A THEOREM OF REILLY FOR THE LINEARIZED OPERATOR OF r^{th} MEAN CURVATURE AND APPLICATIONS TO STABILITY

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In this note, we describe an estimate for the first eigenvalue of the linearized operator of the r^{th} mean curvature of immersed closed hypersurfaces of R^{m+1} and H^{m+1}; we apply this to conclude that the stable closed hypersurfaces of R^{m+1}, of constant curvature H_{r+1} are the round spheres.

Reilly's Theorem: Let M^n be a closed submanifold of R^{m+1} and let H_1, λ_1 be the mean curvature and first eigenvalue of the Laplacian of M respectively. R. Reilly proved [R1]:

$$\frac{\lambda_1}{n} \leq \frac{1}{\mathrm{Vol}(M)} \int_M H_1^2 \tag{1}$$

and equality occurs precisely when M is minimally immersed in a sphere of R^{m+1}. Hence when $n = m$, equality means M is a sphere.

Reilly's theorem extends easily to immersions in the unit sphere S^{m+1} by applying (1) to the immersion $M \to S^{m+1} \subset R^{m+2}$:

$$\frac{\lambda_1}{n} - 1 \leq \frac{1}{\mathrm{Vol}(M)} \int_M H_1^2. \tag{2}$$

For immersions of M^m in H^{m+1}, the situation is more subtle. E. Heintze obtained some results [H] and the best result was obtained by A. El Soufi and S. Ilias [S-I]:

$$\frac{\lambda_1}{n} + 1 \leq \frac{1}{\mathrm{Vol}(M)} \int_M H_1^2,\ m \geq 2, \tag{3}$$

and equality occurs precisely when M is minimally immersed in a geodesic sphere of radius arch $\sqrt{\frac{m}{\lambda_1}}$.

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We pursue the study of $\lambda_1 = \lambda_1(L_r)$, where L_r is the linearised operator of $S_{r+1} = \begin{pmatrix} m \\ r+1 \end{pmatrix} H_{r+1}$ arising from normal variations of an immersed hypersurfaces M^m in R^{m+1}. Here S_r is the r^{th} symmetric function of the eigenvalues of the shape operator A. Details concerning L_r can be found in [R2], [A-C-C], and [Ro]. Briefly, $L_r(f) = \operatorname{div}(T_r \nabla f)$, where T_r is the r'th Newton transformation arising from A:

$$T_0 = I,\ T_r = S_r I - A T_{r-1} (\mathrm{so} L_0 = \Delta).$$

Our Theorem: In R^{m+1} we are able to generalize Reilly's result in the best possible way:

$$\lambda_1^{L_r} \int_M H_r \le C(r) \int_M H_{r+1}^2, \tag{4}$$

where M is an immersed closed hypersurface in R^{m+1} with $H_{r+1} > 0$ and $C(r) = (m-r)\begin{pmatrix} m \\ r \end{pmatrix}$. Equality holds precisely when M is a sphere.

We also prove that if M extends to an isometric immersion of $\Omega^{m+1} \to R^{m+1}$, $\partial\Omega = M$, then

$$\lambda_1^{L_r} \le \frac{C(r)}{(m+1)^2} \cdot \frac{V(M)}{V(\Omega)^2} \int_M H_r$$

and equality holds precisaly when M is a sphere.

Using this we prove such an immersion is r-stable if and only if M is a sphere. Here stability means M is a critical point of the funcional $\int_M H_r + b\overline{V}(M)$ and the second derivative of this functional at M is non negative. Here b is a suitable constant and $\overline{V}(M)$ is the (algebraic) volume bounded by M. This generalizes the theorems of Barbosa do Carmo [B-C] (stability of a constant mean curvature immersion means M is a sphere) and the theorem of Alencar, do Carmo and Colares [A-C-C] (for scalar curvature).

In H^{m+1} we obtain an extrinsic upper bound for $\lambda_1^{L_r}$ but it is not the best possible. Consequently our result does not yield stability here. The stability problem has been solved for scalar curvature in S^{m+1} [A-C-C] but this is not known in H^{m+1}.

The techniques we use are properties of the operator L_r (when it is elliptic, formulae for L_r of particular functions), integral geometry, and inequalities involving the mean curvatures. Details will appear elsewhere.

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