic ($k_1 = k_2 = k_3$), H-tori ($k_1 = k_2 \neq k_3$) or the so-called Cartan hypersurfaces (k_1, k_2 and k_3 distinct). The principal curvatures can be explicitly computed, and in each of the above three families there is one minimal hypersurface.

For $H \neq 0$, there is the recent result of Almeida & Brito who generalized the result of Peng & Terng to constant mean curvature.

Theorem 3. (Almeida & Brito, 1990). *Let M^3 be a compact and let $f \colon M^3 \to S^4(1)$ be an immersion with constant mean curvature H. Assume that $|\phi|^2 = \text{const.}$ and that $|\phi|^2 \leq 6 + 6H^2$. Then M^3 is an isoparametric hypersurface of $S^4(1)$. Furthermore, if $4 + 6H^2 \leq |\phi|^2 \leq 6 + 6H^2$ then $|\phi|^2 = 6 + 6H^2$ and M^3 is a Cartan hypersurface.*

The result of Almeida & Brito solves the above question for $n = 3$ and $|\phi|^2 \leq 6 + 6H^2$. It also throws some light on what happens to some $H(r)$-tori that are missing in Theorem 1, namely, those for which $r^2 > \frac{2}{3}$: they all lie in the interval $B_H < |\phi|^2 < 4 + 6H^2$.

An approximate graphic representation of all these results appear in Fig. 1.*

3. We now pass to another question where the map ϕ appears naturally; this time the ambient space is the euclidean space R^{n+1}.

Let M^n be a complete noncompact hypersurface of R^{n+1}. Let $p \in M$ and denote by $B_R(p) = B_R$ the geodesic ball of center p and radius R in the induced metric. If M is minimal, we say that M is *stable* if for all piecewise smooth functions $f \colon M \to R$ with compact support, we have that

$$\int_M |\nabla f|^2 dM \geq \int_M |A|^2 f^2 dM; \tag{1}$$

here ∇f is the gradient of f in the induced metric and $|A|^2$ is the square of the norm of the second fundamental form. The following is a kind of Bernstein theorem, where we have replaced graphs by stable and added a growth condition on $|A|^2$.

Theorem B. (do Carmo & Peng, 1982). *Let M^n be a minimal complete noncompact hypersurface of R^{n+1}. Assume that M is stable and that*

$$\lim_{R\to\infty} \frac{\int_{B_R} |A|^2 dM}{R^{2+2q}} = 0, \qquad q < \sqrt{\frac{2}{n}}.$$

*After this paper was written, a paper by (Chang, 1993) showed that the area in Fig. 1 labeled with unknown is actually void.

Then M^n is a hyperplane of R^{n+1}.

To extend the above theorem to hypersurfaces with constant mean curvature H, we replace A by ϕ and take into consideration the fact that stability for constant mean curvature means either condition (1) (strong stability) or that condition (1) holds only for those compactly supported f that satisfy $\int_M f dM = 0$ (weak stability). This causes some additional complications in the proof of the desired extension. At any rate, the following theorem can be proved.

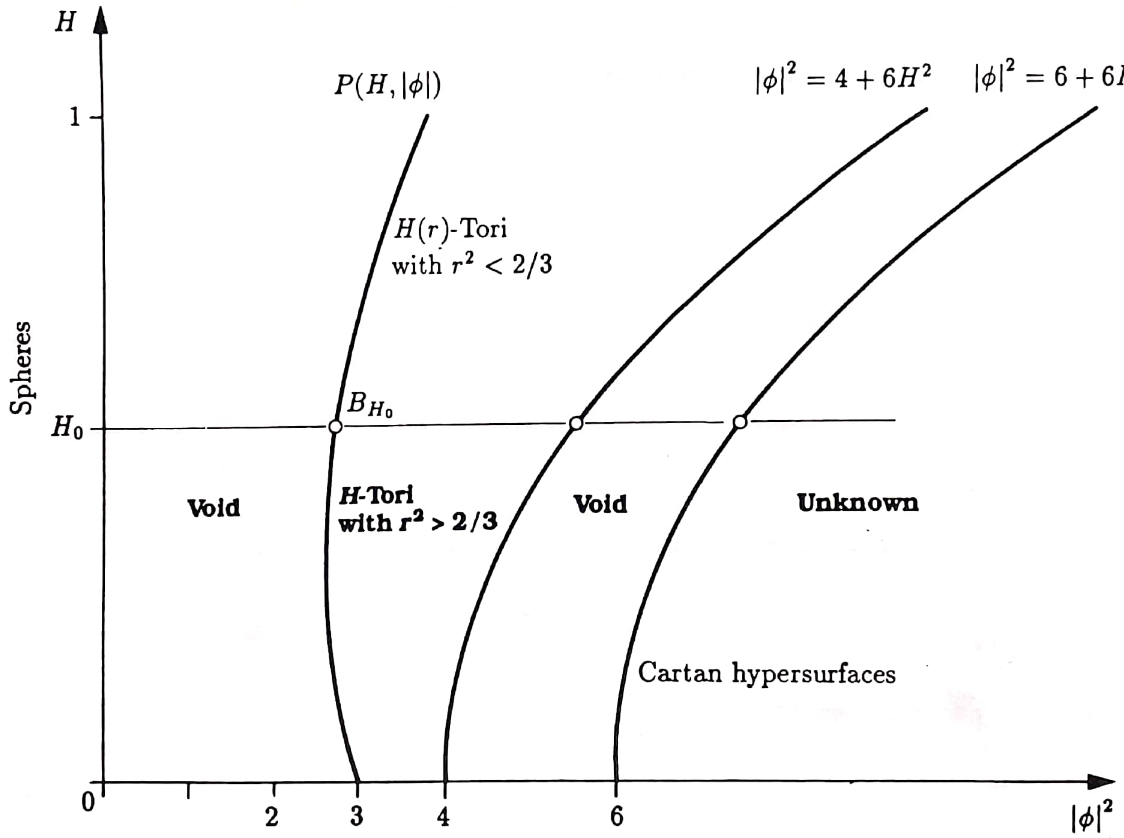

Fig. 1

THEOREM 4. *Let M^n, $n \leq 5$, be a complete noncompact hypersurface of R^{n+1} with constant mean curvature H. Assume that M is strongly stable and that*

$$\lim_{R\to\infty} \frac{\int_{B_R} |\phi|^2 dM}{R^{2+2q}} = 0, \qquad q \leq \frac{2}{6n+1}.$$

Then M^n is a hyperplane of R^{n+1}.

PROOF. We first need a version of Simmon's inequality for the tensor ϕ rather than the tensor A.

Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M that diagonalizes ϕ at each point, i.e., $\phi e_i = \mu_i e_i$. Then (cf. Alencar, do Carmo, 1994).

(2) $$\frac{1}{2}\Delta|\phi|^2 = \sum_{ijk} \phi_{ijk}^2 + \sum_i \mu_i (tr\phi)_{ii} + \frac{1}{2}\sum_{ij} R_{ijij}(\mu_i - \mu_j)^2.$$

Here Δ is the Laplacian in the induced metric of M, ϕ_{ijk} are the components of the covariant derivative of the tensor ϕ, and R_{ijij} is the sectional curvature of the plane $\{e_i, e_j\}$. Since $\mu_i = H - k_i$, where k_i are the principal curvatures of the hypersurface, we obtain that $tr\phi = \sum \mu_i = 0$. By Gauss' formula, we conclude that the last term in (2) is given by

$$\begin{aligned}\tfrac{1}{2}\textstyle\sum_{ij} R_{ijij}(\mu_i - \mu_j)^2 \;&=\; \tfrac{1}{2}\textstyle\sum_{ij} \mu_i\mu_j(\mu_i - \mu_j)^2 \\ &-\tfrac{H}{2}\textstyle\sum_{ij}(\mu_i + \mu_j)(\mu_i - \mu_j)^2 + \tfrac{H^2}{2}\textstyle\sum_{ij}(\mu_i - \mu_j)^2.\end{aligned} \tag{3}$$

Now, since $\sum \mu_i = 0$, it is easily checked that

$$\sum_{ij}(\mu_i - \mu_j)^2 = 2n|\phi|^2, \tag{4}$$

$$\sum_{ij}(\mu_i + \mu_j)(\mu_i - \mu_j)^2 = 2n\sum_i \mu_i^3, \tag{5}$$

$$\sum_{ij}\mu_i\mu_j(\mu_i - \mu_j)^2 = \left(\sum_i \mu_i^2\right)^2 = -2|\phi|^4. \tag{6}$$

From (2)-(6), it follows that

$$\frac{1}{2}\Delta|\phi|^2 = |\phi|\Delta|\phi| + |\nabla|\phi||^2 = \sum_{ijk}\phi_{ijk}^2 - |\phi|^4 - nH\sum_i \mu_i^3 + nH^2|\phi|^2. \tag{7}$$

Again, since $\sum_i \mu_i = 0$, it follows by an argument in (do Carmo-Peng, 1982 cf. Eqs.(2.3) and (2.4)) that

$$\sum_{ijk}\phi_{ijk}^2 \geq \frac{2}{n}|\nabla|\phi||^2 + |\nabla|\phi||^2. \tag{8}$$

Furthermore, by using a lemma of Okumura (for a proof, see Alencar, do Carmo, 1991, Lemma 2.6), we have

$$\sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}|\phi|^3. \tag{9}$$

Finally, by putting together (7), (8) and (9), we obtain the following version of Simmons' inequality

$$|\phi||\nabla|\phi|| + |\phi|^4 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3 - nH^2|\phi|^2 \geq \frac{2}{n}|\nabla|\phi||^2. \tag{10}$$

Now, introduce $f|\phi|^{1+q}$ in the stability inequality (1). We obtain noticing that $|A|^2 = |\phi|^2 + nH^2$,

$$\begin{aligned}&\int_M (|\phi|^{4+2q} + nH^2|\phi|^{2+2q})f^2 \\ &\leq (1+q)^2\int_M |\phi|^{2q}|\nabla|\phi||^2 f^2 + 2(1+q)\int_M |\phi|^{2q+1}f(\nabla f\cdot\nabla|\phi|) \\ &+ \int_M |\phi|^{2q+2}|\nabla f|^2.\end{aligned} \tag{11}$$

Inequalities (10) and (11) will be the geometric informations that we need to prove Theorem (4).

The proof now follows essentially the pattern of (do Carmo, Peng, 1982) taking into account the presence of terms that contain H. We only stress those points which may lead to some differences.

Multiplying (10) by $|\phi|^{2q}f^2$ and integrating over M, we obtain

$$\frac{2}{n}\int_M |\phi|^{2q}f^2|\nabla|\phi||^2 \leq \int_M |\phi|^{2q+1}f^2\Delta|\phi| + \int_M |\phi|^{4+2q}f^2$$

$$+\frac{n(n-2)}{\sqrt{n(n-1)}}H\int_M |\phi|^{2q+3}f^2 - nH^2\int_M |\phi|^{2+2q}f^2.$$

Since

$$\int_M |\phi|^{2q+1}f^2\Delta|\phi| = -\int_M \nabla(|\phi|^{2q+1}f^2)\cdot\nabla|\phi|$$

$$= -(2q+1)\int_M |\phi|^{2q}f^2|\nabla|\phi||^2 - 2\int_M |\phi|^{2q+1}f(\nabla|\phi|\cdot\nabla f),$$

we have, by using the above inequality and multiplying it by $(1+q)$,

$$(1+q)\left(\frac{2}{n}+2q+1\right)\int_M |\phi|^{2q}f^2|\nabla|\phi||^2$$

$$\tag{12} \leq (1+q)\int_M |\phi|^{4+2q}f^2 - 2(1+q)\int_M |\phi|^{2q+1}f(\nabla f\cdot\nabla|\phi|)$$

$$+\frac{n(n-2)}{\sqrt{n(n-1)}}(1+q)H\int_M |\phi|^{2q+3}f^2 - nH^2(1+q)\int_M |\phi|^{2+2q}f^2.$$

Now, sum up (12) and (11) to obtain, after simplification,

$$(1+q)\left(\frac{2}{n}+q\right)\int_M |\phi|^{2q}f^2|\nabla|\phi||^2$$

$$\tag{13} \leq q\int_M |\phi|^{4+2q}f^2 + \int_M |\phi|^{2+2q}|\nabla f|^2$$

$$+(1+q)\frac{n(n-2)}{\sqrt{n(n-1)}}H\int_M |\phi|^{3+2q}f^2 - (2+q)nH^2\int_M |\phi|^{2+2q}f^2.$$

By using in the middle term of the right hand side of (11) the fact that

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2, \quad \text{for all} \quad \varepsilon > 0,$$

with $a = |f\nabla|\phi||$, $b = |\phi||\nabla f|$, we obtain that the stability inequality can be written as

$$\int_M (|\phi|^{4+2q} + nH^2|\phi|^{2+2q})f^2$$

$$\tag{14} \leq (1+q)(1+q+\varepsilon)\int_M |\phi|^{2q}|\nabla|\phi||^2f^2 + \left(1+\frac{1+q}{\varepsilon}\right)\int_M |\phi|^{2+2q}|\nabla f|^2.$$

By introducing (13) into (14), simplifying, and collecting terms, we obtain finally

$$\tag{15} \int_M f^2|\phi|^{2+2q}\{A|\phi|^2 - B|\phi| + C\} \leq D\int_M |\phi|^{2q+2}|\nabla f|^2,$$

where

$$A = 1 - (1+q+\varepsilon)\left(\frac{2}{n}+q\right)^{-1} q,$$

$$B = (1+q+\varepsilon)\left(\frac{2}{n}+q\right)^{-1}(1+q)\frac{n(n-2)}{\sqrt{n(n-1)}}H,$$

$$C = \left(1 + (1+q+\varepsilon)\left(\frac{2}{n}+q\right)^{-1}(2+q)\right) nH^2$$

$$D = (1+q+\varepsilon)\left(\frac{2}{n}+q\right)^{-1} + 1 + \frac{1+q}{\varepsilon}.$$

By using Young's inequality in (15) in the same way as it was used in (do Carmo, Peng, 1982, Eq. (2.11)), we obtain

$$\int_M f^2|\phi|^{2+2q}\{A|\phi|^2 - B|\phi| + C\} \leq \delta \int_M f^2|\phi|^{4+2q} + \beta_1 \int_M \frac{|\phi|^2|\nabla f|^{2(1+q)}}{f^{2q}}, \tag{16}$$

where $\beta_1 > 0$ is a constant (depending on n, ε and q) and $\delta > 0$ can be made arbitrarily small.

Now we have to proceed somewhat differently from (do Carmo, Peng, 1982). Set $\varepsilon = \frac{2}{6n+1}$. By using the fact that $q \leq \frac{2}{6n+1}$, we can easily show that $A > 0$. If, in addition, $n \leq 5$, we claim that $B^2 - 4AC < 0$.

To see that, we first show by a long but straightforward computation that

$$\begin{aligned}
\Delta_n &= B^2 - 4AC \\
&= \frac{nH^2}{(n-1)(2+nq)^2}\{n^4q^4 + 2n^4(\varepsilon + 2n^4(\varepsilon+2)q^3 + n^2(n^2\varepsilon^2 + 6n^2\varepsilon + 6n^2 - 16n + 16)q^2 \\
&+ 2n[n^3\varepsilon^2 + (3n^2 - 8n + 8)n\varepsilon + 2(n^3 - 4n^2 - 4n + 8)]q \\
&+ [n^2(n-2)^2\varepsilon^2 + 2n(n^3 - 4n^2 - 4n + 8)\varepsilon + n^4 - 4n^3 - 12n^2 + 16]\}.
\end{aligned}$$

We first observe that

$$n^4 - 4n^3 - 12n^2 + 16$$

is negative for $2 \leq n \leq 5$ and positive for $n \geq 6$. Further $n^3 - 4n^2 + 8$ is positive for $n \leq 6$. Thus if $n \geq 6$ and q is sufficiently small, the above polynomial in q is positive. It follows that it suffices to check that $\Delta_n < 0$, for $n = 2, 3, 4, 5$. This can easily be done numerically and completes the proof of our claim.

Now set $\overline{A} = A - \delta$ and choose δ small enough so that we still have $B^2 - 4\overline{A}C < 0$ and $\overline{A} > 0$. It follows from (16) that

$$\int_M f^2|\phi|^{2+2q} \leq \beta_2 \int_M \frac{|\phi|^2|\nabla f|^{2(1+q)}}{f^{2q}}. \tag{17}$$

By changing f into f^{1+q} in (17), we obtain our final estimate

$$\int_M f^{2+2q}|\phi|^{2+2q} \leq \beta_3 \int_M |\phi|^2|\nabla f|^{2+2q},$$

The proof now follows exactly as in (do Carmo Peng, 1982).

Remark. The stronger result that Theorem 4 holds for weakly stable hypersurfaces is probably true. We can prove it with the additional assumption that $T(R) = \int_{B_R} \phi dM$ has polynomial growth, i.e., there exist positive numbers c and α such that $T(R) \leq cR^\alpha$. We will come back to that in a future paper.

4. We will describe still another question in which the tensor ϕ appears in a natural way, this time in the hyperbolic space. The results here are incomplete and we hope that this may be looked upon as an interesting question.

Let M^2 be a (two-dimensional) surface immersed in the hyperbolic space $H^3(-1)$ of constant sectional curvature -1, and assume that it has constant mean curvature H. Assume furthermore that M is complete and consider the Morse Index $\mathrm{Ind}(M)$ of M. Then the following results are known:

1) If $H^2 > 1$ (i.e., in the "euclidean range") then

$$\mathrm{Ind}(M) < \infty \leftrightarrow M \,\text{compact}.$$

2) If $H^2 = 1$, then

$$\mathrm{Ind}(M) < \infty \leftrightarrow \int_M |\phi|^2 < \infty.$$

(1) is a result of (A. Silveira, 1987). (2) generalizes the well known result of (Fischer-Colbrie, 1985) and can be found in (do Carmo & Silveira, 1990). It can be shown, by examples, that if $H^2 < 1$, Ind(M) $< \infty$ does not imply that $\int_M |\phi|^2 dM < \infty$ (actually there are examples of stable surfaces with constant H in $H^3(-1)$, $H^2 < 1$, that have $\int_M |\phi|^2 dM = \infty$, (see Silveira, 1987 p. 635). The question is whether the converse holds, i.e., assuming $H^2 < 1$, does $\int_M |\phi|^2 dM < \infty$ implies that Ind $M < \infty$?

For $H = 0$, the above has been proved to be true by Geraldo de Oliveira in his Paris thesis (Oliveira, 1990). If the general case turns out to be true, it will give a nice picture for the behaviour of the Morse Index of surfaces in hyperbolic 3-space.*

Let me conclude with a related question. We know of no example of a complete noncompact surface with constant mean curvature $H \neq 0$ in R^3 with $\int_M |\phi|^2 dM < \infty$. Are there such examples, or is it true that, in this case, $\int_M |\phi|^2 dM < \infty$ implies that M is compact?

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