Surfaces of $M_k^2 \times \mathbb{R}$ invariant under a one-parameter group of isometries

Hilário Alencar · Manfredo do Carmo · Renato Tribuzy

Abstract We assume that an immersed constant mean curvature surface $\Sigma \looparrowright M_k \times \mathbb{R}$ satisfies a relation involving the mean curvature, the Gaussian curvature and the angle that the unit vector of the factor $\mathbb{R}$ makes with the normal to the surface. This relation, although given initially in its pointwise form, can be shown to be equivalent to an integral relation. From the assumed relation, it follows that Σ is invariant under a one-parameter group of isometries of $M_k^2 \times \mathbb{R}$ which are induced by the isometries of M_k^2. An application is made to describe qualitatively those surfaces for which the Abresch-Rosenberg complex quadratic form vanishes.

Keywords Product space · Hyperbolic plane · Isometry · One-parameter group

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1 Introduction

We consider orientable surfaces Σ immersed in $M_k^2 \times \mathbb{R}$, where M_k^2 is a 2-dimensional, complete, simply-connected Riemannian manifold with constant sectional curvature k. The unit vector of the factor $\mathbb{R}$ is denoted by ξ. We choose a unit normal vector field e_3 and a positive orthonormal frame $\{e_1, e_2, e_3\}$ in $M_k^2 \times \mathbb{R}$ so that, along Σ, $\{e_1, e_2\}$ is a basis for $T\Sigma$. We define θ as the angle given by $\cos\theta = \langle \xi, e_3\rangle$, and choose e_1 as the unit vector of the projection of ξ onto $T\Sigma$; thus, $\xi = \sin\theta e_1 + \cos\theta e_3$. The frame $\{e_1, e_2\}$ is globally defined, except at points where $\sin\theta = 0$. It will be called a *canonical frame*. We denote by α the second fundamental form of the immersion and will use the notation $\alpha_{ij} = \alpha(e_i, e_j)$, $i, j = 1, 2$. We also assume that the surfaces considered have nonzero constant mean curvature H.

It should be remarked that, since H is constant, the immersion is real analytic. Thus, if f is a function on Σ defined in terms of the immersion, then either f vanishes identically or f is zero only on a closed set $F \subset \Sigma$ with no interior points. Since the canonical frame only fails to be defined at points where $\sin\theta = 0$, this will only occur if either Σ is a *slice*, that is, a surface of the form $M_k^2 \times \{t_0\}$, or $\sin\theta = 0$ in a set F as above; in this last case, given a point $p_0 \in F$, there exists a sequence of points in the complement $\complement F \subset \Sigma$ of F that converges to p_0. Equalities among geometric quantities that are proved or assumed in $\complement F$ hold everywhere in Σ, except when Σ is a slice, which can be treated as a special case.

From now on, we assume that Σ is not a slice.

2 The results

We will first prove that, in the canonical frame $\{e_1, e_2\}$, Gauss equation can be written as (see Sect. 3, Eq. (8))

$$K = -|\mathrm{d}\theta|^2 + k\cos^2\theta + 2H\,\mathrm{d}\theta(e_1).$$

The hypothesis of our theorem is a slight modification of the Gauss equation.

Theorem 1 *Let $\Sigma \looparrowright M_k^2 \times \mathbb{R}$ be an orientable surface with H a nonzero constant. Choose an orientation so that $H > 0$. Assume that, at $p \in \Sigma$,*

$$K = -|\mathrm{d}\theta|^2 + k\cos^2\theta \pm 2H|\mathrm{d}\theta|, \tag{1}$$

holds. Then, Σ is invariant under a 1-parameter group of isometries of the ambient space that fixes an axis ℓ_0 parallel to ξ and passes through one or two fixed point of the closures $\overline{M}_k^2$ of M_k^2.

If $k > 0$, the isometries are generated by rotations of M_k^2 around a point $p_0 \in M_k^2$ whose trajectories are circles.

If $k < 0$, the isometries are elliptic, parabolic or hyperbolic depending on whether the trajectories are circles, horocycles or hypercycles, respectively. In the two last cases, there are one or two fixed points in ∂M_k^2.

The method of proof of the Theorem can be used to give a proof of an interesting property of surfaces in the euclidean space $\mathbb{R}^3$.

Corollary 1 (of the proof of the Theorem). *Let $\Sigma^2 \looparrowright \mathbb{R}^3$ be an immersion of a compact surface Σ in $\mathbb{R}^3$ with the mean curvature H constant and positive. Assume that*

$$K = -|\mathrm{d}\theta|^2 \pm 2H|\mathrm{d}\theta|, \tag{1'}$$

where θ is the angle which the unit normal vector e_3 to the surface Σ makes with a fixed axis in $\mathbb{R}^3$ (for instance, the coordinate axis $0z$). Then, Σ is isometric to the canonical sphere S^2 and the immersion is the standard embedding of S^2 into $\mathbb{R}^3$.

It will be shown in the proof of the Theorem that the condition (1) implies, but it is not equivalent to, that $\alpha(e_1, e_2) = 0$ in Σ, that is, the canonical frame diagonalizes the second fundamental form. The same occurs with condition (1′). Now, recall a theorem due to H. Hopf that if Σ has genus zero and the complex quadratic form on Σ

$$\alpha^{(2,0)} = \big\{(\alpha(e_1, e_1) - \alpha(e_2, e_2)) - 2i\alpha(e_1, e_2)\big\}\mathrm{d}z^2$$

vanishes identically, that is, $\alpha_{11} = \alpha_{22}$ and $\alpha_{12} = 0$, then Σ is isometric to a sphere. It is surprising that we can obtain the same conclusion from the weaker condition $\alpha_{12} = 0$.

Remark Surfaces Σ with the property that the projection of ξ onto $T\Sigma$ is a principal direction appeared in our paper [2] and, independently in [5], where an explicit description of such surfaces is given.

Corollary 1 can be given an interesting integral form, namely,

Corollary 2 *Let a compact surface Σ be immersed in $\mathbb{R}^3$ with mean curvature constant and positive. Assume that*

$$\int_{\Sigma} \big(- |\mathrm{d}\theta|^2 \pm 2H|\mathrm{d}\theta|\big) = 2\pi\chi(\Sigma), \tag{1''}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Then, Σ is isometric to the standard sphere and the immersion is the standard embedding in $\mathbb{R}^3$.

We can also treat the case of Corollary 1 when Σ is complete and non-compact.

Corollary 3 *Replace in Corollary 1 the condition that Σ is compact by the condition that Σ is complete, non-compact. Assume that (1′) holds. Then, Σ is a Delaunay surface.*

Finally, we present an integral form of our Theorem.

Corollary 4 *Let $\Sigma \looparrowright M_k^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature $H > 0$. Assume that there exists a geodesic triangle T in Σ with interior angles $\beta_1, \beta_2, \beta_3$ which satisfies*

$$\pi - \sum_{i=1}^{3} \beta_i = \int_R -|\mathrm{d}\theta|^2 + k\,\cos^2\theta \pm 2H|\mathrm{d}\theta|,$$

where R is the region bounded by T. Then, the conclusions of Theorem 1 hold for Σ.

An interesting application of the Theorem is a characterization of Abresch-Rosenberg surfaces. They were completely described in the seminal paper [1] and are defined as those surfaces Σ for which the complex quadratic form

$$Q\,\mathrm{d}z^2 = \big\{(\widetilde{Q}(e_1, e_1) - \widetilde{Q}(e_2, e_2)) - 2i\,\widetilde{Q}(e_1, e_2)\big\}\mathrm{d}z^2$$

vanishes identically. Here, z is a complex parameter on Σ, and $\widetilde{Q}$ is a real quadratic form

$$\widetilde{Q}(X, Y) = 2H\alpha(X, Y) - k\langle\xi, X\rangle\langle\xi, Y\rangle, \tag{2}$$

where $X, Y \in T\Sigma$. By setting $X = e_1$, $Y = e_2$, and using that $\xi = \sin\theta\, e_1 + \cos\theta\, e_3$, we obtain from (2) that

$$Q\,dz^2 \equiv 0 \Leftrightarrow \begin{cases} \alpha_{12} = 0, \\ \alpha_{11} - \alpha_{22} = \dfrac{k \sin^2\theta}{2H}, \end{cases} \tag{3}$$

so they are all included in Theorem 1, more precisely, among those surfaces with $\alpha_{12} = 0$. In Sect. 5, we will distinguish them.

Remark We want to thank Harold Rosenberg for conversations on this paper.

3 Preliminaries

We choose a canonical frame $\{e_1, e_2\}$ for a surface $\Sigma \looparrowright M_k^2 \times \mathbb{R}$ as in the Introduction, and we want to write the Gauss equation of Σ in this frame. Recall that $\xi = \sin\theta\, e_1 + \cos\theta\, e_3$, where e_3 is a unit normal vector to Σ.

For future use, we will establish the Gauss equation for the more general situation where the ambient space $E^3(k, \tau)$, $k^2 + \tau^2 \neq 0$, $k - 4\tau^2 \neq 0$, is a 3-dimensional Riemannian fibration over M_k^2. The fibers of this fibration are geodesics of $E^3(k, \tau)$ whose unit tangent vectors are denoted by ξ. Recall that τ is given by

$$\widetilde{\nabla}_X \xi = \tau(X \times \xi),$$

where $\widetilde{\nabla}$ is the Riemannian connection in $E^3(k, \tau)$, X is a vector field in $E^3(k, \tau)$ and $\times$ the cross product in the tangent space of $E^3(k, \tau)$.

If $\tau = 0$, $E^3(k, \tau)$ are the Riemannian products $S_k^2 \times \mathbb{R}$ and $H_k^2 \times \mathbb{R}$, where S_k^2 is a 2-sphere with curvature k and H_k^2 is a hyperbolic plane with curvature k, $k < 0$. For further details on the spaces $E^3(k, \tau)$, we refer to [3].

We now start the computations for the Gauss formula. Notice that for the definition of e_1, we need to assume that $\sin\theta \neq 0$. As we will see in a moment, we also need to assume that $\cos\theta \neq 0$. The second fundamental form, which is a bilinear form on $T\Sigma$, will be denoted by α. We will denote $\alpha(e_i, e_j)$, $i, j = 1, 2$, by α_{ij}.

By the definition of θ, $\langle e_1, \xi\rangle = \sin\theta$. Differentiating both sides of this equality along the tangent vector X, we obtain

$$\langle \widetilde{\nabla}_X e_1, \xi\rangle + \langle e_1, \widetilde{\nabla}_X \xi\rangle = \cos\theta\, d\theta(X),$$

hence, by using that $\widetilde{\nabla}_X^T e_1 \perp \xi$ and that $\widetilde{\nabla}_X \xi = \tau(X \times \xi)$,

$$\langle \alpha(X, e_1) e_3, \xi\rangle + \langle e_1, \tau(X \times \xi)\rangle = \cos\theta\, d\theta(X). \tag{4}$$

Now, set $X = e_1$ in (4) to obtain

$$\langle \alpha_{11} e_3, \xi\rangle + \langle e_1, \tau(e_1 \times \xi)\rangle = \cos\theta\, d\theta(e_1).$$

Since $e_1 \times \xi = -\cos\theta\, e_2$, $\langle e_3, \xi\rangle = \cos\theta$, and $\cos\theta \neq 0$, we have

$$\alpha_{11} = d\theta(e_1). \tag{5}$$

Next, set $X = e_2$ in (4) to obtain

$$\langle \alpha_{12} e_3, \xi\rangle + \langle e_1, \tau(e_2 \times \xi)\rangle = \cos\theta\, d\theta(e_2).$$

But $\langle e_2 \times \xi \rangle = \langle e_2 \times (\sin\theta\, e_1 + \cos\theta\, e_3) \rangle = -\sin\theta\, e_3 + \cos\theta\, e_1$. Thus,

$$\alpha_{12}\cos\theta + \langle e_1, \tau\cos\theta\, e_1 \rangle = \cos\theta\, \mathrm{d}\theta(e_2),$$

hence, since $\cos\theta \neq 0$,

$$\alpha_{12} = \mathrm{d}\theta(e_2) - \tau. \tag{6}$$

From [3], we know that the Gauss formula for $E^3(k,\tau)$ is

$$K = \det A + k\cos^2\theta + \tau^2(1 - 4\cos^2\theta), \tag{7}$$

where K is the Gauss curvature of Σ, and A is the self-adjoint linear map of Σ that is associated with the second fundamental form α. Let us compute $(\det A)$ using (5) and (6).

$$\begin{aligned}\det A &= \alpha_{11}\alpha_{22} - \alpha_{12}^2 = \alpha_{11}(2H - \alpha_{11}) - \alpha_{12}^2 \\ &= -\alpha_{11}^2 + 2H\alpha_{11} - \alpha_{12}^2 \\ &= -(\mathrm{d}\theta(e_1))^2 + 2H(\mathrm{d}\theta(e_1)) - (\mathrm{d}\theta(e_2) - \tau)^2 \\ &= -(\mathrm{d}\theta(e_1))^2 - (\mathrm{d}\theta(e_2))^2 - \tau^2 + 2\tau\,\mathrm{d}\theta(e_2) + 2H\,\mathrm{d}\theta(e_1).\end{aligned}$$

By using the above value of $\det A$ in (7), we obtain

$$K = -|\mathrm{d}\theta|^2 + k\cos^2\theta + 2H\,\mathrm{d}\theta(e_1) + 2\tau\,\mathrm{d}\theta(e_2) - 4\tau^2\cos^2\theta, \tag{8}$$

which holds under the assumption that $\cos\theta \neq 0$ and $\sin\theta \neq 0$. This is an expression of the Gauss formula in terms of H, K and θ, and the constants k and τ. Another useful expression of (8) is

$$K = -(\mathrm{d}\theta(e_1))^2 - (+\mathrm{d}\theta(e_2) - \tau)^2 + 2H\,\mathrm{d}\theta(e_1) + k\cos^2\theta + \tau^2(1 - 4\cos^2\theta). \tag{8$'$}$$

In this paper, we use (8) or (8′) with $\tau = 0$.

To close this section, we will give a simple proof of the fact that for vertical cylinders in $E(k,\tau)$, that is, surfaces for which $\langle e_3, \xi \rangle = \cos\theta \equiv 0$, the Gaussian curvature K vanishes identically.

The proof follows from the fact that since $\widetilde{\nabla}_X \xi = \tau(X \times \xi)$ has no tangent component, for all $X \in T\Sigma$, then ξ is a parallel vector field along Σ, and this means that the Gaussian curvature K of Σ vanishes identically. □

4 Proof of Theorem 1

We will start with Gauss equation (8). We first show that, although it has been proved under the conditions $\sin\theta \neq 0$ and $\cos\theta \neq 0$, if the mean curvature H is constant, it will hold in Σ at all points where $\sin\theta \neq 0$.

This follows from the fact that, in this case, the immersion is real analytic. Thus, if $\cos\theta = 0$ in an open set of Σ, this will be so everywhere, and the Gauss equation

$$K = -|\mathrm{d}\theta|^2 + k\cos^2\theta + 2H\,\mathrm{d}\theta(e_1)$$

holds trivially on Σ except where e_1 is not defined.

Now, consider the sets F_1 where $\cos\theta = 0$ and F_2 where $\sin\theta = 0$. By analyticity, both are closed sets in Σ with no interior points. Let $W = \mathcal{C}(F_1 \cup F_2)$. Then, the Gauss formula is well defined in W, and the canonical frame $\{e_1, e_2\}$ is well defined in $\mathcal{C}\,F_1 \subset \Sigma$. It follows, by continuity, that the Gauss formula holds in $\mathcal{C}\,F_1$, as we claimed.

From now on, along the proof of Theorem 1, we will work, without further mention, in the complement of the set F_1. Since F_1 is a closed set without interior points, its complement $\mathcal{C}\, F_1 \subset \Sigma$ is open and dense in Σ. The geometric conclusions that we obtain can then be passed to the limit to hold in Σ.

Since $H > 0$, we obtain, from the Gauss formula, that

$$-|\mathrm{d}\theta|^2 + k\,\cos^2\theta - 2H|\mathrm{d}\theta| \le K \le -|\mathrm{d}\theta|^2 + k\,\cos^2\theta + 2H|\mathrm{d}\theta| \tag{9}$$

with equality holding in (9) if and only if $e_1 = \pm \dfrac{\operatorname{grad}\theta}{|\operatorname{grad}\theta|}$. This implies that $\mathrm{d}\theta(e_2) = 0$, hence $\alpha_{12} = \mathrm{d}\theta(e_2) - \tau = 0$, since in our case $\tau = 0$.

Notice that the fact that $\alpha_{12} = \mathrm{d}\theta(e_2) = 0$ means that θ is constant along the trajectories of e_2. One of our main concerns will be to determine precisely the trajectories of e_2 that will sometimes be denoted by a capital C.

Now, assume the hypothesis of Theorem, namely, that equality holds in (9). We will need a number of Lemmas. Some of these Lemmas (not all, though) have appeared in a more general context in [2]. In the present situation, we have simpler proofs that we find convenient to present here. We will denote by $\widetilde{\nabla} = \nabla + \nabla^{\perp}$ the Riemannian connection in $M_k^2 \times \mathbb{R}$, where ∇ and $\nabla^{\perp}$ are its tangent and normal components, respectively, along Σ. Recall that the equalities below are proved to hold in a set $\complement\, F_1 \subset \Sigma$, where F_1 is a closed set without interior points where $\sin\theta$ is allowed to be zero.

Lemma 1 $\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0$.

Proof Since $\xi = \sin\theta\, e_1 + \cos\theta\, e_3$ is a parallel vector field in the connection $\widetilde{\nabla}$,

$$0 = \widetilde{\nabla}_{e_1}\xi = \cos\theta\,\mathrm{d}\theta(e_1)e_1 - \sin\theta\,\mathrm{d}\theta(e_1)e_3 + \sin\theta\,\nabla_{e_1} e_1 + \sin\theta\,\alpha_{11}\, e_3 - \cos\theta\, A(e_1) \tag{10}$$

where A is the Weingarten operator. Since both the tangent and normal components of (10) must vanish, we obtain, for the tangent component,

$$-\cos\theta\, A(e_1) + \cos\theta\,\mathrm{d}\theta(e_1)e_1 + \sin\theta\,\nabla_{e_1} e_1 = 0.$$

Since $\alpha_{12} = 0$, $\{e_1, e_2\}$ diagonalizes A. Furthermore, $\nabla_{e_1} e_2$ is a multiple of e_2. It follows that at the points where $\sin\theta \ne 0$, we have that $\nabla_{e_1} e_1 = 0$.

So, we have shown that $\nabla_{e_1} e_1 = 0$. Since $\langle e_1, e_2\rangle = 0$, $\nabla_{e_1} e_2 = -\nabla_{e_1} e_1 = 0$, and this proves Lemma 1. □

Lemma 2 *The function α_{22} is constant along the trajectories of e_2.*

Proof Since H is a constant,

$$e_2(\alpha_{22}) = -e_2(\alpha_{11}) = -\nabla_{e_2}\alpha(e_1, e_1) = -(\nabla_{e_2}\alpha)(e_1, e_1) + 2\alpha(\nabla_{e_2} e_1, e_1).$$

But $\nabla_{e_2} e_1$ is a multiple of e_2 and $\alpha(e_2, e_1) = 0$. Thus, the last summand in the last equality vanishes, and

$$e_2(\alpha_{22}) = (\nabla_{e_2}\alpha)(e_1, e_1) = (\nabla_{e_1}\alpha)(e_2, e_1) + (\widetilde{R}(e_2, e_1)e_1)^{\perp},$$

where we have used the Codazzi equation, and $\widetilde{R}$ denotes the curvature of the connection $\widetilde{\nabla}$. Since, by [2], $\left(\widetilde{R}(e_2, e_1(e_1)\right)^{\perp} = 0$, we obtain

$$e_2(\alpha_{22}) = -(\nabla_{e_1}\alpha)(e_2, e_1) = e_1(\alpha(e_2, e_1)) - \alpha(\nabla_{e_1} e_2, e_1) - \alpha(e_2, \nabla_{e_1} e_1) = 0,$$

where we have used that $\alpha_{12} = 0$ and that $\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0$ by Lemma 1. This completes the proof of Lemma 2. □

Lemma 3 *The function b defined by $\nabla_{e_2} e_2 = b\, e_1$ is given by $g = -\mathrm{cotan}\,\theta\,\alpha_{22}$ which is then constant along the trajectories of e_2.*

Proof Since $\langle \xi, e_2\rangle = 0$, we have that

$$0 = e_2\langle \xi, e_2\rangle = \langle \widetilde{\nabla}_{e_2}\xi, e_2\rangle + \langle \xi, \widetilde{\nabla}_{e_2} e_2\rangle.$$

But $\widetilde{\nabla}_{e_2}\xi = 0$, since in our case $\tau = 0$. Thus,

$$0 = e_2\langle \xi, e_2\rangle = \langle \sin\theta\, e_1 + \cos\theta\, e_3, \nabla_{e_2} e_2 + \alpha_{22}\, e_3\rangle = \sin\theta\, b + \cos\theta\,\alpha_{22}\,.$$

It follows that

$$b = -\mathrm{cotan}\,\theta\,\alpha_{22}\,,$$

as we stated. Since θ and α_{22} are constant along the trajectories of e_2, so is b. This proves Lemma 3. □

Lemma 4 *The trajectories of e_2 have constant geodesic curvature k_g given by*

$$k_g^2 = \alpha_{22}^2\,\frac{1}{\sin^2\theta}\,\cdot$$

Proof We first observe that since e_2 is perpendicular to ξ, the trajectories of e_2 are curves in M_k^2. We know that

$$\widetilde{\nabla}_{e_2} e_2 = \nabla_{e_2} e_2 + \widetilde{\nabla}^{\perp}_{e_2} e_2 = b\, e_1 + \alpha_{22}\, e_3\,. \tag{11}$$

To compute the geodesic curvature of a trajectory of e_2, we should project $\widetilde{\nabla}_{e_2} e_2$ into the tangent plane TM_k^2. But, since M_k^2 is totally geodesic, $\widetilde{\nabla}$ agrees with the connection of M_k^2, so no projection is necessary, and we obtain

$$k_g^2 = b^2 + \alpha_{22}^2 = \frac{\cos^2\theta}{\sin^2\theta}\,\alpha_{22}^2 + \alpha_{22}^2 = \frac{\alpha_{22}^2}{\sin^2\theta}\,\cdot$$

□

We have now concluded the preparatory lemmas and will start the proof itself. We need some notation.

Let $\widetilde{\nabla}$ be the connection on $M_k^2\times\mathbb{R}$ and let $\widetilde{\nabla} = \widehat{\nabla} + \nabla^{\mathbb{R}}$ be its decomposition into the connection $\widehat{\nabla}$ of M_k^2 and the connection $\nabla^{\mathbb{R}}$ of $\mathbb{R}$. Thus, if V and W are vector fields in $M_k^2\times\mathbb{R}$, we have

$$\widetilde{\nabla}_Z W = \widehat{\nabla}_{\mathrm{d}\pi(Z)}\left(\mathrm{d}\pi(W)\right) + \nabla^{\mathbb{R}}_{\mathrm{d}P(Z)}\left(\mathrm{d}P(W)\right),$$

where $\pi\colon M_k^2\times\mathbb{R}\to M_k^2$ is the canonical projection and $\mathrm{d}P\colon T(M_k^2\times\mathbb{R})\to\mathbb{R}$ is given by $\mathrm{d}P(W) = \langle W, \xi\rangle\xi$, so that

$$\nabla^{\mathbb{R}}_{\mathrm{d}P(Z)}\left(\mathrm{d}P(W)\right) = \left\langle \nabla^{\mathbb{R}}_{\mathrm{d}P(Z)}\,\mathrm{d}P(W), \xi\right\rangle\xi.$$

Now let $\beta(s)$ be a trajectory of e_1 in $M_k^2\times\mathbb{R}$, that is,

$$\beta(s) = \big(\gamma(s), t(s)\big),\quad \gamma(s)\in M_k^2\,,\quad t(s)\in\mathbb{R}.$$

Notice that, by construction, $\mathrm{d}\pi(e_2) = e_2$. We want to show that e_2 is parallel along $\gamma(s)$ in the connection $\widehat{\nabla}$. Indeed,

$$\widehat{\nabla}_{\gamma'(s)}\, e_2\big(\gamma(s)\big) = \widehat{\nabla}_{\mathrm{d}\pi(e_1)}\, \mathrm{d}\pi(e_2) = \widehat{\nabla}_{\mathrm{d}\pi(e_1)}\, e_2 = \widetilde{\nabla}_{e_1} e_2 = 0, \tag{12}$$

where we have used Lemma 1 and that $\alpha_{12} = 0$ in the last equality and the fact that $\mathrm{d}P(e_2) = 0$, in the equality preceding the last one. The above equality proves our claim.

We will now prove that the curves $\gamma(s)$ are geodesics in M_k^2 and that they constitute the orthogonal family of the trajectories of e_2 in M_k^2.

To see this, we first notice that $\gamma'(s)$ is orthogonal to $e_2\big(\gamma(s)\big)$. Then,

$$0 = \gamma'(s)\big\langle \gamma'(s), e_2\big(\gamma(s)\big)\big\rangle = \big\langle \widehat{\nabla}_{\gamma'(s)}\, \gamma'(s), e_2\big\rangle + \big\langle \gamma'(s), \widehat{\nabla}_{\gamma'(s)}\, e_2\big\rangle.$$

It follows from (12) and the above equality that $\langle \widehat{\nabla}_{\gamma'(s)}\, \gamma'(s), e_2\rangle = 0$. Thus, $\widehat{\nabla}_{\gamma'(s)}\, \gamma'(s)$ has only components in the direction of $\gamma'(s)$, that is, the curves $\gamma(s)$ are reparameterizations of geodesics in M_k^2. Since $\langle \gamma'(s), e_2\rangle = 0$, they are orthogonal to the trajectories of e_2 in M_k^2, as we claimed.

For the case where $k > 0$, the trajectories of e_2 in the 2-sphere $M_k^2 = S_k^2$ are geodesic circles with constant geodesic curvatures $k_g \geq 0$. Fix one of these curves, say C. Since the trajectories of the projections of e_1 onto S_k^2 are geodesics normal to C, they all meet at one point $p_0 \in S_k^2$. It follows that the other trajectories of e_2 are geodesic circles with center p_0. Thus, Σ is invariant under a one-parameter group of isometries that fix an axis ℓ_0 which is parallel to ξ and passes through p_0.

In the case that $k < 0$, that is, when M_k^2 is the hyperbolic plane $\mathbb{H}_k^2$, the trajectories of e_2 in $\mathbb{H}_k^2$ are curves with constant geodesic curvatures k_g. Thus, they are either geodesic circles (if $k_g^2 > -k$), horocycles (if $k_g^2 = \dfrac{\alpha_{22}^2}{\sin^2\theta} = -k$) or hypercycles (if $k_g^2 < -k$). In the case of circles and horocycles, we can proceed as in the case $k > 0$. One fixes one such curve, say C. Since the projections by π of the integral curves of e_1 onto $\mathbb{H}_k^2$ are geodesics of $\mathbb{H}_k^2$ that are perpendicular to the fixed curve C, all such geodesics meet in a common point p_0 that belongs to $\mathbb{H}_2^k$ in the case C is a circle, or to the boundary $\partial\mathbb{H}_l^2$ in case C is a horocycle. It follows that all the other trajectories of e_2 are circles with center p_0 if $p_0 \in \mathbb{H}_k^2$ or horocycles passing through p_0 if $p_0 \in \partial\mathbb{H}_k^2$. Thus, Σ is foliated either by circles or by horocycles; in addition, Σ is invariant under a one-parameter group of isometries that fix an axis ℓ_0 which is parallel to ξ and passes through p_0.

Finally, we come to the case of the hypercycles. In this case, it is convenient to use more explicitly the model of the upper half plane for $\mathbb{H}_k^2$. So, let

$$\mathbb{R}_+^2 = \Big\{(x, y) \in \mathbb{R}^2;\ y > 0\Big\}, \quad \partial\mathbb{R}_+^2 = \Big\{(x, y) \in \mathbb{R}^2;\ y = 0\Big\}$$

denote the model of the hyperbolic space and its boundary.

In this model, fix a trajectory γ_0 of the projection of e_1 onto $\mathbb{R}_+^2$. Being a geodesic, γ_0 can be chosen as the intersection of $\mathbb{R}_+^2$ with an euclidean semi-circle with center $p_0 \in \partial\mathbb{R}_+^2$. This geodesic must meet orthogonally all the trajectories of e_2, and such trajectories, having constant geodesic curvatures, are equidistant curves from a fixed geodesic $\overline{\gamma}$ which meets γ_0 orthogonally. Furthermore, $\overline{\gamma}$ meets $\partial\mathbb{R}_+^2$ precisely at the point p_0 and at some other point that we denote by q_0. Thus, all the trajectories of e_2 (curves equidistant from $\overline{\gamma}$) pass through p_0 and q_0, meet orthogonally γ_0 and, by construction, also meet orthogonally all the other geodesics that are normal to $\overline{\gamma}$.

It follows that Σ is invariant under a one-parameter group of isometries G. The projections of the isometries of G onto $\mathbb{H}^2_k$ are the isometries of $\mathbb{H}^2_k$ that fix two points in the boundary of $\mathbb{H}^2_k$ (the so-called *hyperbolic isometries* of $\mathbb{H}^2_k$) and leave invariant the geodesic $\overline{\gamma}$ (in the sense that of $g \in G$, $g(\overline{\gamma}) = \overline{\gamma}$) that joins these two points.

This concludes the case of hypercycles, hence the proof of Theorem 1. □

5 The Abresch–Rosenberg surfaces

As we mentioned before, these are the surfaces in $M^2_k \times \mathbb{R}$ for which the (2, 0)-part of the quadratic form

$$\widetilde{Q}(X, Y) = 2H\,\alpha(X, Y) - k\langle \xi, X\rangle\langle \xi, Y\rangle,$$

namely

$$Q\,\mathrm{d}z^2 = \big\{(\widetilde{Q}(e_1, e_1) - \widetilde{Q}(e_2, e_2)) - 2i\,\widetilde{Q}(e_1, e_2)\big\}\mathrm{d}z^2,$$

vanishes identically. They satisfy Eq. (3) in Sect. 2, so they are included in the surfaces with $\alpha_{12} = 0$. From Eq. (3), and the fact that $2H = \alpha_{11} + \alpha_{22}$, one obtains that

$$\alpha_{11} = H + \frac{k\sin^2\theta}{4H}, \quad \alpha_{22} = H - \frac{k\sin^2\theta}{4H} \tag{13}$$

It follows that the extrinsic curvature K_e of an Abresch-Rosenberg surface is

$$K_e = \alpha_{11}\,\alpha_{22} = (16\,H^4 - k^2\sin^4\theta)\big/16\,H^2.$$

We will distinguish two cases in the classification of the Abresch-Rosenberg surfaces.

(A) $4H^2 + k\sin^2\theta > 0$. This only makes sense if $k < 0$; otherwise, it always holds. In case (A) with $k < 0$, we have that

$$16\,H^4 - k^2\sin^4\theta > 0 \;\Rightarrow\; K_e > 0.$$

(B) $4H^2 + k\sin^2\theta \le 0$. Again, this requires $k < 0$; otherwise, it is never true. In case (B), with $k < 0$, we have

$$16\,H^2 - k^2\sin^4\theta \le 0 \;\Rightarrow\; K_e \le 0.$$

Let us consider each case separately:

CASE A By a theorem of Espinar–Rosenberg [4], $K_e > 0$ implies that Σ is convex. Assume first that Σ is compact. Then, it is foliated by circles and meets the rotation axis; otherwise, it would have points of non-convexity. By compactness, it meets the axis twice, and by convexity, it is embedded and homeomorphic to a sphere. In the notation of [1], it is S_H, of a surface of *spherical type.*
Assume now that Σ is a complete, non-compact surface. By the theorem of Espinar–Rosenberg, it is the graph of a convex function over a slice. In the notation of [1], it is D_H, of a surface of *disk-type.*

CASE B In this case, $K_e \le 0$. Notice that, since $k < 0$, $\alpha_{22} > 0$ and $\alpha_{11} \le 0$ by (13).

We first consider the case where $\alpha_{11} < 0$. Then, $K_e < 0$. Thus, Σ is a complete saddle surface that looks like a catenoid. In the notation of [1], it is C_H, or a *catenoid-type* surface.

We now consider the case where $\alpha_{11} \equiv 0$. We claim that, in this case,

$$k_g^2 = -k. \tag{14}$$

Let us prove (14) using the value of α_{22} in (13).

$$\begin{aligned} k_g^2 = \alpha_{22}^2 / \sin^2\theta &= \left(\frac{H}{\sin\theta} - \frac{k\sin\theta}{4H}\right)^2 \\ &= \frac{H^2}{\sin^2\theta} + \frac{k^2\sin^2\theta}{16\,H^2} - \frac{2H\,k\,\sin\theta}{4H\,\sin\theta} \end{aligned}$$

Thus,

$$k_g^2 + k = \frac{H^2}{\sin^2\theta} + \frac{k^2\sin^2\theta}{16\,H^2} - \frac{k}{2} + k.$$

But, from

$$\alpha_{11} = 0 = \frac{4\,H^2 + k\sin^2\theta}{4\,H}$$

we have that $4\,H^2 = -k\sin^2\theta$, hence

$$k_g^2 + k = -\frac{k}{4} - \frac{k}{4} + \frac{k}{2} = 0,$$

as we claimed.

It follows that Σ is foliated by horocycles. In the notation of [1], it is P_H.

Notice that, by analyticity and (13), if $K_e \le 0$, we have that either $\alpha_{11} \equiv 0$ or $\alpha_{11} < 0$ everywhere in Σ. So, these two situations exhaust case B.

6 Proofs of the Corollaries

Proof of Corollary 1 Let the unit vector of the coordinate axis $0z$ or any other choice of axis be denoted by ξ. The notation has been chosen so that by setting e_1 to be the unit vector of the projection of ξ onto $T\Sigma$, we have again that $\xi = \sin\theta\, e_1 + \cos\theta\, e_3$. By setting $e_2 \perp [e_1, e_2]$, we obtain that $\{e_1, e_2\}$ is a canonical frame on Σ in which the Gauss formula reads

$$K = -|\mathrm{d}\theta|^2 + 2H\,\mathrm{d}\theta(e_1). \tag{8''}$$

We recall that (8″) holds on Σ except at a closed set $F \subset \Sigma$ with no interior points ($\sin\theta = 0$ in F).

Proceeding as in the proof of Theorem 1, we obtain from (8″) that

$$-|\mathrm{d}\theta|^2 - 2H|\mathrm{d}\theta| \le K \le -|\mathrm{d}\theta|^2 + 2H|\mathrm{d}\theta| \tag{9'}$$

and that equality holds if and only if $e_1 = \pm\dfrac{\operatorname{grad}\theta}{|\operatorname{grad}\theta|}$ what implies that $\alpha_{12} = 0$. From that point on, the proof is the same as in Theorem 1, and Σ turns out to be a rotation surface with constant mean curvature in $\mathbb{R}^3$. Being compact, it is the canonical sphere. □

Proof of Corollary 2 It is obvious that equality in the right-hand side of (9′) implies that

$$\int_\Sigma K\,\mathrm{d}\sigma = \int_\sigma (-|\mathrm{d}\theta|^2 + 2H|\mathrm{d}\theta|)\sigma,$$

where $\mathrm{d}\sigma$ is the element of area of Σ (notice that in the present situation $k = 0$). For the converse, we observe that if, at some point $p \in \Sigma$, we had

$$K < -|\mathrm{d}\theta|^2 + 2H|\mathrm{d}\theta| \tag{15}$$

this would be so in a neighborhood U of p. Since, we always have in Σ that

$$K \leq -|\mathrm{d}\theta|^2 + 2H\,|\mathrm{d}\theta|;$$

Equation (15) in U would contradict the above integral, thus proving the converse, hence the equivalence of the above integral and the right-hand side of (9′). A similar argument applies to the left hand side.

Since $\int_\Sigma K\,\mathrm{d}\sigma = 2\pi\,\chi(\Sigma)$, we can continue as in Corollary 1. □

Proof of Corollary 3 The proof is similar to the proof of Theorem 1. The only possible new point is that the trajectories of e_2 can have curvature zero. This cannot occur.

To see that, notice that, since Σ is complete, the above e_2-trajectories are straight lines. On the other hand, the projection onto the plane P orthogonal to ξ of the e_1-trajectories are geodesics in P, hence straight lines. Fix one of these lines, say r, in P. The projection onto P of some e_2-trajectory is orthogonal to r. Since the same is true for all such projections, they are all parallel straight lines in P. Thus, the e_2-trajectories themselves are parallel straight lines in $\mathbb{R}^3$. This implies that Σ is a plane, a contradiction, since we have assumed that $H \neq 0$.

It follows that the trajectories of e_2 are circles, and the rest of the proof is as in Theorem 1. □

Proof of Corollary 4 By Gauss geodesic triangle theorem, $\pi - \sum_{i=1}^{3}\beta_i = \int_R K\,\mathrm{d}\sigma$, where R is the region bounded by T. Thus, we obtain

$$\begin{aligned}\int_R \left(-|\mathrm{d}\theta|^2 + k\,\cos^2\theta - 2H|\mathrm{d}\theta|\right)\mathrm{d}\sigma &\leq \int_R K\,\mathrm{d}\sigma \\ &\leq \int_R \left(-|\mathrm{d}\theta|^2 + k\,\cos^2\theta + 2H|\mathrm{d}\theta|\right)\mathrm{d}\sigma\end{aligned}$$

Now, with an argument similar to that of Corollary 2, we obtain that the inequalities in (9) are equivalent to the above integral inequalities provided we are in an open set if Σ, namely, the interior of the region R. By analyticity, (9) holds everywhere in Σ. The conclusions of Theorem 1 follow. □

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