UPPER BOUNDS FOR THE FIRST EIGENVALUE OF THE OPERATOR $\mathbf{L_r}$ AND SOME APPLICATIONS

HILÁRIO ALENCAR, MANFREDO DO CARMO, AND FERNANDO MARQUES

Abstract. We obtain upper bounds for the first eigenvalue of the linearized operator L_r of the r-mean curvature of a compact manifold immersed in a space of constant curvature δ. By the same method, we obtain an upper bound for the first eigenvalue of the stability operator associated to L_r when $\delta < 0$.

1. Introduction

Let M^n be a compact, connected, orientable Riemannian manifold, isometrically immersed in a simply connected space form $\overline{M}^{n+1}(\delta)$, with constant sectional curvature δ. We obtain upper bounds for the first eigenvalue of some elliptic operators defined on M (see below). In 1988, Heintze [H] proved that

$$\lambda_1^\Delta \leq n\delta + n \max H_1^2$$

for manifolds immersed in a hyperbolic space ($\delta < 0$) and that

$$\lambda_1^\Delta \leq n\delta + \frac{n}{\operatorname{vol} M} \int_M H_1^2$$

for manifolds immersed in a sphere ($\delta > 0$), contained in a convex ball of radius $r \leq \frac{\pi}{4\sqrt{\delta}}$ (if $\delta > 0$). Here λ_1^Δ denotes the first eigenvalue of the Laplacian on M and H_1 denotes the mean curvature. (In fact, Heintze considered as ambient spaces Riemannian manifolds with curvature bounded above by δ.) The latter inequality was obtained by Reilly [R] in 1977 for manifolds immersed in Euclidean space ($\delta = 0$), and generalized to arbitrary δ by El Soufi and Ilias [ESI] in 1992. In all of these estimates, equality holds precisely when M is a geodesic sphere of $\overline{M}$. Both El Soufi and Ilias [ESI] and Heintze [H] applied these bounds to obtain the stability theorems of Barbosa and do Carmo [B-dC] for immersions in $\mathbb{R}^{n+1}$, and of Barbosa, do Carmo,

2000 *Mathematics Subject Classification.* 53C42, 53A10.
The authors were partially supported by CNPq. The second author was partially supported by FAPERJ.

and Eschenburg [B-dC-E] for immersions in S^{n+1} and H^{n+1}. (The restriction $r \leq \frac{\pi}{4\sqrt{\delta}}$ in the case of S^{n+1} in [H] is stronger than that in [B-dC-E].)

We first introduce some notation. Consider the elementary symmetric functions S_r $(r = 1, \ldots, n)$ of the principal curvatures and the r-mean curvatures

$$H_r = \frac{S_r}{\binom{n}{r}}.$$

Let A be the second fundamental form associated to a globally defined normal unit vector field N. We define an operator L_r by

$$L_r f = \operatorname{div}(P_r \nabla_M f),$$

where $\nabla_M f$ stands for the gradient of f in M and P_r denotes the classical Newton transformation defined inductively by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - A P_{r-1}. \end{aligned}$$

Each P_r is a self-adjoint operator whose trace is $c(r)H_r$, where $c(r) = (n - r)\binom{n}{r}$ (see [B-C, Lemma 2.1]). Note that $L_0 = \Delta$.

In general, the operator L_r is not elliptic and some conditions are necessary to ensure the presence of ellipticity. However, in the theorems below, the hypotheses will guarantee that L_r is elliptic; see, for instance, the remarks made at the beginning of the proofs of Theorems 1.1 and 1.2. Thus we can consider the first eigenvalue $\lambda_1^{L_r}$ of L_r. This is the object we study here.

Assume that S_{r+1} is constant. Following [B-C] we say that the immersion is r-stable if $I_r(f) \geq 0$ for any $f\colon M \to \mathbb{R}$ satisfying $\int_M f\, dM = 0$, where

$$I_r(f) = -\int_M f \left\{ L_r(f) + \left[\frac{n}{r+1} c(r) H_1 H_{r+1} - c(r+1) H_{r+2} + \delta c(r) H_r \right] f \right\}.$$

In 1993, Alencar, do Carmo, and Rosenberg [A-dC-R, Theorem 1.1] proved that if H_{r+1} is positive (but not necessarily constant) on M, then

$$\lambda_1^{L_r} \int_M H_r \leq c(r) \int_M H_{r+1}^2,$$

for manifolds immersed in Euclidean space, and equality holds if and only if M is a sphere. They applied this result to obtain the theorem of Barbosa and do Carmo [B-dC] and a theorem of Alencar, do Carmo, and Colares [A-dC-C]. They also proved that an immersion of a hypersurface in $\mathbb{R}^{n+1}$ wth H_{r+1} constant is r-stable if and only if M is a sphere. In 1995, Grosjean [G1] obtained sharp integral bounds for $\lambda_1^{L_r}$ of immersions in any space form $\overline{M}(\delta)$, under the additional hypothesis of convexity of the immersion.

In this paper, inspired by Heintze's work [H], we obtain sharp upper bounds for $\lambda_1^{L_r}$ without the convexity hypothesis, in both the hyperbolic and spherical spaces. We prove the following theorems:

Theorem 1.1. *Let M^n be a compact manifold isometrically immersed in $\overline{M}^{n+1}(\delta)$, with $\delta < 0$. If $H_{r+1} > 0$ on M, then*

$$\lambda_1^{L_r} \le \delta c(r) \min H_r + c(r) \frac{\max H_{r+1}^2}{\min H_r},$$

and equality holds if and only if M is a geodesic sphere.

Theorem 1.2. *Let M^n be a compact manifold isometrically immersed in $\overline{M}^{n+1}(\delta)$, with $\delta > 0$, and suppose M is contained in a convex ball of radius $\frac{\pi}{4\sqrt{\delta}}$. If $H_{r+1} > 0$ on M, then*

$$\lambda_1^{L_r} \le \left(\delta c(r) + c(r) \frac{\max H_{r+1}^2}{\min H_r^2} \right) \frac{\int H_r}{\operatorname{vol} M},$$

and equality holds if and only if M is a geodesic sphere.

Furthermore, using the same techniques, we obtain an upper bound for the first eigenvalue of the operator $L_r - q$, where

$$q = c(r+1) H_{r+2} - \frac{nc(r)}{r+1} H_1 H_{r+1} - \delta c(r) H_r.$$

From this the r-stability theorem for manifolds immersed in a hyperbolic space, proved in 1997 by Barbosa and Colares [B-C], will follow. (In fact, in [B-C] the r-stability theorem was proved for any manifold $\overline{M}^{n+1}(\delta)$.)

To state this theorem, we need some notation. We denote by s_δ the solution of the differential equation $y'' + \delta y = 0$, with the initial conditions $y(0) = 0$, $y'(0) = 1$. Set $c_\delta = s'_\delta$; then $c'_\delta = -\delta s_\delta$ and $c_\delta^2 + \delta s_\delta^2 = 1$. We have $s_\delta(t) = \frac{1}{\sqrt{-\delta}} \sinh(\sqrt{-\delta} t)$ and $c_\delta(t) = \cosh(\sqrt{-\delta} t)$ in the case $\delta < 0$, and $s_\delta(t) = \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta} t)$, $c_\delta(t) = \cos(\sqrt{\delta} t)$ in the case $\delta > 0$. Note that if $\delta = 0$, $s_\delta(t) = t$.

We will prove the following result:

Theorem 1.3. *If $\delta < 0$ and $H_{r+1} > 0$, there exists a point $p_0 \in \overline{M}$ such that, if $d = d(p_0, \cdot)$ denotes the distance function from p_0 in $\overline{M}$, then*

$$\lambda_1(L_r - q) \le c(r) \frac{\displaystyle\int_M \left(\frac{\max H_{r+1}^2}{\min H_r} - H_1 H_{r+1} \right) s_\delta^2(d)}{\displaystyle\int_M s_\delta^2(d)},$$

and equality holds precisely when M is a geodesic sphere.

As a consequence, we obtain:

Corollary 1.4 ([B-C]). *The only r-stable compact immersed hypersurfaces in a hyperbolic space, with constant $H_{r+1} > 0$, are the geodesic spheres.*

REMARK 1.5. After writing this paper, we received a preprint of J.F. Grosjean (see [G2]) who obtained, independently, the results presented here.

We want to thank the referee for useful suggestions.

2. Preliminaries

Throughout this paper, we use the notations defined in the Introduction. Let M^n and $\overline{M}^{n+1}(\delta)$ be as before, let $p_0 \in \overline{M}$, and let $d = d(p_0, \cdot)$ be the distance function from p_0 in $\overline{M}$. Let x_i $(i = 1, \dots, n+1)$ be the normal coordinates centered in p_0, with respect to some orthonormal basis in $T_{p_0}\overline{M}$. We denote by ∇ and ∇_M the gradients taken in $\overline{M}$ and M, respectively.

LEMMA 2.1. *Suppose $x \in \overline{M}$. Assume $x \in B\left(p_0, \frac{\pi}{2\sqrt{\delta}}\right)$ in the case $\delta > 0$. If $u, v \in T_x\overline{M}$ and v is orthogonal to ∇d, then*

$$\frac{s_\delta^2(d)}{d^2} \sum_{i=1}^{n+1} (\langle \nabla x_i, u\rangle \langle \nabla x_i, v\rangle) = \langle u, v\rangle.$$

Proof. The map

$$L_{\tilde{x}} = \left(d \exp_{p_0}\right)_{\tilde{x}} \colon T_{p_0}\overline{M} \to T_x\overline{M},$$

where $\exp_{p_0}(\tilde{x}) = x$, is a linear isomorphism. Using $\langle \nabla x_i, u\rangle = (L_{\tilde{x}}^{-1}u)(x_i)$, we obtain

$$\frac{s_\delta^2(d)}{d^2} \sum_{i=1}^{n+1} (\langle \nabla x_i, u\rangle \langle \nabla x_i, v\rangle) = \frac{s_\delta^2(d)}{d^2} \langle L_{\tilde{x}}^{-1}(u), L_{\tilde{x}}^{-1}(v)\rangle.$$

Since $L_{\tilde{x}}$ is a radial isometry, $L_{\tilde{x}}^{-1}(v)$ is tangent to the sphere of radius $|\tilde{x}|$ in $T_{p_0}\overline{M}$. Further, $L_{\tilde{x}}^{-1}(u) = \tilde{w} + \tilde{r}$, where $\tilde{w}$ is tangent and $\tilde{r}$ is orthogonal to this sphere in $T_{p_0}\overline{M}$. Hence, $\langle L_{\tilde{x}}^{-1}(u), L_{\tilde{x}}^{-1}(v)\rangle = \langle \tilde{w}, \tilde{v}\rangle$, where $\tilde{v} = L_{\tilde{x}}^{-1}(v)$.

Let $\gamma \colon [0, d] \to \overline{M}$ be the normalized geodesic with

$$\gamma(0) = p, \quad \gamma(d) = x, \quad \gamma'(0) = \frac{\tilde{x}}{|\tilde{x}|},$$

where $|\tilde{x}| = d$. Let $J_v(t)$, $J_w(t)$ be Jacobi fields along γ such that

$$J_v(0) = J_w(0) = 0 \quad J_v'(0) = \frac{\tilde{v}}{|\tilde{v}|}, \quad J_w'(0) = \frac{\tilde{w}}{|\tilde{w}|}.$$

Since $\overline{M}$ has constant sectional curvature,

$$\langle J_v(d), J_w(d)\rangle = \frac{s_\delta^2(d)}{|\tilde{v}||\tilde{w}|} \langle \tilde{v}, \tilde{w}\rangle. \tag{1}$$

Recall also that

$$J_v(t) = \left(d \exp_{p_0}\right)_{t\frac{\tilde{x}}{d}} \left(t \frac{\tilde{v}}{|\tilde{v}|}\right), \quad J_w(t) = \left(d \exp_{p_0}\right)_{t\frac{\tilde{x}}{d}} \left(t \frac{\tilde{w}}{|\tilde{w}|}\right).$$

Hence, $\langle J_v(d), J_w(d)\rangle = \frac{d^2}{|\tilde{v}||\tilde{w}|}\langle v, w\rangle$, where $w = L_{\tilde{x}}(\tilde{w})$, and using (1), we obtain $\langle v, w\rangle = \frac{s_\delta^2}{d^2}\langle \tilde{v}, \tilde{w}\rangle$. Thus

$$\frac{s_\delta^2}{d^2}\sum_{i=1}^{n+1}\langle \nabla x_i, u\rangle\langle \nabla x_i, v\rangle = \frac{s_\delta^2}{d^2}\langle \tilde{v}, \tilde{w}\rangle = \langle v, w\rangle = \langle v, u\rangle,$$

which concludes the proof. □

Define the position vector X of M^n in $\overline{M}^{n+1}(\delta)$ with respect to p_0 by $X = s_\delta(d)\nabla d$. Denote by X^T the component of X tangent to M; i.e., $X^T = s_\delta(d)\nabla_M d$. Observe that

$$\nabla_M c_\delta = -\delta X^T.$$

Lemma 2.2. *With the above notation,*

$$\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right), \nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle + \delta\langle P_r X^T, X^T\rangle = c(r)H_r.$$

Proof. Using the fact that P_r is self-adjoint, we obtain

$$\begin{aligned}\left\langle P_r\nabla_M\frac{s_\delta}{d}x_i, \nabla_M\frac{s_\delta}{d}x_i\right\rangle = {}& \frac{x_i^2}{d^2}\left(c_\delta - \frac{s_\delta}{d}\right)^2\langle P_r\nabla_M d, \nabla_M d\rangle\\ &+ 2\frac{x_i s_\delta}{d^2}\left(c_\delta - \frac{s_\delta}{d}\right)\langle P_r\nabla_M d, \nabla_M x_i\rangle\\ &+ \frac{s_\delta^2}{d^2}\langle P_r\nabla_M x_i, \nabla_M x_i\rangle.\end{aligned}$$

Since $\sum_i x_i\nabla x_i = d\nabla d$ and

$$\langle P_r\nabla_M d, \nabla_M x_i\rangle = \langle P_r\nabla_M d, \nabla x_i\rangle,$$

because $P_r\nabla_M d$ is tangent to M, we have

$$\begin{aligned}&\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\frac{s_\delta}{d}x_i, \nabla_M\frac{s_\delta}{d}x_i\right\rangle\\ &\qquad = \left(c_\delta^2 - \frac{s_\delta^2}{d^2}\right)\cdot\langle P_r\nabla_M d, \nabla_M d\rangle + \frac{s_\delta^2}{d^2}\sum_{i=1}^{n+1}\langle P_r\nabla_M x_i, \nabla_M x_i\rangle.\end{aligned}$$

Further

$$\delta\langle P_r X^T, X^T\rangle = \delta s_\delta^2\langle P_r\nabla_M d, \nabla_M d\rangle.$$

Thus

$$\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\frac{s_\delta}{d}x_i,\nabla_M\frac{s_\delta}{d}x_i\right\rangle+\delta\langle P_rX^T,X^T\rangle$$
$$=\frac{s_\delta^2}{d^2}\sum_{i=1}^{n+1}\langle P_r\nabla_Mx_i,\nabla_Mx_i\rangle+\left(1-\frac{s_\delta^2}{d^2}\right)\langle P_r\nabla_Md,\nabla_Md\rangle.$$

Now let $e_1,\ldots,e_n\in T_pM$ be an orthonormal basis such that e_n lies in the direction of ∇_Md (if $\nabla_Md\neq 0$). Then there are numbers λ and μ satisfying $e_n=\lambda\nabla d+\mu e_n^*$, where e_n^* is a unit vector orthogonal to ∇d. From this we easily obtain $\nabla_Md=\lambda e_n$ and $(e_n^*)^T=\mu e_n$.

A simple calculation gives

$$\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right),\nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle+\delta\langle P_rX^T,X^T\rangle$$
$$=\frac{s_\delta^2}{d^2}\sum_{j=1}^{n-1}\sum_{i=1}^{n+1}\left(\langle\nabla x_i,P_re_j\rangle\langle\nabla x_i,e_j\rangle\right)+\frac{s_\delta^2}{d^2}\sum_{i=1}^{n+1}\langle P_r\nabla_Mx_i,e_n\rangle\langle\nabla_Mx_i,e_n\rangle$$
$$+\left(1-\frac{s_\delta^2}{d^2}\right)\lambda^2\langle P_re_n,e_n\rangle.$$

Since e_j is orthogonal to ∇d for all $j=1,\ldots,n-1$, we can apply Lemma 2.1 and obtain

$$\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right),\nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle+\delta\langle P_rX^T,X^T\rangle \tag{2}$$
$$=\sum_{j=1}^{n-1}\langle P_re_j,e_j\rangle+\frac{s_\delta^2}{d^2}\sum_{i=1}^{n+1}\langle P_r\nabla_Mx_i,e_n\rangle\langle\nabla_Mx_i,e_n\rangle$$
$$+\left(1-\frac{s_\delta^2}{d^2}\right)\lambda^2\langle P_re_n,e_n\rangle.$$

Observing that, for any $x\in\overline{M}$ and any $u\in T_x\overline{M}$,

$$\sum_{i}^{n+1}\langle\nabla x_i,u\rangle\langle\nabla x_i,\nabla d\rangle=\langle u,\nabla d\rangle,$$

we obtain, after some manipulation,

$$\sum_{i=1}^{n+1}\langle P_r\nabla_Mx_i,e_n\rangle\langle\nabla_Mx_i,e_n\rangle=\lambda\langle P_re_n,\nabla d\rangle+\sum_{i=1}^{n+1}\langle\nabla x_i,P_re_n\rangle\langle\nabla x_i,\mu e_n^*\rangle.$$

Applying Lemma 2.1 again, because e_n^* is orthogonal to ∇d, the right-hand side of the last equation is equal to

$$\lambda\langle P_r e_n, \nabla d\rangle + \frac{d^2}{s_\delta^2(d)}\mu\langle P_r e_n, e_n^*\rangle.$$

Substituting this in (2), we obtain

$$\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right), \nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle + \delta\langle P_r X^T, X^T\rangle$$
$$= \sum_{j=1}^{n}\langle P_r e_j, e_j\rangle = \operatorname{trace}(P_r) = c(r)H_r. \qquad \square$$

Lemma 2.3. *If $H_{r+1} > 0$ and $c_\delta \geq 0$, then*

$$\frac{\int_M c_\delta}{\int_M s_\delta} \leq \frac{\max H_{r+1}}{\min H_r}.$$

Proof. Recall that if $H_{r+1} > 0$, then $H_j > 0$, where $1 \leq j \leq r$ (see [B-C, Proposition 3.2]). Recall also Minkowski's formula (see [A-C])

$$\int_M [H_r c_\delta + H_{r+1}\langle X, N\rangle] = 0.$$

Then, by the Cauchy-Schwarz inequality,

$$\int_M H_r c_\delta = -\int_M H_{r+1}\langle X, N\rangle \leq \int_M H_{r+1}|X|$$
$$= \int_M H_{r+1}s_\delta \leq (\max H_{r+1})\int_M s_\delta.$$

Since $c_\delta \geq 0$, we also have $(\min H_r)\int_M c_\delta \leq \int_M H_r c_\delta$. From these inequalities we obtain

$$(\min H_r)\int_M c_\delta \leq (\max H_{r+1})\int_M s_\delta.$$

This concludes the proof. $\square$

3. Proofs of Theorems 1.1, 1.2, 1.3 and Corollary 1.4

We are now in a position to prove the upper bounds for the first eigenvalue of L_r. Our proofs use the Rayleigh quotient, applied with suitable test functions.

In all proofs, $p_0 \in \overline{M}$ will be a point such that

$$\int_M \frac{s_\delta(d)}{d}x_i = 0 \quad (i = 1, \ldots, n+1),$$

where $d = d(p_0, \cdot)$. The existence of such a point, assuming that M lies in a convex ball of $\overline{M}$, can be verified by a standard argument. Namely, if M lies in a convex ball B, then

$$Y_q = \int_M \frac{s_\delta(d(q,p))}{d(q,p)} \exp_q^{-1}(p)dp \in T_q\overline{M}$$

defines a vector field in a neighborhood of B which, at the boundary, points towards the interior of B. Thus, Y has a zero in B, and if we take p_0 as this zero, then p_0 has the required property. Note that if B has radius less than $\frac{\pi}{4\sqrt{\delta}}$, then M lies in a ball of radius $< \frac{\pi}{2\sqrt{\delta}}$ around p_0. As a consequence, $c_\delta \geq 0$.

Proof of Theorem 1.1. Since $H_{r+1} > 0$ and M is compact, L_r is elliptic (see [B-C, Proposition 3.2]).

Using the Rayleigh quotient with the test functions $\frac{s_\delta(d)}{d}x_i$, we obtain

$$\begin{aligned}(3)\qquad \lambda_1^{L_r}\int_M s_\delta^2 &= \lambda_1^{L_r}\int_M \sum_{i=1}^{n+1}\left(\frac{s_\delta}{d}x_i\right)^2 \\ &\leq \int_M \sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right), \nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle \\ &= c(r)\int_M H_r - \delta\int_M \langle P_rX^T, X^T\rangle,\end{aligned}$$

where the last equality follows from Lemma 2.2.

From Stokes' theorem it follows that

$$\int_M fL_rg + \langle P_r\nabla_M f, \nabla_M g\rangle = 0.$$

Applying this with $f = g = c_\delta$ and using the relation $\nabla_M c_\delta = -\delta X^T$, we obtain

$$\delta\int_M \langle P_rX^T, X^T\rangle = -\frac{1}{\delta}\int_M c_\delta L_r(c_\delta).$$

Hence,

$$\int_M \sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right), \nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle = c(r)\int_M H_r + \frac{1}{\delta}\int_M c_\delta L_r(c_\delta).$$

It is known that (see [A-C, Lemma 1])

$$L_r(c_\delta) = -\delta\left[c(r)H_rc_\delta + c(r)\langle X, N\rangle H_{r+1}\right].$$

From this and the inequality

$$-\langle X, N\rangle \leq |X| = s_\delta,$$

we obtain

$$\begin{aligned}\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right),\nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle &\le \delta c(r)\int_M s_\delta^2 H_r - c(r)\int_M c_\delta\langle X,N\rangle H_{r+1}\\ &\le \delta c(r)\int_M s_\delta^2 H_r + c(r)\int_M c_\delta s_\delta H_{r+1}\\ &\le \delta c(r)\int_M s_\delta^2 H_r + c(r)\max H_{r+1}\int_M c_\delta s_\delta.\end{aligned}$$

If $\delta \le 0$, it is also known that (see Lemma 2.8 in [H])

$$\int_M s_\delta \int_M s_\delta c_\delta \le \left(\int_M s_\delta^2\right)\int_M c_\delta.$$

Using this inequality and Lemma 2.2, we have

$$\begin{aligned}(4)\qquad &\sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right),\nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle\\ &\qquad\le \delta c(r)\int_M s_\delta^2 H_r + c(r)\frac{(\max H_{r+1})^2}{\min H_r}\int_M s_\delta^2\\ &\qquad\le \delta c(r)(\min H_r)\int_M s_\delta^2 + c(r)\frac{(\max H_{r+1}^2)}{\min H_r}\int_M s_\delta^2.\end{aligned}$$

By applying (3), we obtain

$$\lambda_1^{L_r}\int_M s_\delta^2 \le \delta c(r)(\min H_r)\int_M s_\delta^2 + c(r)\frac{\max H_{r+1}^2}{\min H_r}\int_M s_\delta^2.$$

Dividing both sides by $\int_M s_\delta^2$ gives the desired estimate.

If equality holds, then we necessarily have

$$-\langle X,N\rangle = |X||N|,$$

and this implies that ∇d is orthogonal to M. Thus, d is constant on M, and therefore M is a geodesic sphere around p_0. □

Proof of Theorem 1.2. Since $H_{r+1} > 0$ and M is contained in a convex ball, L_r is again an elliptic operator (see [B-C, Proposition 3.2]). Put

$$c = \frac{1}{\operatorname{vol} M}\int_M c_\delta, \quad \text{so} \quad \int_M \frac{(c_\delta - c)}{\sqrt{\delta}} = 0.$$

Recall that $s_\delta(d) = \frac{1}{\sqrt{\delta}}\sin(\sqrt{\delta}d)$ and $c_\delta(d) = \cos(\sqrt{\delta}d)$, so $|c| < 1$.

Using the Rayleigh quotient with $\frac{s_\delta(d)}{d}x_i$ and $\frac{c_\delta - c}{\sqrt{\delta}}$ as test functions, we obtain

$$\begin{aligned}
&\lambda_1^{L_r} \int_M \left[s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right] \\
&\quad \leq \int_M \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \frac{s_\delta}{d} x_i, \nabla_M \frac{s_\delta}{d} x_i \right\rangle + \int_M \left\langle P_r \nabla_M \left(\frac{c_\delta - c}{\sqrt{\delta}} \right), \nabla_M \left(\frac{c_\delta - c}{\sqrt{\delta}} \right) \right\rangle \\
&\quad = \int_M \left[\sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle + \delta \langle P_r X^T, X^T \rangle \right] = c(r) \int H_r,
\end{aligned}$$

where the last equality follows from Lemma 2.2.

Further, a direct calculation gives

$$\int_M \left[s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right] = \frac{1}{\delta} (\operatorname{vol} M)(1 - c^2).$$

Thus,

$$\lambda_1^{L_r} \leq \left(\frac{1}{1 - c^2} \right) \delta \frac{c(r)}{\operatorname{vol} M} \int_M H_r. \tag{5}$$

We next prove that

$$\frac{1}{1 - c^2} \leq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2}.$$

By Lemma 2.3 we have

$$c^2 = \frac{1}{(\operatorname{vol} M)^2} \left(\int_M c_\delta \right)^2 \leq \frac{1}{(\operatorname{vol} M)^2} \left(\frac{\max H_{r+1}}{\min H_r} \right)^2 \left(\int_M s_\delta \right)^2,$$

and the Cauchy-Schwarz inequality gives

$$\left(\int_M s_\delta \right)^2 \leq \left(\int_M s_\delta^2 \right) \operatorname{vol} M.$$

Therefore

$$\begin{aligned}
&(1 - c^2) \left(1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} \right) \\
&\quad \geq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} - \frac{1}{\operatorname{vol} M} \frac{\max H_{r+1}^2}{\min H_r^2} \int_M s_\delta^2 - c^2 \cdot \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} \\
&\quad \geq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} - \frac{1}{\operatorname{vol} M} \frac{\max H_{r+1}^2}{\min H_r^2} \int_M s_\delta^2 - \left(\frac{1}{\operatorname{vol} M} \int_M c_\delta^2 \right) \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} \\
&\quad = 1 + \frac{\max H_{r+1}^2}{\min H_r^2} \left(\frac{1}{\delta} - \frac{1}{\delta \operatorname{vol} M} \int_M \left(\delta s_\delta^2 + c_\delta^2 \right) \right) = 1,
\end{aligned}$$

where the last inequality follows from Cauchy-Schwarz inequality. Hence,

$$\frac{1}{1-c^2} \leq 1 + \frac{1}{\delta}\frac{\max H_{r+1}^2}{\min H_r^2}.$$

From (5), we have

$$\lambda_1^{L_r} \leq \left(\delta + \frac{\max H_{r+1}^2}{\min H_r^2}\right) c(r)\frac{\int H_r}{\operatorname{vol} M}.$$

If equality holds, we also have equality in Lemma 2.3, so $-\langle X, N\rangle = |X||N|$ and therefore ∇d is orthogonal do M. Hence, d is constant on M, and therefore M is a geodesic sphere around p_0. □

Proof of Theorem 1.3. Using the Rayleigh quotient for the operator $L_r - q$, with $\frac{s_\delta}{d}x_i$ as test functions, we obtain

$$\text{(6)}\quad \lambda_1(L_r - q)\int_M s_\delta^2 \leq \int_M \sum_{i=1}^{n+1}\left(\frac{s_\delta}{d}x_i\right)\left[-L_r\left(\frac{s_\delta}{d}x_i\right) + q\left(\frac{s_\delta}{d}x_i\right)\right]$$
$$= \int_M \sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right), \nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle + \int_M q s_\delta^2.$$

By (4) we have

$$\int_M \sum_{i=1}^{n+1}\left\langle P_r\nabla_M\left(\frac{s_\delta}{d}x_i\right), \nabla_M\left(\frac{s_\delta}{d}x_i\right)\right\rangle$$
$$\leq \delta c(r)\int_M s_\delta^2 H_r + c(r)\frac{\max H_{r+1}^2}{\min H_r}\int_M s_\delta^2.$$

Applying this to (6), we obtain

$$\lambda_1(L_r - q)\int_M s_\delta^2 \leq c(r)\left(\int_M s_\delta^2\right)\frac{\max H_{r+1}^2}{\min H_r}$$
$$+ c(r+1)\int_M H_{r+2}s_\delta^2 - \frac{nc(r)}{r+1}\int_M H_1 H_{r+1} s_\delta^2.$$

Since $H_{r+2} \leq H_1 H_{r+1}$, with equality at umbilical points (see [A-dC-R, p. 392]), we obtain

$$\lambda_1(L_r - q)\int_M s_\delta^2 \leq c(r)\left(\int_M s_\delta^2\right)\frac{\max H_{r+1}^2}{\min H_r} - c(r)\int_M H_1 H_{r+1} s_\delta^2,$$

because $c(r+1) - \frac{nc(r)}{r+1} = -c(r)$.

Dividing both terms by $\int_M s_\delta^2$, the desired inequality follows. The case when equality holds is handled in the same way as in the previous cases. □

Proof of Corollary 1.4. Let M^n be a compact hypersurface, immersed in $\overline{M}^{n+1}(\delta)$, where $\delta < 0$, with $H_{r+1} > 0$ and constant. Suppose M is r-stable, that is to say,

$$\int_M f(-L_r + q)(f) \geq 0 \quad \text{when} \quad \int_M f = 0.$$

Taking for f an eigenfunction of $L_r - q$ belonging to $\lambda_1^{L_r - q}$, we obtain

$$\lambda_1^{L_r - q} \int_M f^2 \geq 0,$$

and thus

$$\lambda_1^{L_r - q} \geq 0. \tag{7}$$

Also, if $H_{r+1} > 0$, then $H_j > 0$ for all $j = 1, \dots, r$ (see [B-C, Proposition 3.1]) and $H_r \geq H_{r+1}^{\frac{r}{r+1}}$ (see [M-R, Lemma 1]). Hence,

$$\frac{\max H_{r+1}^2}{\min H_r} \leq \frac{H_{r+1}^2}{H_{r+1}^{\frac{r}{r+1}}} = H_{r+1}^{\frac{r+2}{r+1}}. \tag{8}$$

Since $H_1 \geq H_{r+1}^{\frac{1}{r+1}}$, it follows that

$$H_{r+1}^{\frac{r+2}{r+1}} - H_1 H_{r+1} \leq 0, \tag{9}$$

with equality at umbilical points.

By (7), (8), (9) and Theorem 1.3, we conclude that $H_{r+1}^{\frac{r+2}{r+1}} - H_1 H_{r+1} = 0$ everywhere, and so M is a geodesic sphere. □

References

[A-C] H. Alencar and A. G. Colares, *Integral formulas for the r-mean curvature linearized operator of a hypersurface*, Ann. Global Anal. Geom. **16** (1998), 203–220.

[A-dC-C] H. Alencar, M. Do Carmo, and A. G. Colares, *Stable hypersurfaces with constant scalar curvature*, Math. Z. **213** (1993), 117–131.

[A-dC-R] H. Alencar, M. Do Carmo, and H. Rosenberg, *On the first eigenvalue of the linearized operator of the r-th mean curvature of a hypersurface*, Ann. Global Anal. Geom. **11** (1993), 387–391.

[B-C] L. Barbosa and A. G. Colares, *Stability of hypersurfaces with constant r-mean curvature*, Ann. Global Anal. Geom. **15** (1997), 277–297.

[B-dC] L. Barbosa and M. Do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **195** (1984), 339–353.

[B-dC-E] L. Barbosa, M. Do Carmo, and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), 123–138.

[ESI] A. El-Soufi and S. Ilias, *Une inegalité du type Reilly pour les sous-varietés de l'space hyperbolique*, Comment. Math. Helv. **67** (1992), 167–181.

[G1] J.-F. Grosjean, *Majoration de la première valeur propre de certains opérateurs elliptiques naturels sur les hypersurfaces des espaces formes et applications*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), 323–326.

[G2] ______, *Estimations extrinséques de la première valeur propre d'opérateurs elliptiques définis sur des sous-variétés*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 807–810.

[H] E. Heintze, *Extrinsic upper bounds for λ_1*, Math. Ann. **280** (1988), 389–402.

[M-R] S. Montiel and A. Ros, *Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures*, Differential Geometry (B. Lawson and K. Tenenblat, eds.), Pitman Monographs, vol. 52, Longman Scientific and Technical, Essex, 1991, pp. 279–296.

[R] R. Reilly, *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. **52** (1977), 525–533.

Hilário Alencar, Departamento de Matemática, Universidade Federal de Alagoas, 57072-900, Maceió - AL, Brazil

E-mail address: `hilario@mat.ufal.br`

Manfredo do Carmo, Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, 22460-320, Rio de Janeiro - RJ, Brazil

E-mail address: `manfredo@impa.br`

Fernando Marques, Department of Mathematics, Cornell University, Ithaca, NY 14853-4089, USA

E-mail address: `marques@polygon.math.cornell.edu`