Monotonicity formula for complete hypersurfaces in the hyperbolic space and applications

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Abstract. In this paper we prove a monotonicity formula for the integral of the mean curvature for complete and proper hypersurfaces of the hyperbolic space and, as consequences, we obtain a lower bound for the integral of the mean curvature and that the integral of the mean curvature is infinity.

1. Introduction and main results

Let $\mathbb{H}^{n+1}(\varkappa)$ be the $(n+1)$-dimensional hyperbolic space with constant sectional curvature $\varkappa<0$. The main result of this paper is the following

Theorem 1.1. (Monotonicity) *Let M^n, $n\geq 3$, be a complete and proper hypersurface of $\mathbb{H}^{n+1}(\varkappa)$ with mean curvature $H>0$. If there exists a constant $\Gamma\geq 0$ such that scalar curvature R satisfies $\varkappa\leq R\leq \frac{\Gamma}{n-1}H+\varkappa$, then the function $\varphi\colon \mathbb{R}\to\mathbb{R}$ defined by*

$$\varphi(r)=\frac{e^{\frac{\Gamma}{2}r}}{(\sinh\sqrt{-\varkappa}r)^{\frac{n-1}{2}}}\int_{M\cap B_r}(\sinh\sqrt{-\varkappa}\rho)H\,dM$$

is monotone non decreasing, where ρ is the geodesic distance function of $\mathbb{H}^{n+1}(\varkappa)$ starting at $p\in\mathbb{H}^{n+1}(\varkappa)$ and $B_r=B_r(p)$ denotes the geodesic open ball of $\mathbb{H}^{n+1}(\varkappa)$ with center $p\in\mathbb{H}^{n+1}(\varkappa)$ and radius r. Moreover, if $\Gamma<(n-3)\sqrt{-\varkappa}$, then

$$\int_M H\,dM=\infty.$$

Hilário Alencar was partially supported by CNPq of Brazil.

The monotonicity of Theorem 1.1 above implies the following estimate for the integral of mean curvature:

Corollary 1.2. *Let M^n, $n\geq 3$, be a complete and proper hypersurface of $\mathbb{H}^{n+1}(\varkappa)$ with mean curvature $H>0$. If there exists a constant $\Gamma\geq 0$ such that the scalar curvature R satisfies $\varkappa\leq R\leq\frac{\Gamma}{n-1}H+\varkappa$, then*

$$\int_{M\cap B_r} H\,dM \geq (\sinh\sqrt{-\varkappa}r)^{\frac{n-3}{2}}\int_{r_0}^{r} Ce^{-\frac{\Gamma}{2}\tau}\,d\tau$$

for all $r>r_0$, where $C=C(r_0,M,p)$ is a constant depending only on r_0, M and p.

Remark 1.3. In this direction, we can cite the following result of H. Alencar, W. Santos and D. Zhou, see [3], proved in the context of higher order curvatures, whose version for mean curvature we state below.

Let $\overline{M}^{n+1}(\varkappa)$ be an $(n+1)$-dimensional, simply connected, complete Riemannian manifold with constant sectional curvature $\varkappa$, and let M^n be a complete, noncompact, properly immersed hypersurface of $\overline{M}^{n+1}(\varkappa)$. Assume there exists a nonnegative constant α such that

$$|R-\varkappa|\leq\alpha H.$$

If $P_1=nHI-A$ is positive semidefinite, where $I\colon TM\to TM$ is the identity map, then for any $q\in M$ such that $H(q)\neq 0$ and any $\mu_0>0$, there exists a positive constant C, depending only on μ_0, q and M such that, for every $\mu\geq\mu_0$,

$$\int_{M\cap\overline{B}_\mu(p)} H\,dM \geq \int_{\mu_0}^{\mu} Ce^{-\alpha\tau}\,d\tau,$$

where $\overline{B}_\mu(p)$ is the closed ball of radius μ and center $q\in\overline{M}^{n+1}(\varkappa)$. In particular, if $\varkappa\leq 0$, $R=\varkappa$, $H\geq 0$ and $H\not\equiv 0$, then $\int_M H\,dM=\infty$.

Acknowledgements. The authors would like to thank the anonymous referee for his/her valuable comments.

2. Preliminary results

Let $\mathbb{H}^{n+1}(\varkappa)$ be the $(n+1)$-dimensional hyperbolic space with constant sectional curvature $\varkappa$.

Let $A\colon TM\to TM$ be the linear operator associated to the second fundamental form of the immersion. The first Newton transformation $P_1\colon TM\to TM$ is defined by

$$P_1=nHI-A,$$

where $I\colon TM\to TM$ is the identity map.

Notice that, since A is self-adjoint, then P_1 is also a self-adjoint linear operator. Denote by $k_1,k_2,...,k_n$ the eigenvalues of the operator A, also called principal curvatures of the immersion. Since P_1 is a self-adjoint operator, we can consider its eigenvalues $\lambda_1,\lambda_2,...,\lambda_n$ given by $\lambda_i=nH-k_i$, $i=1,2,...,n$.

If $H>0$ and $R\geq\varkappa$, then P_1 is semi-positive definite. This fact is known, and can be found in [1], Remark 2.1, p. 552. We include a proof here for the sake of completeness. If $R\geq\varkappa$, then $(nH)^2=|A|^2+n(n-1)(R-\varkappa)\geq k_i^2$, for all $i=1,2,...,n$. Thus $0\leq(nH)^2-k_i^2=(nH-k_i)(nH+k_i)$ which implies that all eigenvalues of P_1 are non-negative, provided $H\geq0$, i.e., P_1 is semi-positive definite. Let us denote by $\overline{\nabla}$ and ∇ the connections of $\mathbb{H}^{n+1}(\varkappa)$ and M, respectively. In order to prove our main theorem we will need the next two results.

Lemma 2.1. *Let $x\colon M^n\to\mathbb{H}^{n+1}(\varkappa)$ be an isometric immersion, $\rho(x)=\rho(p,x)$ be the geodesic distance function of $\mathbb{H}^{n+1}(\varkappa)$ starting at $p\in\mathbb{H}^{n+1}(\varkappa)$, and $\overline{X}=\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\overline{\nabla}\rho$ the position vector of $\mathbb{H}^{n+1}(\varkappa)$, where $\overline{\nabla}\rho$ denotes the gradient of ρ on $\mathbb{H}^{n+1}(\varkappa)$. Then, for every $q\in M$,*

$$\operatorname{tr}\big(E\longmapsto P_1\big((\overline{\nabla}_E\overline{X})^T\big)\big)(q)=n(n-1)H(q)\big(\cosh\sqrt{-\varkappa}\rho(q)\big).$$

Proof. Let γ be the only unit geodesic of $\mathbb{H}^{n+1}(\varkappa)$ going from p to q. Let $\{e_1(q),e_2(q),...,e_n(q)\}$ a basis of T_qM made by eigenvectors of P_1, i.e., $P_1(e_i(q))=\lambda_i(q)e_i(q)$, where λ_i, $i=1,...,n$, are the eigenvalues of P_1. Writing $e_i=b_i\gamma'+c_iY_i$, where $\|Y_i\|=1$ and $\langle\gamma',Y_i\rangle=0$, we have $b_i^2+c_i^2=1$, and

$$\begin{aligned}\sum_{i=1}^n\big\langle\overline{\nabla}_{e_i}\overline{X},P_1(e_i)\big\rangle&=\sum_{i=1}^n\lambda_i\langle\overline{\nabla}_{e_i}\overline{X},e_i\rangle=\sum_{i=1}^n\lambda_i\langle\overline{\nabla}_{b_i\gamma'+c_iY_i}\overline{X},b_i\gamma'+c_iY_i\rangle\\&=\sum_{i=1}^n\lambda_i\big[b_i^2\langle\overline{\nabla}_{\gamma'}\overline{X},\gamma'\rangle+b_ic_i\langle\overline{\nabla}_{\gamma'}\overline{X},Y_i\rangle\\&\quad+b_ic_i\langle\overline{\nabla}_{Y_i}\overline{X},\gamma'\rangle+c_i^2\langle\overline{\nabla}_{Y_i}\overline{X},Y_i\rangle\big].\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle\overline{\nabla}_{\gamma'}\overline{X},\gamma'\rangle &= \left\langle\overline{\nabla}_{\gamma'}\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\gamma'\right),\gamma'\right\rangle\\ &= \left\langle\gamma'\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\gamma'+\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\overline{\nabla}_{\gamma'}\gamma',\gamma'\right\rangle\\ &= (\cosh\sqrt{-\varkappa}\rho)\langle\gamma',\gamma'\rangle=\cosh\sqrt{-\varkappa}\rho,\end{aligned}$$

$$\begin{aligned}\langle\overline{\nabla}_{\gamma'}\overline{X},Y_i\rangle &= \left\langle\overline{\nabla}_{\gamma'}\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\gamma'\right),Y_i\right\rangle\\ &= (\cosh\sqrt{-\varkappa}\rho)\langle\gamma',Y_i\rangle+\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\langle\overline{\nabla}_{\gamma'}\gamma',Y_i\rangle\\ &= 0,\end{aligned}$$

$$\begin{aligned}\langle\overline{\nabla}_{Y_i}\overline{X},\gamma'\rangle &= \left\langle\overline{\nabla}_{Y_i}\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\gamma'\right),\gamma'\right\rangle\\ &= Y_i\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\langle\gamma',\gamma'\rangle+\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\langle\overline{\nabla}_{Y_i}\gamma',\gamma'\rangle\\ &= \left\langle\overline{\nabla}\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right),Y_i\right\rangle+\frac{1}{2}\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)Y_i\langle\gamma',\gamma'\rangle\\ &= \left\langle\overline{\nabla}\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right),Y_i\right\rangle\\ &= (\cosh\sqrt{-\varkappa}\rho)\langle\gamma',Y_i\rangle=0,\end{aligned}$$

$$\begin{aligned}\langle\overline{\nabla}_{Y_i}\overline{X},Y_i\rangle &= \left\langle\overline{\nabla}_{Y_i}\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\gamma'\right),Y_i\right\rangle\\ &= Y_i\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\langle\gamma',Y_i\rangle+\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\langle\overline{\nabla}_{Y_i}\gamma',Y_i\rangle\\ &= \left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)\langle\overline{\nabla}_{Y_i}\overline{\nabla}\rho,Y_i\rangle.\end{aligned}$$

Since

$$\langle\overline{\nabla}_U\overline{\nabla}\rho,V\rangle=\sqrt{-\varkappa}(\coth\sqrt{-\varkappa}\rho)\big(\langle U,V\rangle-\langle\overline{\nabla}\rho,U\rangle\langle\overline{\nabla}\rho,V\rangle\big),$$

for any vector fields $U, V \in T\mathbb{H}^{n+1}(\varkappa)$, see [4], p. 713, and [2], p. 6, we have

$$\begin{aligned}
\sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} \overline{X}, P_1(e_i) \rangle &= \sum_{i=1}^{n} \lambda_i \left[b_i^2 (\cosh \sqrt{-\varkappa}\rho) + c_i^2 \left(\frac{\sinh \sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}} \right) \langle \overline{\nabla}_{Y_i} \overline{\nabla}\rho, Y_i \rangle \right] \\
&= \sum_{i=1}^{n} \lambda_i b_i^2 (\cosh \sqrt{-\varkappa}\rho) \\
&\quad + \sum_{i=1}^{n} \lambda_i c_i^2 \frac{\sinh \sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}} \sqrt{-\varkappa} (\coth \sqrt{-\varkappa}\rho) \big(\langle Y_i, Y_i \rangle \\
&\quad + \langle \overline{\nabla}\rho, Y_i \rangle \langle \overline{\nabla}\rho, Y_i \rangle \big) \\
&= (\cosh \sqrt{-\varkappa}\rho) \sum_{i=1}^{n} \lambda_i \left[b_i^2 + c_i^2 \right] = (\cosh \sqrt{-\varkappa}\rho) \sum_{i=1}^{n} \lambda_i \\
&= n(n-1) H (\cosh \sqrt{-\varkappa}\rho). \quad \square
\end{aligned}$$

Proposition 2.2. *Let $x\colon M^n \to \mathbb{H}^{n+1}(\varkappa)$ be an isometric immersion, $\rho(x) = \rho(p, x)$ be the geodesic distance function of $\mathbb{H}^{n+1}(\varkappa)$ starting at $p \in \mathbb{H}^{n+1}(\varkappa)$, and $\overline{X} = \frac{\sinh \sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}} \overline{\nabla}\rho$ the position vector of $\mathbb{H}^{n+1}(\varkappa)$, where $\overline{\nabla}\rho$ denotes the gradient of ρ on $\mathbb{H}^{n+1}(\varkappa)$. If $f\colon M \to \mathbb{R}$ is any smooth function, then*

$$\operatorname{div}\big(P_1\big(f X^T\big)\big) = \langle \overline{X}, P_1(\nabla f) \rangle + n(n-1) f H (\cosh \sqrt{-\varkappa}\rho) + n(n-1)(R - \varkappa) f \langle \overline{X}, \eta \rangle,$$

where ∇f denotes the gradient of f on M, $X^T = \overline{X} - \langle \overline{X}, \eta \rangle \eta$ is the component of $\overline{X}$ tangent to M and η is the unit normal vector field of the immersion.

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be an adapted orthonormal frame tangent to M. Since A and $P_1 = nHI - A$ are self-adjoint, we have

$$\begin{aligned}
\operatorname{tr}\big(E \longmapsto P_1\big((\overline{\nabla}_E f \overline{X})^T\big)\big) &= \sum_{i=1}^{n} \big\langle P_1\big((\overline{\nabla}_{e_i} f \overline{X})^T\big), e_i \big\rangle = \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} (f \overline{X}), P_1(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} \big(f X^T\big) + \overline{\nabla}_{e_i} (\langle f \overline{X}, \eta \rangle \eta), P_1(e_i) \big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} \big(f X^T\big), P_1(e_i) \big\rangle - \big\langle f \overline{X}, \eta \big\rangle \sum_{i=1}^{n} \big\langle \eta, \overline{\nabla}_{e_i} \big(P_1(e_i)\big) \big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i} \big(f X^T\big), P_1(e_i) \big\rangle - f \big\langle \overline{X}, \eta \big\rangle \sum_{i=1}^{n} \langle A(e_i), P_1(e_i) \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \big\langle \nabla_{e_i}\big(fX^T\big), P_1(e_i)\big\rangle - f\big\langle \overline{X}, \eta\big\rangle \operatorname{tr}(A\circ P_1) \\
&= \sum_{i=1}^{n} \big\langle P_1\big(\nabla_{e_i}\big(fX^T\big)\big), e_i\big\rangle - f\big\langle \overline{X}, \eta\big\rangle \operatorname{tr}(A\circ P_1) \\
&= \sum_{i=1}^{n} \big\langle \nabla_{e_i}\big(P_1\big(fX^T\big)\big), e_i\big\rangle - \sum_{i=1}^{n} \big\langle (\nabla_{e_i}P_1)\big(fX^T\big), e_i\big\rangle \\
&\quad - f\langle \overline{X}, \eta\rangle \operatorname{tr}(A\circ P_1) \\
&= \operatorname{div}\big(P_1\big(fX^T\big)\big) - (\operatorname{div} P_1)\big(fX^T\big) - f\langle \overline{X}, \eta\rangle \operatorname{tr}(A\circ P_1).
\end{aligned}
$$

By using Gauss equation, we have

$$
\operatorname{tr}(A\circ P_1) = \operatorname{tr}\big(nHA - A^2\big) = nH \operatorname{tr} A - \operatorname{tr} A^2 = n^2H^2 - |A|^2 = n(n-1)(R-\varkappa)
$$

and, since $\operatorname{div} P_1 \equiv 0$, see [5], p. 470 and [6], p. 225, we have

$$
\operatorname{tr}\big(E \longmapsto P_1\big((\overline{\nabla}_E \overline{X})^T\big)\big) = \operatorname{div}\big(P_1\big(fX^T\big)\big) - n(n-1)(R-\varkappa) f\langle \overline{X}, \eta\rangle. \tag{2.1}
$$

On the other hand, by using Lemma 2.1, we have

$$
\begin{aligned}
\operatorname{tr}\big(E \longmapsto P_1\big((\overline{\nabla}_E f\overline{X})^T\big)\big) &= \sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}(f\overline{X}), P_1(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \big\langle e_i(f)\overline{X} + f\overline{\nabla}_{e_i}\overline{X}, P_1(e_i)\big\rangle \\
&= \sum_{i=1}^{n} \big\langle \overline{X}, P_1\big(e_i(f)e_i\big)\big\rangle + f\sum_{i=1}^{n} \big\langle \overline{\nabla}_{e_i}\overline{X}, P_1(e_i)\big\rangle \\
&= \big\langle \overline{X}, P_1(\nabla f)\big\rangle + f \operatorname{tr}\big(E \longmapsto P_1\big((\overline{\nabla}_E \overline{X})^T\big)\big) \\
&= \big\langle \overline{X}, P_1(\nabla f)\big\rangle + n(n-1)Hf(\cosh\sqrt{-\varkappa}\rho).
\end{aligned} \tag{2.2}
$$

Replacing (2.2) in (2.1) we obtain the result. $\square$

Lemma 2.3. *Let $x\colon M^n \to \mathbb{H}^{n+1}(\varkappa)$, $n\geq 3$, be a proper isometric immersion. Suppose $H>0$ and $R\geq\varkappa$. Let $\rho=\rho(p,\cdot)$ be the geodesic distance function of $\mathbb{H}^{n+1}(\varkappa)$ starting at $p\in\mathbb{H}^{n+1}(\varkappa)$. Let $h\colon \mathbb{R}\to\mathbb{R}$ be a smooth function such that $h(t)=0$ for $t\leq 0$ and $h(t)$ is increasing for $t>0$. If $f\colon M\to\mathbb{R}$ is any non negative locally integrable,*

$\mathcal{C}^1$ *function, then for all* $t>s>0$,

$$\begin{aligned}
&\frac{1}{(\sinh\sqrt{-\varkappa}t)^{\frac{n-1}{2}}}\int_M h(t-\rho)(\sinh\sqrt{-\varkappa}\rho)fH\,dM\\
&\qquad-\frac{1}{(\sinh\sqrt{-\varkappa}s)^{\frac{n-1}{2}}}\int_M h(s-\rho)(\sinh\sqrt{-\varkappa}\rho)fH\,dM\\
&\geq\frac{1}{2}\int_s^t\frac{1}{(\sinh\sqrt{-\varkappa}r)^{\frac{n-1}{2}}}\int_M h(r-\rho)(\sinh\sqrt{-\varkappa}\rho)\Big\langle\overline{\nabla}\rho,\frac{1}{n}P_1(\nabla f)\\
&\qquad+(n-1)(R-\varkappa)f\eta\Big\rangle\,dM\,dr.
\end{aligned}$$

Proof. Applying Proposition 2.2 to $h(r-\rho(x))f(x)$, we have

$$\begin{aligned}
\operatorname{div}\big(P_1\big(h(r-\rho)fX^T\big)\big)=&-h'(r-\rho)f\big\langle\overline{X},P_1(\nabla\rho)\big\rangle+h(r-\rho)\big\langle\overline{X},P_1(\nabla f)\big\rangle\\
&+n(n-1)h(r-\rho)fH(\cosh\sqrt{-\varkappa}\rho)\\
&+n(n-1)(R-\varkappa)h(r-\rho)f\langle\overline{X},\eta\rangle.
\end{aligned}\tag{2.3}$$

Since $h(r-\rho)fX^T$ is supported in $M\cap B_r$ and M is proper, then $h(r-\rho)fX^T$ is compactly supported on M. Thus, by using divergence theorem, we have

$$\int_M\operatorname{div}\big(P_1\big(h(r-\rho)fX^T\big)\big)\,dM=0.\tag{2.4}$$

Integrating (2.3) and by using (2.4) above we have

$$\begin{aligned}
\int_M h'(r-\rho)f\big\langle\overline{X},P_1(\nabla\rho)\big\rangle\,dM=&\int_M h(r-\rho)\big\langle\overline{X},P_1(\nabla f)\big\rangle\,dM\\
&+n(n-1)\int_M h(r-\rho)fH(\cosh\sqrt{-\varkappa}\rho)\,dM\\
&+n(n-1)\int_M h(r-\rho)f(R-\varkappa)\langle\overline{X},\eta\rangle\,dM.
\end{aligned}\tag{2.5}$$

Let $k_1,\ k_2,\dots,k_n$ be the principal curvatures of the immersion and $\lambda_i=nH-k_i$ the eigenvalues of P_1. From $H>0$ and $R\geq\varkappa$, P_1 is semi-positive definite, that is, $\lambda_i\geq0$ $(i=1,2,\dots,n)$. Since

$$\begin{aligned}
\lambda_i=nH-k_i&\leq nH+|k_i|\leq nH+\sqrt{k_1^2+k_2^2+\dots+k_n^2}\\
&\leq nH+|A|\leq nH+\sqrt{n^2H^2-n(n-1)(R-\varkappa)}\\
&\leq 2nH,
\end{aligned}$$

we have

$$\int_M h'(r-\rho) f\langle \overline{X}, P_1(\nabla\rho)\rangle\, dM = \int_M h'(r-\rho) f \frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}\langle\overline{\nabla}\rho, P_1(\nabla\rho)\rangle\, dM$$
$$\leq 2n\int_M h'(r-\rho) f\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}} H\, dM$$
$$= 2n\frac{d}{dr}\left(\int_M h(r-\rho) f\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}} H\, dM\right). \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$2n\frac{d}{dr}\left(\int_M h(r-\rho) f\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}} H\, dM\right)$$
$$\geq \int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}\langle\overline{\nabla}\rho, P_1(\nabla f)\rangle\, dM$$
$$+n(n-1)\int_M h(r-\rho) f H(\cosh\sqrt{-\varkappa}\rho)\, dM$$
$$+n(n-1)\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}(R-\varkappa)\langle\overline{\nabla}\rho, \eta\rangle\, dM.$$

Since $\coth x$ is a decreasing function, we can estimate the second integral in the right hand side of inequality above by

$$\int_M h(r-\rho) f H(\cosh\sqrt{-\varkappa}\rho)\, dM$$
$$> \sqrt{-\varkappa}(\coth\sqrt{-\varkappa}r)\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}\, Hf\, dM,$$

which implies

$$\frac{d}{dr}\left(\int_M h(r-\rho) f\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}} H\, dM\right)$$
$$\geq \frac{n-1}{2}\sqrt{-\varkappa}(\coth\sqrt{-\varkappa}r)\int_M h(r-\rho) f\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}} H\, dM$$
$$+\frac{1}{2}\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}\left\langle\overline{\nabla}\rho, \frac{1}{n}P_1(\nabla f)+(n-1)(R-\varkappa) f\eta\right\rangle dM.$$

Since

$$\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)^{\frac{n-1}{2}}\frac{d}{dr}\left(\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)^{-\frac{n-1}{2}}\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}Hf\,dM\right)$$
$$=-\frac{n-1}{2}\sqrt{-\varkappa}(\coth\sqrt{-\varkappa}r)\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}Hf\,dM$$
$$+\frac{d}{dr}\left(\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}Hf\,dM\right),$$

we have

$$\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)^{\frac{n-1}{2}}\frac{d}{dr}\left(\left(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}}\right)^{-\frac{n-1}{2}}\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}Hf\,dM\right)$$
$$\geq\frac{1}{2}\int_M h(r-\rho)\frac{(\sinh\sqrt{-\varkappa}\rho)}{\sqrt{-\varkappa}}\left\langle\overline{\nabla}\rho,\frac{1}{n}P_1(\nabla f)+(n-1)(R-\varkappa)f\eta\right\rangle dM.$$

Dividing expression above by $(\frac{\sinh\sqrt{-\varkappa}\rho}{\sqrt{-\varkappa}})^{\frac{n-1}{2}}\times(\sqrt{-\varkappa})^{\frac{n-3}{2}}$ and integrating on r from s to t we obtain the result

$$\frac{1}{(\sinh\sqrt{-\varkappa}t)^{\frac{n-1}{2}}}\int_M h(t-\rho)(\sinh\sqrt{-\varkappa}\rho)fH\,dM$$
$$-\frac{1}{(\sinh\sqrt{-\varkappa}s)^{\frac{n-1}{2}}}\int_M h(s-\rho)(\sinh\sqrt{-\varkappa}\rho)fH\,dM$$
$$\geq\frac{1}{2}\int_s^t\frac{1}{(\sinh\sqrt{-\varkappa}r)^{\frac{n-1}{2}}}\int_M h(r-\rho)(\sinh\sqrt{-\varkappa}\rho)\left\langle\overline{\nabla}\rho,\frac{1}{n}P_1(\nabla f)\right.$$
$$\left.+(n-1)(R-\varkappa)f\eta\right\rangle dM\,dr.\quad\square$$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Choosing $f\equiv 1$ in the inequality of Lemma 2.3, we have, for every $t>s>0$,

$$\frac{1}{(\sinh\sqrt{-\varkappa}t)^{\frac{n-1}{2}}}\int_M h(t-\rho)(\sinh\sqrt{-\varkappa}\rho)H\,dM$$
$$-\frac{1}{(\sinh\sqrt{-\varkappa}s)^{\frac{n-1}{2}}}\int_M h(s-\rho)(\sinh\sqrt{-\varkappa}\rho)H\,dM$$

$$\geq \frac{1}{2}\int_s^t \frac{1}{(\sinh\sqrt{-\varkappa}r)^{\frac{n-1}{2}}}\int_M h(r-\rho)(\sinh\sqrt{-\varkappa}\rho)\big\langle\overline{\nabla}\rho,(n-1)(R-\varkappa)\eta\big\rangle\,dM\,dr$$

$$\geq -\frac{1}{2}\int_s^t \frac{1}{(\sinh\sqrt{-\varkappa}r)^{\frac{n-1}{2}}}\int_M h(r-\rho)(\sinh\sqrt{-\varkappa}\rho)(n-1)(R-\varkappa)\,dM\,dr$$

$$\geq -\frac{\Gamma}{2}\int_s^t \frac{1}{(\sinh\sqrt{-\varkappa}r)^{\frac{n-1}{2}}}\int_M h(r-\rho)(\sinh\sqrt{-\varkappa}\rho)H\,dM\,dr.$$

Letting $g(r)=\frac{1}{(\sinh\sqrt{-\varkappa}r)^{\frac{n-1}{2}}}\int_M h(r-\rho)(\sinh\sqrt{-\varkappa}\rho)H\,dM$, inequality above becomes

$$g(t)-g(s)\geq -\frac{\Gamma}{2}\int_s^t g(r)\,dr,$$

which implies

$$g'(t)\geq -\frac{\Gamma}{2}g(t),$$

i.e.,

$$\frac{d}{dt}\big(e^{\frac{\Gamma}{2}t}g(t)\big)\geq 0$$

and thus

$$e^{\frac{\Gamma}{2}r}g(r)=e^{\frac{\Gamma}{2}r}(\sinh\sqrt{-\varkappa}r)^{-\frac{n-1}{2}}\int_M h(r-\rho)(\sinh\sqrt{-\varkappa}\rho)H\,dM$$

is monotone non-decreasing. Now, let us apply this result to the sequence of smooth functions $h_m\colon\mathbb{R}\to\mathbb{R}$ such that $h_m(t)=0$ for $t\leq 0$, $h_m(t)=1$ for $t\geq\frac{1}{m}$ and h_m is increasing for $t\in(0,\frac{1}{m})$. Taking $m\to\infty$, sequence h_m tends to the characteristic function of $(0,\infty)$ and the first part of the theorem follows.

To prove that $\int_M H\,dM=\infty$ for $\Gamma<(n-3)\sqrt{-\varkappa}$, notice that monotonicity of $\varphi(r)$ implies

$$\int_{M\cap B_r}(\sinh\sqrt{-\varkappa}\rho)H\,dM$$
$$\geq e^{\frac{\Gamma}{2}(r_0-r)}\left(\frac{\sinh\sqrt{-\varkappa}r}{\sinh\sqrt{-\varkappa}r_0}\right)^{\frac{n-1}{2}}\int_{M\cap B_{r_0}}(\sinh\sqrt{-\varkappa}\rho)H\,dM,$$

for all $r>r_0>0$. Since $\sinh x$ is an increasing function, we have

$$\int_{M\cap B_r}(\sinh\sqrt{-\varkappa}\rho)H\,dM\leq(\sinh\sqrt{-\varkappa}r)\int_{M\cap B_r}H\,dM,$$

which implies

$$\int_{M\cap B_r} H\,dM \geq \frac{(\sinh\sqrt{-\varkappa}r)^{\frac{n-3}{2}}}{e^{\frac{\Gamma}{2}r}} \times \frac{e^{\frac{\Gamma}{2}r_0}}{(\sinh\sqrt{-\varkappa}r_0)^{\frac{n-1}{2}}} \int_{M\cap B_{r_0}} (\sinh\sqrt{-\varkappa}\rho) H\,dM.$$

Since $\sinh\sqrt{-\varkappa}r=\frac{1}{2}(e^{\sqrt{-\varkappa}r}-e^{-\sqrt{-\varkappa}r})$, taking $r\to\infty$, and by using that $\Gamma<(n-3)\sqrt{-\varkappa}$, we obtain $\int_M H\,dM=\infty$. $\square$

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