O(m) × O(n)-Invariant Minimal Hypersurfaces in $\mathbb{R}^{m+n}$

HILÁRIO ALENCAR[1], ABDÊNAGO BARROS[2], OSCAR PALMAS[3],
J. GUADALUPE REYES[4] and WALCY SANTOS[5]

[1,*] *Universidade Federal de Alagoas, Departamento de Matemática, 57072-900 Maceió AL, Brazil. e-mail: hilario@mat.ufal.br*

[2] *Universidade Federal do Ceará, Departamento de Matemática, 60455-760 Fortaleza CE, Brazil. e-mail: abbarros@mat.ufc.br*

[3] *Universidad Nacional Autónoma de México, Facultad de Ciencias, 04510 México DF, México. e-mail: opv@hp.fcieneias.unam.mx*

[4] *Universidad Autónoma Metropolitana–Iztapalapa, Departamento de Matemáticas, Av. San Rafael Atlixco 186, Col. Vicentina, 09340 México DF, México. e-mail: revg@xanum.uam.mx*

[5] *Universidade Federal do Rio de Janeiro, Instituto de Matemática, Av. Brigadeiro Trompowsky, s/n Cidade Universitária, Ilha do Fundão, Caixa Postal 68530, 21945-970 Rio de Janeiro RJ, Brazil. e-mail: walcy@im.ufrj.br*

Abstract. We classify the nonextendable immersed O(m) × O(n)-invariant minimal hypersurfaces in the Euclidean space $\mathbb{R}^{m+n}$, $m, n \geqslant 3$, analyzing also whether they are embedded or stable. We show also the existence of embedded, complete, stable minimal hypersurfaces in $\mathbb{R}^{m+n}$, $m + n \geqslant 8$, $m, n \geqslant 3$ not homeomorphic to $\mathbb{R}^{m+n-1}$ that are O(m) × O(n)-invariant.

Mathematics Subject Classifications (2000): 53A10, 53C42.

Key words: mean curvature, O(m) × O(n)-invariant hypersurfaces.

1. Introduction and Statement of Results

The study of constant mean curvature hypersurfaces in Euclidean spaces, particularly the minimal surfaces, has a very long history. One important issue in this area is the construction of examples providing a testing ground for conjectures and theorems, since the work due to Hsiang and Lawson [1].

G-invariant constant mean curvature hypersurfaces, that is, invariant under the action of some isometry group G, have proved to be manageable and useful. We may quote the classic work [2], studying rotational (i.e., O(2)-invariant) constant mean curvature surfaces, or the analysis and classification of O(n)-invariant minimal hypersurfaces in space forms carried out in [3].

Following the classification of low cohomogeneity isometry groups established by Hsiang and Lawson [1], the next step was to study the O(m) × O(n)-invariant hypersurfaces with constant mean curvature. For example, Hsiang *et al.* [4] constructed a family of such hypersurfaces for $m = n$.

Techniques developed by Bombieri *et al.* [5] allowed them to show the existence of complete $O(m)\times O(n)$-invariant minimal hypersurfaces. Using these techniques, Alencar [6] analyzed these hypersurfaces in the case $m = n$ and classified them for $m \leqslant 3$. It is worth noting that these ideas have been successfully applied also to the null scalar curvature case; see [7, 8].

The aim of this paper is to extend the classification theorems proved in [6] to arbitrary m, n. To state our results we must fix some notations.

We will use the standard action of $O(m) \times O(n)$ over $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. In this case, the orbit space can be identified with $Q = \{(x, y); x \geqslant 0, y \geqslant 0\}$, in such a way that every interior point of Q corresponds to a principal orbit given as the product of spheres $\mathbb{S}^{m-1}(x) \times \mathbb{S}^{n-1}(y)$. We define a hypersurface M of $\mathbb{R}^{m+n}$ invariant under this action by giving a generating *profile curve* $\gamma(t) = (x(t), y(t))$ contained in Q, so that M is parametrized by

$$\begin{aligned} &\bar{x}(t, \phi_1, \dots, \phi_{m-1}, \psi_1, \dots, \psi_{n-1}) \\ &\quad = (x(t)\Phi(\phi_1, \dots, \phi_{m-1}), y(t)\Psi(\psi_1, \dots, \psi_{n-1})), \end{aligned} \tag{1}$$

where Φ and Ψ are orthogonal parametrizations of a unit sphere of the corresponding dimension.

Figures 1 and 2 exhibit all cases of profile curves associated to complete immersed $O(m) \times O(n)$-invariant minimal hypersurfaces, characterized in Theorems 1.1 and 1.2.

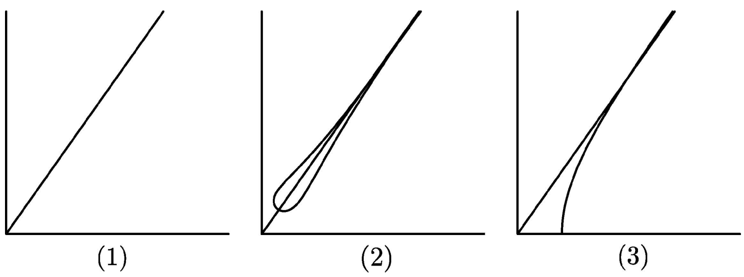

Figure 1. Some examples of profile curves for $m + n \leqslant 7$. Numbering corresponds to that in Theorem 1.1.

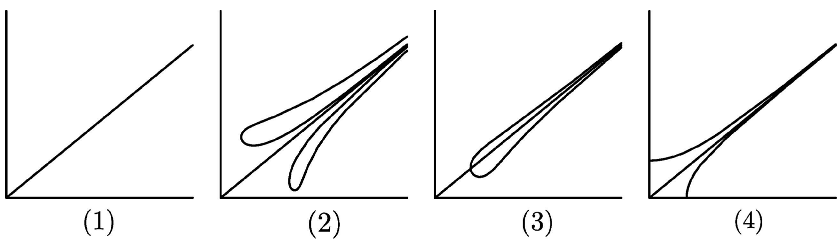

Figure 2. Some examples of profile curves for $m + n \geqslant 8$. Numbering corresponds to that in Theorem 1.2.

THEOREM 1.1. *Given integers* $m, n \geqslant 3$ *such that* $m + n \leqslant 7$, *every non-extendable* $O(m) \times O(n)$*-invariant minimal hypersurface* $M \subset \mathbb{R}^{m+n}$ *falls in only one of the following types*:

(1) *M is a cone* $\mathcal{C}_{m,n}$ *with vertex at the origin, generated by a ray* $y = \sqrt{\frac{n-1}{m-1}}\, x$.
(2) *M is an immersed complete hypersurface which intersects itself and* $\mathcal{C}_{m,n}$ *infinitely countable times*, *approaching this cone asymptotycally.*
(3) *M is an embedded complete hypersurface intersecting* $\mathcal{C}_{m,n}$ *infinitely countable times, approaching this cone asymptotycally and intersecting orthogonally* $\mathbb{R}^m \times \{0\}$ *or* $\{0\} \times \mathbb{R}^n$.

THEOREM 1.2. *Given integers* $m, n \geqslant 3$ *such that* $m + n \geqslant 8$, *every non-extendable* $O(m) \times O(n)$*-invariant minimal hypersurface* $M \subset \mathbb{R}^{m+n}$ *falls in only one of the following types*:

(1) *M is a cone* $\mathcal{C}_{m,n}$ *with vertex at the origin, generated by a ray* $y = \sqrt{\frac{n-1}{m-1}}\, x$.
(2) *M is an immersed complete hypersurface which does not intersect* $\mathcal{C}_{m,n}$, *being doubly asymptotic to this cone*.
(3) *M is an embedded complete hypersurface which intersects* $\mathcal{C}_{m,n}$ *once*, *being doubly asymptotic to this cone*.
(4) *M is an embedded complete hypersurface which does not intersect* $\mathcal{C}_{m,n}$, *being asymptotic to this cone and intersecting orthogonally* $\mathbb{R}^m \times \{0\}$ *or* $\{0\} \times \mathbb{R}^n$.

In the last part of this paper we discuss the stability of these hypersurfaces, obtaining the following results.

THEOREM 1.3. *Let* $m, n \geqslant 3$ *and* $m + n \leqslant 7$. *Any complete* $O(m) \times O(n)$*-invariant minimal hypersurface M in* $\mathbb{R}^{m+n}$ *has infinite index.*

THEOREM 1.4. *Let* $m, n \geqslant 3$ *and* $m + n \geqslant 8$. *The unique stable complete* $O(m) \times O(n)$*-invariant minimal hypersurfaces are those of the type* (4) *given in Theorem* 1.2.

As a consequence of these classification results, we obtain examples of complete, stable minimal hypersurfaces homeomorphic to $\mathbb{R}^m \times \mathbb{S}^{n-1}$ or to $\mathbb{S}^{m-1} \times \mathbb{R}^n$. Our following existence result should be compared with the theorem on the structure of this kind of hypersurfaces obtained by (see [9]).

THEOREM 1.5. *There exist embedded*, *complete*, *stable minimal hypersurfaces in* $\mathbb{R}^N$, *for* $N \geqslant 8$, *not homeomorphic to* $\mathbb{R}^{N-1}$.

2. The Minimal Hypersurface Equation

Using the parametrization $\bar{x}$ given in (1) and the normal vector

$$N(t, \phi_1, \ldots, \phi_{m-1}, \psi_1, \ldots, \psi_{n-1}) = (-y'(t)\Phi(\phi_1, \ldots, \phi_{m-1}), x'(t)\Psi(\psi_1, \ldots, \psi_{n-1})),$$

it can be shown that the principal curvatures λ_0, λ_i, λ_j, $i = 1, \dots, m-1$, $j = m, \dots, m+n-2$ associated to M are

$$\lambda_0 = \frac{-x''y' + x'y''}{[(x')^2 + (y')^2]^{3/2}},$$

$$\lambda_i = \frac{y'}{x\sqrt{(x')^2 + (y')^2}}, \quad i = 1, 2, \dots, m-1,$$

$$\lambda_j = \frac{-x'}{y\sqrt{(x')^2 + (y')^2}}, \quad j = m, \dots, m+n-2.$$

The mean curvature of the hypersurface is given by

$$nH = \sum_{k=0}^{m+n-2} \lambda_k = \frac{-x''y' + y''x'}{[(x')^2 + (y')^2]^{3/2}} + \frac{(m-1)y'}{x[(x')^2 + (y')^2]^{1/2}} - \frac{(n-1)x'}{y[(x)^2 + (y')^2]^{1/2}}.$$

Thus, to obtain a minimal hypersurface of this kind is equivalent to solve the following second-order differential equation:

$$\frac{-x''y' + y''x'}{(x')^2 + (y')^2} + \frac{(m-1)y'}{x} - \frac{(n-1)x'}{y} = 0. \tag{2}$$

Note that every curve $\gamma(t) = (x(t), y(t))$ solving this last equation generates a whole family of minimal hypersurfaces, since every homothetic curve $\gamma_c(t) = (c\,x(t), c\,y(t))$ is also a solution.

Since γ is a regular curve we may parametrize it by arc length. From now on we do that. Then (2) becomes

$$-x''y' + y''x' + \frac{(m-1)y'y - (n-1)x'x}{xy} = 0. \tag{3}$$

Note that x'' and y'' may be expressed explicitly in terms of x, x', y, y', as follows: since $x'^2 + y'^2 = 1$, we obtain $x'x'' + y'y'' = 0$. This equation and (3) may be seen as a system of two linear equations in x'' and y'' with nonzero determinant and solutions given by

$$x'' = -\frac{(m-1)y'y - (n-1)x'x}{xy} y'$$

and

$$y'' = -\frac{(m-1)y'y - (n-1)x'x}{xy} x'.$$

To close this section, we may suppose that $y = y(x)$ in Equation (3), which becomes

$$\frac{d^2y}{dx^2} = -\frac{(m-1)y\frac{dy}{dx} - (n-1)x}{xy}\left(1 + \left(\frac{dy}{dx}\right)^2\right). \tag{4}$$

On the other hand, if $x = x(y)$, Equation (3) can be written as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}y^2} = -\frac{(m-1)y - (n-1)\frac{\mathrm{d}x}{\mathrm{d}y}x}{xy}\left(1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2\right). \tag{5}$$

Expressions (4) and (5) show that our profile curves do not have singularities. In the next section we will perform a useful transformation on this equation.

3. The Associated Vector Field

Following [5, 6] (see also [7, 8]), we define the Bombieri–de Giorgi–Gusti coordinate transformation $(x, y) \mapsto (u, v)$ given by

$$\tan u = \frac{y}{x}, \qquad \tan v = \frac{y'}{x'}, \tag{6}$$

defined for $(u, v) \in \bar{D}$, where

$$D = \left(0, \frac{\pi}{2}\right) \times (-\pi, \pi).$$

It is easy to see that

$$u' = \frac{y'x - x'y}{x^2 + y^2} \quad \text{and} \quad v' = -x''y' + y''x'.$$

Multiplying (3) by $\frac{x^2y^2}{(x^2+y^2)^{3/2}}\, u'$ and using this change of coordinates, we get

$$v'[-\cos u \sin u \sin(u - v)] + u'[(m-1)\sin v \sin u - (n-1)\cos v \cos u] = 0.$$

This last equation provides us with a system of ordinary differential equations for u, v and a vector field $X(u, v) = (X_1(u, v), X_2(u, v))$ defined in $\bar{D}$ given by

$$\begin{aligned} X_1(u, v) &= u' = \cos u\ \sin u \sin(u - v), \\ X_2(u, v) &= v' = (m-1)\ \sin v\ \sin u - (n-1)\cos v\ \cos u. \end{aligned} \tag{7}$$

LEMMA 3.1. *The vector field X has the following properties*:

(1) *X_1 vanishes along the lines $u = 0$, $u = \pi/2$, $v = u$ and $v = u - \pi$.*
(2) *X_2 vanishes along the graphs of the smooth functions*

$$v_1(u) = \arctan\left(\frac{n-1}{m-1}\cot u\right),$$
$$v_2(u) = \arctan\left(\frac{n-1}{m-1}\cot u\right) - \pi;$$

defined for $u \in [0, \pi/2]$. Moreover, v_1 and v_2 are decreasing and we have

(a) $\lim_{u\to 0} v_1(u) = \pi/2$ *and* $\lim_{u\to\frac{\pi}{2}} v_1(u) = 0$.
(b) $\lim_{u\to 0} v_2(u) = -\pi/2$ *and* $\lim_{u\to\frac{\pi}{2}} v_2(u) = -\pi$.
(c) *If* $n < m$ $(n = m, n > m)$, *then* v_1 *and* v_2 *are concave up* (*linear*, *concave down*).
(d) $v_1'(0) = v_2'(0) = -\frac{m-1}{n-1}$ *and* $v_1'(\frac{\pi}{2}) = v_2'\left(\frac{\pi}{2}\right) = -\frac{n-1}{m-1}$.

Proof. The proof is a straightforward calculation, solving directly $X_1 = 0$ and $X_2 = 0$ in (7). For (c) and (d), note that for $i = 1, 2$ we have

$$\frac{\mathrm{d}v_i}{\mathrm{d}u} = -\frac{\frac{n-1}{m-1}}{\sin^2 u + \left(\frac{n-1}{m-1}\right)^2 \cos^2 u} < 0$$

and

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}u^2} = \frac{2\frac{n-1}{m-1}\sin u \cos u}{\left(\sin^2 u + \left(\frac{n-1}{m-1}\right)^2 \cos^2 u\right)^2}\left(1 - \left(\frac{n-1}{m-1}\right)^2\right). \qquad \square$$

The graphs of the possible types of v_1 and v_2 are shown in Figure 3.

We obtain the singular points of X by intersecting the graphs of the functions $v = u$, $v = u - \pi$ with those of $v_1(u)$ and $v_2(u)$; namely,

COROLLARY 3.2. *The singular points of the vector field* X *in* $\bar{D} = [0, \frac{\pi}{2}] \times [-\pi, \pi]$ *are* $p_1 = (0, -\frac{\pi}{2})$, $p_2 = (0, \frac{\pi}{2})$, $p_3 = (\frac{\pi}{2}, -\pi)$, $p_4 = (\frac{\pi}{2}, 0)$, $p_5 = (\frac{\pi}{2}, \pi)$, *and* $p_6 = (\alpha, \alpha)$, $p_7 = (\alpha, \alpha - \pi)$, *where* $\alpha = \arctan\sqrt{\frac{n-1}{m-1}} \in (0, \pi/2)$.

Due to the continuity of the vector field X we have the next result.

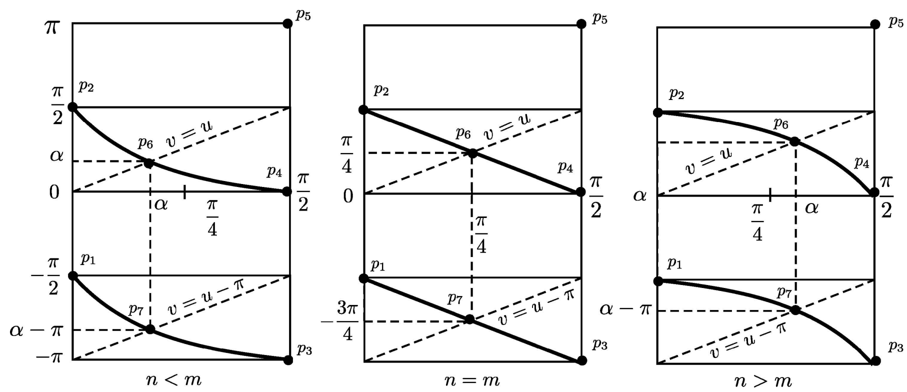

Figure 3. The graphs of the functions $v_1 = v_1(u)$ and $v_2 = v_2(u)$.

COROLLARY 3.3. *The vector field X also satisfies*

$$X(0,v)=\begin{cases}(0,X_2^+) & \textit{for } v\in\left(-\pi,-\dfrac{\pi}{2}\right);\\ (0,X_2^-) & \textit{for } v\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right);\\ (0,X_2^+) & \textit{for } v\in\left(\dfrac{\pi}{2},\pi\right).\end{cases}$$

$$X\left(\frac{\pi}{2},v\right)=\begin{cases}(0,X_2^-) & \textit{for } v\in(-\pi,0);\\ (0,X_2^+) & \textit{for } v\in(0,\pi).\end{cases}$$

$$X(u,u)=\begin{cases}(0,X_2^-) & \textit{for } 0<u<\alpha;\\ (0,X_2^+) & \textit{for } \alpha<u<\dfrac{\pi}{2}.\end{cases}$$

$$X(u,u-\pi)=\begin{cases}(0,X_2^+) & \textit{for } 0<u<\alpha;\\ (0,X_2^-) & \textit{for } \alpha<u<\dfrac{\pi}{2}.\end{cases}$$

$$X\,(u,\ v_1(u))=\begin{cases}(X_1^-,0) & \textit{for } 0<u<\alpha;\\ (X_1^+,0) & \textit{for } \alpha<u<\dfrac{\pi}{2}.\end{cases}$$

$$X\,(u,v_2(u))=\begin{cases}(X_1^-,\ 0) & \textit{for } 0<u<\alpha;\\ (X_1^+,\ 0) & \textit{for } \alpha<u<\dfrac{\pi}{2}.\end{cases}$$

where $X_i^+>0$, *while* $X_i^-<0$.

PROPOSITION 3.4. *For any integers* $m,n>0$, *the singular points* p_1, p_2, p_3, p_4 *and* p_5 *of the vector field X are saddle points. If* $m+n\leqslant 7$, p_6 *is an unstable* (*repulsor*) *focus and* p_7 *is a stable* (*attractor*) *focus. If* $m+n\geqslant 8$ *then* p_6 *is an unstable node and* p_7 *is a stable node.*

Proof. The linear part of X is given by $DX_p=(a_{ij})$, where

$$\begin{aligned}a_{11}&=\cos 2u\sin(u-v)+\cos u\sin u\cos(u-v),\\ a_{12}&=-\cos u\sin u\cos(u-v),\\ a_{21}&=(m-1)\sin v\cos u+(n-1)\cos v\sin u,\\ a_{22}&=(m-1)\cos v\sin u+(n-1)\sin v\cos u,\end{aligned}$$

and, therefore,

– $DX_{p_1}=\begin{pmatrix}-1 & 0\\ m-1 & n-1\end{pmatrix}$

– $DX_{p_2}=\begin{pmatrix}1 & 0\\ m-1 & -(n-1)\end{pmatrix}$

– $DX_{p_3} = \begin{pmatrix} 1 & 0 \\ -(n-1) & -(m-1) \end{pmatrix}$,

– $DX_{p_4} = \begin{pmatrix} -1 & 0 \\ n-1 & m-1 \end{pmatrix}$,

– $DX_{p_5} = \begin{pmatrix} 1 & 0 \\ -(n-1) & -(m-1) \end{pmatrix}$.

These equalities prove the first claiming of the proposition. For $p_6 = (\alpha, \alpha)$, $\alpha \in (0, \pi/2)$, we have $\cos\alpha > 0$, $\sin\alpha > 0$ and

$$DX_{p_6} = \sin\alpha\cos\alpha \begin{pmatrix} 1 & -1 \\ m+n-2 & m+n-2 \end{pmatrix}.$$

If we set $\beta = m + n - 2$, then it suffices to calculate the eigenvalues of

$$A = \begin{pmatrix} 1 & -1 \\ \beta & \beta \end{pmatrix}$$

that are given by

$$\mu_1 = \tfrac{1}{2}\big[\beta + 1 + \sqrt{(\beta+1)^2 - 8\beta}\big]$$

and

$$\mu_2 = \tfrac{1}{2}\big[\beta + 1 - \sqrt{(\beta+1)^2 - 8\beta}\big].$$

This expression shows that p_6 is a repulsor point.

Similarly, for $p_7 = (\alpha, \alpha - \pi)$ we have

$$DX_{p_7} = \sin\alpha\ \cos\alpha \begin{pmatrix} -1 & 1 \\ -\beta & -\beta \end{pmatrix}.$$

The eigenvalues of the matrix in this expression are given by

$$\nu_1 = -\tfrac{1}{2}\big[\beta + 1 + \sqrt{(\beta+1)^2 - 8\beta}\big]$$

and

$$\nu_2 = -\tfrac{1}{2}\big[\beta + 1 - \sqrt{(\beta+1)^2 - 8\beta}\big].$$

This proves that p_7 is an attractor point.

To finish the proof, we note that the discriminant $(\beta+1)^2 - 8\beta$ is negative if and only if $\beta < 3 + 2\sqrt{2}$. Therefore, $\mu_{1,2}$ are real numbers if and only if $m + n \geqslant 8$, which ends the proof of the proposition. □

We now analyze the behavior of X in the regions D_1^+, D_2^+, D_1^-, D_2^- defined by

$$D_1^+ = \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right),$$
$$D_2^+ = \left(0, \frac{\pi}{2}\right) \times \left(\frac{\pi}{2}, \pi\right),$$
$$D_1^- = \left(0, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, 0\right),$$
$$D_2^- = \left(0, \frac{\pi}{2}\right) \times \left(-\pi, -\frac{\pi}{2}\right).$$

LEMMA 3.5. *For any integers $m,n \geqslant 3$, X does not have periodic orbits in any of the regions D_1^+, D_2^-, D_1^-, D_2^+.*

Proof. First note that in D_1^+, the functions $\sin u, \cos u, \sin v$, and $\cos v$ are positive. Calculating the divergence of the vector field X, we have

$$\operatorname{Div} X = [3\cos^2 u + (m-2)]\sin u \cos v + [3\sin^2 u + (n-1)]\cos u \sin v.$$

If $m, n \geqslant 3$, then the sign of Div depends on $f(u, v) = \sin u \cos v$ and $g(u, v) = \cos u \sin v$. Therefore, in D_1^+ we have

$$\operatorname{Div} X > 0.$$

Hence, the Bendixson criterion implies the claiming on D_1^+.

On the other hand, on D_2^- the functions $\sin v$ and $\cos v$ are negative, so that

$$\operatorname{Div} X < 0.$$

Using again the Bendixson criterion, our assertion on D_2^- follows.

Using a continuity argument, we see that $X_2(u, v) > 0$ for

$$\arctan\left(\frac{n-1}{m-1}\cot u\right) < v < \pi$$

and $X_2(u, v) < 0$ for

$$-\frac{\pi}{2} < v < \arctan\left(\frac{n-1}{m-1}\cot u\right).$$

Then, X does not have periodic orbits in the region

$$D_1^- = \left\{(u, v) \middle| -\frac{\pi}{2} < v < 0\right\}.$$

The earlier argument proves also a similar claiming for X in D_2^+ and ends the proof. $\square$

Therefore the Poincaré–Bendixson Theorem implies the following results.

COROLLARY 3.6. *For any $m, n \geqslant 3$, the set $(0, \frac{\pi}{2}) \times [-\frac{\pi}{2}, \pi]$ is contained in the unstable manifold $W^u(p_6)$ of the singular point p_6.*

COROLLARY 3.7. *For any $m,n \geqslant 3$, the set $(0, \frac{\pi}{2}) \times [-\pi, 0]$ is contained in the stable manifold $W^s(p_7)$ of the singular point p_7.*

Now, we study the behavior of the part of the stable manifold $W^s(p_2)$ contained in $W^u(p_6)$. Since along the vertical axis we have

$$X(0, v) = \begin{cases} (0, X_2^-) & \text{for } v \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ (0, X_2^+) & \text{for } v \in \left(\dfrac{\pi}{2}, \pi\right), \end{cases}$$

and the linear part of X in p_2 is

$$DX_{p_2} = \begin{pmatrix} -1 & 0 \\ m-1 & n-1 \end{pmatrix},$$

it follows that the unstable manifold $W^u(p_2)$ is contained in the y-axis.

We have the following result for the stable manifold $W^s(p_2)$.

PROPOSITION 3.8. *The vector field X has a unique integral curve $\phi(t)$ defined for all $t \in \mathbb{R}$ and such that*

$$\lim_{t\to-\infty} \phi(t) = p_6 = (\alpha, \alpha)$$

and

$$\lim_{t\to\infty} \phi(t) = p_2 = \left(0, \frac{\pi}{2}\right).$$

That is, $\{\phi(t)\} \subset W^s(p_2)$.

Proof. The stable manifold theorem in dimension two implies the existence of one-dimensional stable manifold for p_2. The tangent space of this manifold is generated by $\xi = (\xi^1, \xi^2)$, an eigenvector associated to the eigenvalue -1 of DX_{p_2}. A direct calculation shows that we may choose

$$\xi = \left(1, -\frac{m-1}{n}\right).$$

As for any $m, n \in \mathbb{N}$ we have the inequality

$$\frac{\mathrm{d}v_1}{\mathrm{d}u}(0) = -\frac{m-1}{n-1} < -\frac{m-1}{n},$$

it follows that $W^s(p_2)$ is transversal to the graph of the function v_1 at p_2; moreover, ξ points to the region above of this graph.

Because of this, the part of $W^s(p_2)$ contained in $(0, \pi/2) \times [-\pi/2, \pi]$ is also contained in $W^u(p_6)$. Since these manifolds are one-dimensional, we get an integral curve $\phi(t)$ of X, which is defined for all $t \in \mathbb{R}$ since X is bounded in $[0, \pi/2] \times [-\pi/2, \pi]$. Hence we have

$$\lim_{t\to-\infty} \phi(t) = p_6 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = p_2$$

and so the proof is complete. □

We may prove in a similar way the existence of one-dimensional manifolds invariant under the flow of X connecting every saddle point p_1, p_2, p_3, p_4, p_5 with p_6 and p_7. From now on, $W^u(p_i)$, $W^s(p_i)$, $i = 1, \dots, 5$, will denote the part of such invariant manifolds contained in $\bar{D} = [0, \frac{\pi}{2}] \times [-\pi, \pi]$.

On the other hand, since $\bar{D}$ is compact and X is continuous, it follows that every orbit is complete, that is, it is defined for all $t \in \mathbb{R}$.

We summarize the earlier results as follows.

PROPOSITION 3.9. *For $m, n \geqslant 3$, every orbit $\{\phi(t)\}$ of X contained in $\bar{D} = [0, \frac{\pi}{2}] \times [-\pi, \pi]$ is defined for all $t \in \mathbb{R}$, and falls into one of the following types:*

(1) *$\{\phi(t)\}$ is contained in the v-axis and has p_1 or p_2 as α-limits or ω-limits. (The singular orbits p_1 and p_2 belong to this case.)*
(2) *$\{\phi(t)\}$ is contained in the line $u = \pi/2$ and has p_3, p_4 or p_5 as α-limits or ω-limits. (The singular orbits p_3, p_4 and p_5 belong to this case.)*
(3) *$\phi(t) \equiv p_6$ or $\phi(t) \equiv p_7$.*
(4) *$\{\phi(t)\} \subset W^u(p_6) \cap W^s(p_7)$. That is,*

$$\lim_{t\to-\infty} \phi(t) = p_6 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = p_7.$$

(5) *$\{\phi(t)\} \subset W^u(p_6) \cap W^s(p_8)$, with $p_8 = (\alpha, \alpha + \pi)$. That is,*

$$\lim_{t\to-\infty} \phi(t) = p_6 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = p_8.$$

(6) *$\{\phi(t)\} \subset W^u(p_9) \cap W^s(p_7)$, where $p_9 = (\alpha, \alpha - 2\pi)$. That is,*

$$\lim_{t\to-\infty} \phi(t) = p_9 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = p_7.$$

(7) *$\{\phi(t)\} = W^s(p_2) \subset W^u(p_6)$. That is,*

$$\lim_{t\to-\infty} \phi(t) = p_6 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = p_2.$$

(8) $\{\phi(t)\} = W^s(p_4) \subset W^u(p_6)$. *That is,*

$$\lim_{t\to-\infty} \phi(t) = p_6 \quad \textit{and} \quad \lim_{t\to\infty} \phi(t) = p_4.$$

(9) $\{\phi(t)\} = W^u(p_1) \subset W^s(p_7)$. *That is,*

$$\lim_{t\to-\infty} \phi(t) = p_1 \quad \textit{and} \quad \lim_{t\to\infty} \phi(t) = p_7.$$

(10) $\{\phi(t)\} = W^u(p_3) \subset W^s(p_7)$. *That is,*

$$\lim_{t\to-\infty} \phi(t) = p_3 \quad \textit{and} \quad \lim_{t\to\infty} \phi(t) = p_7.$$

We now describe all nonvertical orbits of X contained in the closure $\overline{W^u(p_6) \cap W^s(p_7)}$. Since the behavior of the singular points in the other invariant two-dimensional manifolds is the same as in $\overline{W^u(p_6) \cap W^s(p_7)}$, the global behavior of X is represented in $\bar{D}$.

LEMMA 3.10. *Every orbit* $\{\phi(t)\} \subset W^u(p_6) \cap W^s(p_7)$ *meets the line* $v = u$ *or the line* $v = u - \pi$.

Proof. Any orbit $\phi(t) = (u(t), v(t))$ contained in $W^u(p_6) \cap W^s(p_7)$ may be reparametrized by $t = t(\tau)$, $\tau \in (0, 1)$, in order to obtain a smooth curve $\psi(\tau) = \phi(t(\tau)) = (\widetilde{u}(\tau), \widetilde{v}(\tau))$ with $\psi(0) = p_6$ and $\psi(1) = p_7$. As $\widetilde{u}(0) = \widetilde{u}(1) = \alpha$, the mean value theorem implies the existence of $\tau_0 \in (0, 1)$ such that $\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau}(\tau_0) = 0$. But $\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau}(\tau_0) = \frac{\mathrm{d}u}{\mathrm{d}t}(t_0)\frac{\mathrm{d}t}{\mathrm{d}\tau}(\tau_0)$, where $t_0 = t(\tau_0)$. This fact implies $\frac{\mathrm{d}u}{\mathrm{d}t}(t_0) = 0$, because $\frac{\mathrm{d}t}{\mathrm{d}\tau}(\tau_0) \neq 0$ and $t = t(\tau)$ is a reparametrization.

In this way, there exists $t_0 \in \mathbb{R}$ such that $u'(t_0) = 0$. But the first coordinate of the field X vanishes in $W^u(p_6) \cap W^s(p_7)$ only along the lines $v = u$ and $v = u - \pi$, which implies $\phi(t_0)$ lies in some of those lines. □

COROLLARY 3.11. *Given* $m, n \geqslant 3$ *integers such that* $m + n \leqslant 7$, *every orbit* $\{\phi(t)\}$ *contained in* $W^u(p_6) \cap W^s(p_7)$ *meets the line* $v = u$ *and the line* $v = u - \pi$ *infinitely countable times. Moreover, every such orbit spirals out of* p_6 *and spirals into* p_7.

Proof. This follows from the fact than p_6 and p_7 are hyperbolic foci. □

Figure 4 shows the flows in the cases $m + n \leqslant 7$.

PROPOSITION 3.12. *For any integers* $m, n \geqslant 3$ *such that* $m + n \geqslant 8$, *every orbit* $\{\phi(t)\}$ *contained in* $W^u(p_6) \cap W^s(p_7)$ *has one and only one of the following properties*:

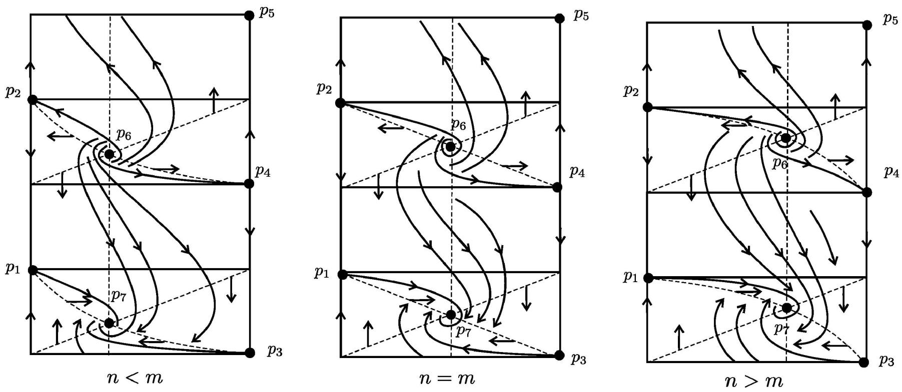

Figure 4. Flows for the cases $m + n \leqslant 7$.

(1) $\{\phi(t)\}$ *meets once the line* $v = u$ *and does not meet the line* $v = u - \pi$. *Moreover,* $\{\phi(t)\}$ *meets once the curve* $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u)$ *and does not meet the curve* $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u) - \pi$.
(2) $\{\phi(t)\}$ *meets once the line* $v = u$ *and once the line* $v = u - \pi$. *Moreover,* $\{\phi(t)\}$ *meets once the curve* $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u)$ *and meets once the curve* $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u) - \pi$.
(3) $\{\phi(t)\}$ *meets once the line* $v = u - \pi$ *and does not meet the line* $v = u$. *Moreover,* $\{\phi(t)\}$ *meets once the curve* $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u) - \pi$ *and does not meet the curve* $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u)$.

Proof. Let $\beta = m + n - 2$ be as in Proposition 3.4. If we define a field $Y = (Y_1(z, w), Y_2(z, w))$ as the linearization of X at the point p_7, we obtain a system of linear differential equations

$$z' = \mu_1 z = Y_1(z, w), \qquad w' = \mu_2 w = Y_2(z, w),$$

where

$$\mu_1 = -\tfrac{1}{2}[\beta + 1 + \sqrt{(\beta + 1)^2 - 8\beta}]$$

and

$$\mu_2 = -\tfrac{1}{2}[\beta + 1 - \sqrt{(\beta + 1)^2 - 8\beta}]$$

are the eigenvalues obtained in Proposition 3.4, associated to the corresponding eigenvectors

$$\xi_1 = \left(\xi_1^1, \xi_1^2\right) = \left(1, \frac{-2\beta}{3\beta + 1 + \sqrt{(\beta + 1)^2 - 8\beta}}\right),$$
$$\xi_2 = \left(\xi_2^1, \xi_2^2\right) = \left(1, \frac{-2\beta}{3\beta + 1 - \sqrt{(\beta + 1)^2 - 8\beta}}\right).$$

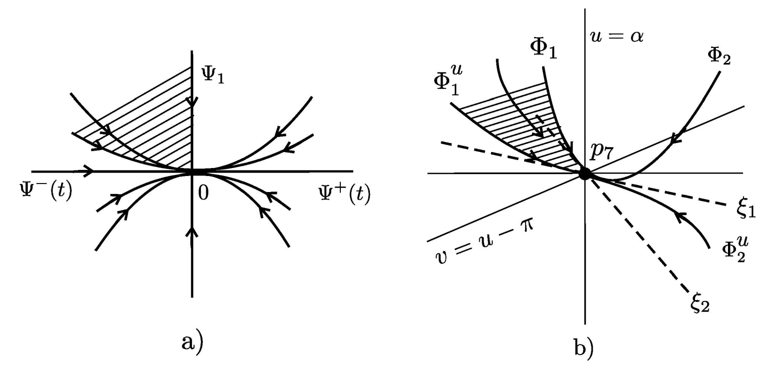

Figure 5. Conjugation of the fields X and Y.

We remark that the second coordinates of these vectors are negative and $\xi_1^2 < \xi_2^2$, which implies that ξ_1 points to the region above of ξ_2.

In the system of coordinates (z, w) relative to the basis $\{\xi_1, \xi_2\}$, the flow of Y around the origin behaves as shown in Figure 5.

Hence, the Grobman–Hartman Theorem for C^1-flows (see [10], p. 127) implies that the field X in a neighborhood of p_7 is C^1-conjugated to the field Y near the origin.

Since the orbits $\psi^-(t)$ and $\psi^+(t)$ on the z-axis are separatrices for the field Y, under the conjugation they correspond locally to the separatrices of X; namely, $\phi_1^u(t) = W^u(p_1)$ and $\phi_2^u(t) = W^u(p_3)$, respectively. These orbits are contained in $\overline{W^u(p_6) \cap W^s(p_7)}$ and converge to p_7 as $t \to \infty$ with the same direction as ξ_1, this is, with slope $-2\beta/(3\beta + 1 + \sqrt{(\beta+1)^2 - 8\beta})$.

The orbit $\psi^-(t)$ converges to zero with slope zero, together with a family of orbits of Y, as shown in Figure 5. Conjugation implies the existence of a corresponding family of orbits of X which together with $\phi_1^u(t)$ converge to p_7 with the same slope $-2\beta/(3\beta + 1 + \sqrt{(\beta+1)^2 - 8\beta})$. Such a family does not meet the line $u = \alpha$ and therefore does not meet the line $v = u - \pi$. From Lemma 3.10, it follows that this family meets only $v = u$.

On the other hand, since X is continuous and transversal to the line $u = \alpha$, it follows the existence of an orbit $\{\phi_1(t)\} \subset W^u(p_6) \cap W^s(p_7)$ bounding the aforementioned family, converging asymptotically to p_7 with direction ξ_2, this is, with slope $-2\beta/(3\beta + 1 - \sqrt{(\beta+1)^2 - 8\beta})$. This orbit ϕ_1 corresponds under conjugation with the vertical orbit ψ_1 of Y on the upper half part of the plane z, w. We observe that ϕ_1 does not meet transversally $u = \alpha$, and therefore belongs to the class of orbits meeting only the line $v = u$.

A linearization near the hyperbolic node p_6 will give that the family of orbits considered earlier will diverge from p_6 with direction ξ_1. In particular, those orbits of the family corresponding to the region to the left of $u = \alpha$ (or the left lower part of the plane z, w) will cross the horizontal line $v = \alpha$ after leaving $p_6 = (\alpha, \alpha)$. This fact implies the existence of a point where $v' = 0$, or $X_2 = 0$. Recalling Lemma 3.1

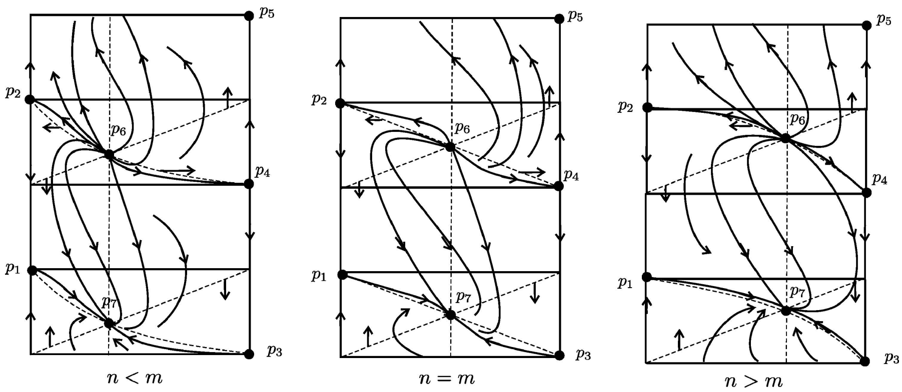

Figure 6. Flows for the cases $m + n \geqslant 8$.

we obtain a point (u, v) of the orbit where $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u)$. As the orbit does not meet $u = \alpha$, it also does not meet $v = v_2(u) = \arctan(\frac{n-1}{m-1}\cot u) - \pi$.

A similar analysis near the hyperbolic node p_6 proves the existence of another family of orbits of X in $W^u(p_6) \cap W^s(p_7)$ such that every such orbit satisfy (3) in this Proposition; moreover, this family will be bounded by an orbit $\{\phi_2(t)\}$ which together with the family converges to p_7 in the direction of ξ_2, this is, with slope $-2\beta/(3\beta + 1 + \sqrt{(\beta+1)^2 - 8\beta})$.

The continuity of X and its transversality relative to the lines $v = u, u = \alpha$ and $v = u - \pi$ imply that every orbit in the region bounded by $\{\phi_1(t)\}$ and $\{\phi_2(t)\}$ satisfy conditions (2), which finish the proof. $\square$

Proposition 3.12 completes our global description of the flow for X. Figure 6 shows the flows in the case $m + n \geqslant 8$.

Remark 3.13. As shown in the proof, Proposition 3.12 implies the existence of an orbit $\{\phi_1(t)\}$ with property (1) separating the curves with property (1) from the curves with property (2). Similarly, there is an orbit $\{\phi_2(t)\}$ with property (3) separating curves with property (3) from those satisfying property (2). They will play an important role in the conclusion of Theorem 1.2.

4. The Profile Curves

The aim of this section will be to translate the behavior of the orbits $\{\phi(t)\}$ of X described in Proposition 3.9 into information about the corresponding profile curves γ.

PROPOSITION 4.1. *For $m, n \geqslant 3$, let $\phi(t) = (u(t), v(t))$ be an orbit of X contained in $\bar{D} = [0, \frac{\pi}{2}] \times [-\pi, \pi]$, defined for all $t \in \mathbb{R}$ and $\gamma(t) = (x(t), y(t))$*

be the corresponding profile curve. Hence γ falls into one of the following types, numbered according to Proposition 3.9:

1. γ *is contained in the x-axis.*
2. γ *is contained in the y-axis.*
3. γ *is a ray* $y = \sqrt{\frac{n-1}{m-1}}\, x$.

4–6. γ *is doubly asymptotic to the ray* $y = \sqrt{\frac{n-1}{m-1}}\, x$.

7–8. γ *is asymptotic to* $y = \sqrt{\frac{n-1}{m-1}}\, x$ *and meets the x-axis orthogonally. Moreover, it is a graph over its projection on the x-axis.*

9–10. γ *is asymptotic to* $y = \sqrt{\frac{n-1}{m-1}}\, x$ *and meets the y-axis orthogonally. Moreover, it is a graph over its projection on the y-axis.*

Proof. First we analyze the singularities of X given in Corollary 3.2. We have $u(t) \equiv 0$ for p_1 and p_2, so Equation (6) shows that $y(t) \equiv 0$. Similarly, $u(t) \equiv \pi/2$ for p_3, p_4, p_5, so that $x(t) \equiv 0$. These correspond to the Cases (1) and (2) in the current Proposition. For p_6 and p_7 we have $u(t) \equiv \alpha$, so that $y(t) = (\tan\alpha)x(t)$. Recalling that $\alpha = \arctan\sqrt{\frac{n-1}{m-1}}$, we have Case (3).

Now we analyze Case (4). From Proposition 3.9 we know that $\{\phi(t)\}$ is contained in $W^u(p_6) \cap W^s(p_7)$, which means that

$$\lim_{t\to-\infty}(u(t), v(t)) = (\alpha, \alpha) \quad \text{and} \quad \lim_{t\to\infty}(u(t), v(t)) = (\alpha, \alpha - \pi).$$

The first fact implies that the profile curve $\gamma(t)$ satisfies

$$\lim_{t\to-\infty}\frac{y(t)}{x(t)} = \tan\alpha \quad \text{and} \quad \lim_{t\to-\infty}\frac{y'(t)}{x'(t)} = \lim_{t\to-\infty}\frac{dy}{dx} = \tan\alpha,$$

so that γ asymptotics $y = (\tan\alpha)x$ at $-\infty$.

On the other hand, we have

$$\lim_{t\to\infty}\frac{y(t)}{x(t)} = \tan\alpha \quad \text{and} \quad \lim_{t\to\infty}\frac{y'(t)}{x'(t)} = \lim_{t\to\infty}\frac{dy}{dx} = \tan(\alpha - \pi).$$

As $\tan\alpha = \tan(\alpha - \pi)$, we have that γ asymptotics $y = (\tan\alpha)x$ at ∞. Hence we derive Case (4); Cases (5) and (6) can be treated similarly.

The asymptotic behavior in Cases (7)–(10) can be treated as earlier. In Case (7) of Proposition 3.9, $\{\phi(t)\} = W^s(p_2) \subset W^u(p_6)$, so that $\lim_{t\to\infty}\phi(t) = p_2 = (0, \frac{\pi}{2})$, which gives

$$\lim_{t\to-\infty}\frac{y(t)}{x(t)} = 0 \quad \text{and} \quad \lim_{t\to-\infty}\frac{y'(t)}{x'(t)} = \lim_{t\to-\infty}\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\frac{\pi}{2} = \infty,$$

these facts imply the desired orthogonality for Case (7). To prove that this profile curve is a graph over its projection on the x-axis, we note that the associated orbit $(u(t), v(t))$ is contained in the set $D_1^+ = (0, \frac{\pi}{2}) \times (0, \frac{\pi}{2})$. Since $\tan v(t) = y'/x' = \mathrm{d}y/\mathrm{d}x$, we have that $\mathrm{d}y/\mathrm{d}x > 0$ for every point of the profile curve. The implicit function theorem implies that the profile curve is a graph.

Cases (8)–(10) may be treated similarly. Therefore we derive the Lemma. □

LEMMA 4.2. *The profile curve* γ *has an inflection point* (*when it is seen as the graph of a function* $y = y(x)$ *or* $x = x(y)$) *if and only if* $\{\phi(t)\}$ *either meets the curve* $v = v_1(u) = \arctan(\frac{n-1}{m-1}\cot u)$ *or meets the curve* $v = v_2(u) = \arctan(\frac{n-1}{m-1}\cot u) - \pi$.

Proof. The proof follows from the fact that d^2y/dx^2 or d^2x/dy^2 in Equations (4) and (5) change sign if and only if

$$(m-1)y\frac{dy}{dx} - (n-1)x = 0 \quad \text{or} \quad (m-1)y - (n-1)\frac{dx}{dy}x = 0.$$

Each one of these equations is equivalent to

$$v_1(u) = \arctan\left(\frac{n-1}{m-1}\cot u\right) \quad \text{or} \quad v_2(u) = \arctan\left(\frac{n-1}{m-1}\cot u\right) - \pi$$

in the transformed plane (u, v). □

Now we will study the cases $m + n \leqslant 7$ and $m + n \geqslant 8$ separately. First we will treat the case $m + n \leqslant 7$.

PROPOSITION 4.3. *Let* $m, n \geqslant 3$ *be integers such that* $m + n \leqslant 7$. *Profile curves corresponding to Cases* 4–10 *in Proposition 3.9 intersect the ray* $y = \sqrt{\frac{n-1}{m-1}}\,x$ *infinitely countable times.*

Proof. By Corollary 3.11, an orbit $\{\phi(t)\}$ of this kind spirals into p_7, which means that the orbit meets infinitely countable times the line $u = \alpha$. As this line corresponds to the ray $y = \sqrt{\frac{n-1}{m-1}}\,x$, the proposition follows. □

Second we will treat the case $m + n \geqslant 8$.

PROPOSITION 4.4. *Let* $m, n \geqslant 3$ *be integers such that* $m + n \geqslant 8$. *The profile curves corresponding to Cases 4–10 in Proposition 3.9 may be classified as follows*:

(1) *Those profile curves whose corresponding orbit* $\{\phi(t)\}$ *satisfies (a) in Proposition* 3.12 *are always below the ray* $y = \sqrt{\frac{n-1}{m-1}}\,x$ *and have only one inflection point.*
(2) *Those profile curves whose corresponding orbit* $\{\phi(t)\}$ *satisfies* (2) *in Proposition* 3.12 *are embedded and have only two inflection points.*
(3) *Those profile curves whose corresponding orbit* $\{\phi(t)\}$ *satisfies* (3) *in Proposition* 3.12 *are always above the ray* $y = \sqrt{\frac{n-1}{m-1}}\,x$ *and have only one inflection point.*

Proof. Statements (1) and (3) are just reformulations of the behavior of the corresponding cases in Proposition 3.12, using also Lemma 4.2.

On the other hand, by Case (2) in Proposition 3.12 and Lemma 4.2 we have that a profile curve γ of type (2) in the current proposition has only two inflection points.

We will prove now that γ does not have selfintersections. Let $\{\phi(t)\}$ an orbit associated to γ. By (3) in Proposition 3.12, there exist $t_0 < t_1$ such that $u'(t_0) = u'(t_1) = 0$ and $u(t_0) < u(t_1)$.

We claim that $\{\phi(t)\}$ restricted to $[t_0, t_1]$ meets once every line $u = \beta$ for $u(t_0) < \beta < u(t_1)$. If this is not the case, there exist β and $t_0 < t_2 < t_3 < t_1$ such that $u(t_2) = u(t_3) = \beta$. The mean value theorem implies that there exists $t^* \in (t_2, t_3)$ such that $u'(t^*) = 0$. Therefore, $\{\phi(t^*)\}$ is in the straight line $v = u$ or in the line $v = u - \pi$, giving a contradiction.

This fact implies that γ restricted to $[t_0, t_1]$ meets once every line $y = (\tan\beta)x$ for $u(t_0) < \beta < u(t_1)$ and thus $\gamma[t_0, t_1]$ is embedded in the plane (x, y). Note that the points $\gamma(t_0)$, $\gamma(t_1)$ lie in opposite sides of the line $y = (\tan\alpha)x$, because $\{\phi(t)\}$ meets only once $u = \alpha$. Similarly, the sets $\gamma(-\infty, t_0]$ and $\gamma[t_1, \infty)$ lie in opposite sides of the line $y = (\tan\alpha)x$.

In this way, if the smooth curve γ has some selfintersection, it must occur in one side of $y = (\tan\alpha)x$. But the existence of such a selfintersection will imply the existence of at least three inflection points in γ, which contradicts what we have just proved. $\square$

5. Classification of the $\mathrm{O}(m) \times \mathrm{O}(n)$-Invariant Minimal Hypersurfaces in $\mathbb{R}^{m+n}$

In this section we finally translate the behavior of the trajectories of the vector field X and that of the profile curves to derive the classification of our hypersurfaces. We will use the following facts concerning a $\mathrm{O}(m) \times \mathrm{O}(n)$-invariant hypersurface M in $\mathbb{R}^{m+n}$: M is embedded if and only if the associated profile curve $\gamma(t)$ is embedded in the orbit space. Moreover, if the orbit of X associated to the profile curve $\gamma(t)$ is defined for all $t \in \mathbb{R}$, then the corresponding hypersurface is complete. (See e.g. [8]).

We will use as main tool our Proposition 3.9. It is clear that the first two cases in that Proposition give rise to 'degenerate' manifolds of dimensions m or n.

Before stating and prove our classification theorems, it is now clear that the cases $m + n \leqslant 7$ and $m + n \geqslant 8$ must be treated separately. We now describe the first case, whose proof follows the same lines as in [6].

THEOREM 5.1 [Theorem 1.1 of Section 1]. *Given integers $m, n \geqslant 3$ such that $m + n \leqslant 7$, every nonextendable $\mathrm{O}(m) \times \mathrm{O}(n)$-invariant minimal hypersurface $M \subset \mathbb{R}^{m+n}$ falls in only one of the following types*:

(1) *M is a cone $\mathcal{C}_{m,n}$ with vertex at the origin, generated by a ray $y = \sqrt{\frac{n-1}{m-1}}\, x$.*
(2) *M is an immersed complete hypersurface which intersects itself and $\mathcal{C}_{m,n}$ infinitely countable times, approaching this cone asymptotically.*
(3) *M is an embedded complete hypersurface intersecting $\mathcal{C}_{m,n}$ infinitely countable times, approaching this cone asymptotically and intersecting orthogonally $\mathbb{R}^m \times \{0\}$ or $\{0\} \times \mathbb{R}^n$.*

Proof. Case 1 follows from the corresponding case in Proposition 4.1. The proof for Case 2 is quite similar to that of Lemma 3.6 (ii) in [6]; we refer the reader to that paper and omit the details.

As for the Case 3, the profile curve corresponds to a separatrix of the singularities of X contained in $(0, \pi/2) \times (-\pi, \pi)$. The claiming follows from Corollary 3.11 and Cases 7–10 of Proposition 4.1. □

Now we will work out the case $m + n \geqslant 8$.

THEOREM 5.2 [Theorem 1.2 of Section 1]. *Given integers $m, n \geqslant 3$ such that $m + n \geqslant 8$, every nonextendable* $\mathrm{O}(m) \times \mathrm{O}(n)$*-invariant minimal hypersurface $M \subset \mathbb{R}^{m+n}$ falls in only one of the following types*:

(1) *M is a cone $\mathcal{C}_{m,n}$ with vertex at the origin, generated by a ray $y = \sqrt{\frac{n-1}{m-1}}\, x$.*
(2) *M is an immersed complete hypersurface which does not intersect $\mathcal{C}_{m,n}$, being asymptotic to this cone.*
(3) *M is an embedded complete hypersurface which intersects $\mathcal{C}_{m,n}$ once, being asymptotic to this cone.*
(4) *M is an embedded complete hypersurface which does not intersect $\mathcal{C}_{m,n}$, being asymptotic to this cone and intersecting orthogonally* $\mathbb{R}^m \times \{0\}$ *or* $\{0\} \times \mathbb{R}^n$.

Proof. Case 1 follows again from the corresponding case in Proposition 4.1.

In Case 2, the profile curves are associated with Cases (1) and (3) given in Proposition 3.12. Proposition 4.4 implies that every such profile curve does not meet the line $y = \sqrt{\frac{n-1}{m-1}}x$, being doubly asymptotic to this line. This fact implies that the hypersurface M is asymptotic to the cone $\mathcal{C}_{m,n}$.

Case 3 corresponds to the class of orbits $\{\phi(t)\}$ satisfying (2) in Proposition 4.4. As noted at the beginning of this section, the fact of these curves being embedded implies that the associated hypersurface are embedded.

As for Case 4, we consider the profile curves of Cases 7–10 in Proposition 4.1. Again, these curves are embedded and so do the corresponding hypersurfaces.

In Cases 2–4, the profile curves are complete in the orbit space and thus the corresponding orbits of X are complete. Therefore the theorem follows. □

6. On the Stability of the O(m) × O(n)-Invariant Minimal Hypersurfaces

In this section we analyze the stability of the hypersurfaces classified in the previous section. We will follow closely [11] (see also [8]).

It is well known that a minimal immersion $\bar{x}\colon M^k \to \mathbb{R}^{k+1}$ is a critical point for the area functional $\int_D dM$ defined in every relatively compact domain $D \subset M^k$. More precisely, given a smooth function $f \in C_c^\infty(D)$, the space of functions $f\colon D \to \mathbb{R}$ with compact support, we use it to define a smooth normal variation $\bar{x}_t$ of $\bar{x} = \bar{x}_0$. If $A(t) = \int_{\bar{x}_t(D)} dM$, then the first variation $A'(0)$ vanishes. The well-known second variation formula reads

$$A''(0) = -\int_D \left(f\Delta f + \|B\|^2 f^2\right) dM, \tag{8}$$

where Δ denotes the Laplace operator and $\|B\|^2$ is the squared norm of the second fundamental form of $\bar{x}$. We say that the immersion $\bar{x}$ is *stable* if and only if $A''(0) \geqslant 0$ for every $f \in C_c^\infty(D)$.

Associated to the second variation formula we have the second order differential operator $T = \Delta + \|B\|^2 I$. T is elliptic and is called the Jacobi operator. The following proposition is proved in ([11], p. 201).

PROPOSITION 6.1. *A minimal immersion $\bar{x}$ is stable if and only if there exists a positive function $h \in C_c^\infty(D)$ defined in M satisfying $Th = 0$.*

It is also known that, given a minimal orientable immersion $\bar{x}$ with normal vector field N, the *support function* $h = \langle \bar{x}, N\rangle$ satisfies $Th = 0$ (see [12], for example). Finally, we recall that the index of T in D is the maximal dimension of a subspace of $C_c^\infty(D)$ where the quadratic form

$$I(f, f) = -\int_D f\,Tf\,dM$$

is negative definite. The index of T in M^k is

$$\mathrm{Ind}(T, M^k) = \sup_{D\subset M^k} \mathrm{Ind}(T, D),$$

the supremum taken over all relatively compact domains D in M^k.

We are ready to state and prove the results on stability of our hypersurfaces.

THEOREM 6.2. *[Theorem 1.3 of Section 1] Let $m, n \geqslant 3$ and $m + n \leqslant 7$. Any complete minimal* $\mathrm{O}(m) \times \mathrm{O}(n)$*-invariant hypersurface M in $\mathbb{R}^{m+n}$ has infinite index.*

Proof. A straightforward calculation shows that the support function h of M, expressed in the parametrization (1), is given by

$$h(t) = -u'(t)(x(t)^2 + y(t)^2)$$

which clearly depends only on the profile curve. So it suffices to analyze the set where u' vanishes.

Since for this case the singular points p_6 and p_7 are hyperbolic foci, every trajectory in $W^u(p_6)$ intersects infinitely countable times the line $u = v$. Thus, there exists an increasing, unbounded sequence of points t_k such that $u'(t_k) = 0$. This fact implies the existence of an increasing sequence of compact sets

$$D_1 \subset D_2 \subset \cdots D_k \subset \cdots \subset M$$

such that $h|_{\partial D_k} = 0$, where ∂D_k is the orbit of $\gamma(t_k)$ under the action of $\mathrm{O}(m)\times\mathrm{O}(n)$. The Morse Index Theorem (see [13]) implies that $\mathrm{Ind}(T, M)$ is infinite. $\square$

THEOREM 6.3 [Theorem 1.4 of Section 1]. *Let $m, n \geqslant 3$ and $m + n \geqslant 8$. The unique stable complete minimal* $\mathrm{O}(m) \times \mathrm{O}(n)$*-invariant hypersurfaces are those of the type (4) given in Theorem 1.2.*

Proof. In this case, the hyperbolic singularities are nodes. The separatrix curves $\phi(t)$ given by Proposition 3.8 never intersect the lines $u = v$ and $v = u - \pi$ and thus $u'(t) \neq 0$ for every t. This implies that the associated support functions never vanish along these curves. By Proposition 6.1, the corresponding hypersurfaces are stable. The uniqueness follows from Propositions 3.8 and 3.12. □

Since the hypersurfaces associated to profile curves of type (4) in Theorem 1.2 are homeomorphic to $\mathbb{R}^m \times \mathbb{S}^{n-1}$ or to $\mathbb{S}^{m-1} \times \mathbb{R}^n$, we obtain:

THEOREM 6.4 [Theorem 1.5 of Section 1]. *There exist embedded, complete, stable minimal hypersurfaces in $\mathbb{R}^{m+n}$, $m + n \geqslant 8$, $m \geqslant 3$, $n \geqslant 3$, not homeomorphic to $\mathbb{R}^{m+n-1}$ that are* $\mathrm{O}(m) \times \mathrm{O}(n)$*-invariant.*

References

1. Hsiang, W. Y. and Lawson, H. B.: Minimal submanifolds of low cohomogeneity, *J. Differential Geom.* **5** (1971), 1–38.
2. Delaunay, C.: Sur la surface de revolution dont la courbure moyenne est constante, *J. Math. Pure Appl.* **16** (1841), 309–321.
3. do Carmo, M. P. and Dajczer, M.: Rotational hypersurfaces in spaces of constant curvature, *Trans. Amer. Math. Soc.* **277**(2) (1983), 685–709.
4. Hsiang, W. Y., Teng, Z. H. and Yu, W. C.: New examples of constant mean curvature immersions of $(2k - 1)$-spheres into Euclidean $2k$-space, *Ann. Math.* **117** (1983), 609–625.
5. Bombieri, E., De Giorgi, E. and Giusti, E.: Minimal cones and the Bernstein problem, *Invent. Math.* **7** (1969), 243–269.
6. Alencar, H.: Minimal hypersurfaces in $\mathbb{R}^{2m}$ invariant by $\mathrm{SO}(m) \times \mathrm{SO}(m)$, *Trans. Amer. Math. Soc.* **337** (1993), 129–141.
7. Palmas, O.: $\mathrm{O}(2) \times \mathrm{O}(2)$-invariant hypersurfaces with zero scalar curvature, *Arch. Math.* **74** (2000), 226–233.
8. Sato, J.: Stability of $\mathrm{O}(p + 1) \times \mathrm{O}(p + 1)$-invariant hypersurfaces with zero scalar curvature in Euclidean space, *Ann. Global Anal. Geom.* **22** (2002), 135–153.
9. Cao, H. D., Shen, Y. and Zhu, S.: The structure of stable minimal hypersurfaces in $\mathbb{R}^{n+1}$, *Math. Res. Lett.* **4**(5) (1997), 637–644.
10. Perko, J.: *Differential equations and Dynamical Systems*, TAM 7, Springer-Verlag, Berlin, 1996.
11. Fischer-Colbrie, D. and Schoen, R.: The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature, *Comm. Pure Appl. Math.* **33** (1980), 199–211.
12. Rosenberg, H.: Hypersurfaces of constant curvature in space forms, *Bull. Sci. Math.* **117** (1993), 211–239.
13. Smale, S.: The Morse index theorem, *J. Math. Mech.* **14** (1965), 1049–1056.