Mini-course in Maceio on embedded constant mean curvature surfaces in $\mathbf{R}^3$

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Based on joint work with Giuseppe Tinaglia.

Some preliminary papers on the lecture material can be found on Tinaglia's web page at Kings College London.

Outline of 3 lectures

1. Lecture 1: Background material, statements of the main results.
2. Lecture 2: Proof of extrinsic curvature estimates for $\mathbf{H}$-disks.
3. Lecture 3: Applications:
    1. Intrinsic curvature and radius estimates for $\mathbf{H}$-disks.
    2. Chord-arc results and $\mathbf{1}$-sided curvature estimates for $\mathbf{H}$-disks.
    3. Curvature estimates for $\mathbf{H}$-annuli.
    4. Classification of $\mathbf{0}$ and $\mathbf{1}$-connected $\mathbf{H}$-surfaces, $\mathbf{H} > 0$.

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- Recently **Meeks** and **Tinaglia** proved that if $\mathbf{\Sigma} \subset \mathbf{R}^3$ is a complete, embedded $\mathbf{H}$-surface with finite topology, then $\mathbf{\Sigma}$ is properly embedded. **(***Proved for* $\mathbf{H} = 0$ *by* **Colding-Minicozzi***, 2008***)**

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- Given a Riemannian surface $\mathbf{M}$, the injectivity radius function $\mathbf{I_M} \colon \mathbf{M} \to (0, \infty]$ is defined by: $\mathbf{I_M}(\mathbf{p}) = \sup\{R > 0 \mid \exp_{\mathbf{p}} \colon B(R) \subset \mathbf{T_p M} \to \mathbf{M}$ is a diffeomorphism.$\}$
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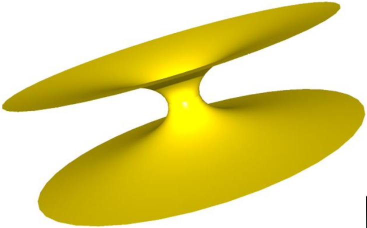
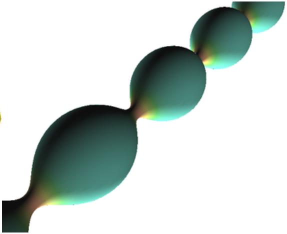
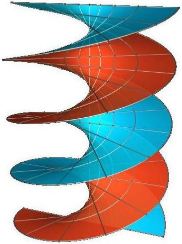
Theorem (Meeks-Tinaglia, based on previous work of Colding-Minicozzi & Meeks-Rosenberg)

- Complete embedded $\mathbf{H}$-surfaces $\mathbf{M} \subset \mathbf{R^3}$ with finite topology have positive injectivity radius.
- Let $\mathbf{M} \subset \mathbf{R^3}$ be a complete, connected embedded $\mathbf{H}$-surface with $\mathbf{H} > 0$ and positive injectivity radius. Then $\mathbf{M}$ has bounded second fundamental form and it is properly embedded in $\mathbf{R^3}$.

- This theorem by **Meeks-Tinaglia** and work of **Meeks-Rosenberg**, **Colding-Minicozzi**, **Collin**, **Lopez-Ros** when $\mathbf{H} = 0$, and **Meeks** and **Korevaar-Kusner-Solomon** when $\mathbf{H} \neq 0$, completes the **classification** of complete, embedded **H**-surfaces of genus **0** with **0**, **1** or **2** ends.
- They are **planes, spheres, catenoids, unduloids, helicoids**.







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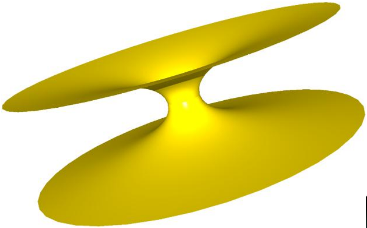
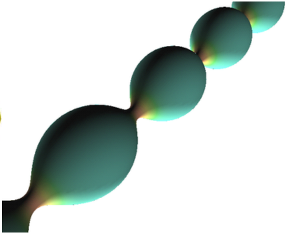
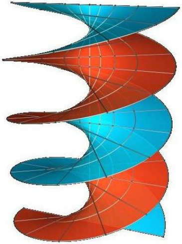

Remark

One **Main Objective** of this course is to present the theory behind this classification for the special case where $\mathbf{H} > 0$.

Theorem (Intrinsic Curvature Estimates for H-Disks, Meeks-Tinaglia)

Fix $\varepsilon > 0$ and $\mathbf{H} = 1$. $\exists$ $\mathbf{C} \geq \pi$ such that for every embedded $\mathbf{1}$-disk $\mathbf{D} \subset \mathbf{R}^3$ and every $p \in \mathbf{D}$ with $\mathbf{dist}_{\mathbf{D}}(p, \partial\mathbf{D}) \geq \varepsilon$,

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Brief idea/ingredients of the proof.

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  Curvature estimate fails for $\mathbf{D}$ = disk with $\partial \mathbf{D} \subset \partial \mathbb{B}(\delta)$ and $\vec{0} \in \mathbf{D}$ is a point of large second fundamental form.

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- **Rescaling arguments** imply **helicoid-type surfaces occur** near $\vec{0}$.
- **Pair of highly-sheeted multigraphs around** $\vec{0}$ **extends** to pair of highly-sheeted multigraphs for a fixed distance proportional to δ, impossible for $\mathbf{H} = 1$. □

Theorem (One-sided curvature estimate for $\mathbf{H}$-disks, Meeks-Tinaglia)

There exist $\varepsilon \in (0, \frac{1}{2})$ and $C \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds. Let $\mathbf{\Sigma}$ be an $\mathbf{H}$-disk such that

$$\mathbf{\Sigma} \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \emptyset \quad \text{and} \quad \partial\mathbf{\Sigma} \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \emptyset.$$

Then:

$$\sup_{x \in \mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}} |\mathbf{A}_{\mathbf{\Sigma}}|(x) \leq \frac{C}{R}. \tag{1}$$

In particular, if $\mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\} \neq \emptyset$, then $\mathbf{H} \leq \frac{C}{R}$.

Theorem (Chord-Arc Theorem, Meeks-Tinaglia)

There exists a positive constant C such that if $\mathbf{\Sigma} \subset \mathbf{R}^3$ is an $\mathbf{H}$-disk, $B_{\mathbf{\Sigma}}(\vec{0}, CR) \subset \mathbf{\Sigma} - \partial\mathbf{\Sigma}$ and $\sup_{\mathbb{B}_{\mathbf{\Sigma}}(\vec{0}, r_0)} |\mathbf{A}_{\mathbf{\Sigma}}| \geq \frac{1}{r_0}$ where $R > r_0$, then for $x \in B_{\mathbf{\Sigma}}(\vec{0}, R)$,

$$\frac{1}{6}\,\mathrm{dist}_{\mathbf{\Sigma}}(x, \vec{0}) < |x| + r_0. \tag{2}$$

Key Preliminary Step.

Theorem (Meeks-Tinaglia)

$\exists\ \varepsilon > 0$ s.t. for $\mathbf{M}$ an $\mathbf{1}$-disk with $\partial\mathbf{M} \subset (\mathbf{R}^3 - \mathbb{B}(\delta))$ with $\delta < \varepsilon$, then every component of $\mathbf{M} \cap \mathbb{B}(\delta)$ has at most $\mathbf{5}$ boundary components.

Brief Sketch of the Proof.

- Arguing by contradiction, $\exists$ a sequence of $\mathbf{1}$-disks $\mathbf{\Sigma}_n$ with $\partial\mathbf{\Sigma}_n \subset (\mathbf{R}^3 - \mathbb{B}(\frac{1}{n}))$, s.t. there is a component $\mathbf{\Delta}$ of $\mathbf{\Sigma}_n \cap \mathbb{B}(\frac{1}{n})$ and $\partial\mathbf{\Delta}$ has at at least $\mathbf{6}$ boundary curves.
- Use the Alexandrov reflection principle as described on the blackboard to obtain a contradiction when $n \to \infty$.

□

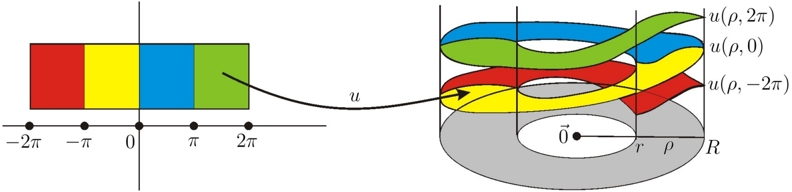

Figure: A 2-valued graph with positive separation.

Definition

- In polar coordinates (ρ, θ) on $\mathbb{R}^2 - \{0\}$ with $\rho > 0$ and $\theta \in \mathbb{R}$, a ***k*-valued graph on an annulus of inner radius *r* and outer radius *R*,** is a single-valued graph of a function $u(\rho, \theta)$ defined over

$$S_{r,R}^{-k,k} = \{(\rho, \theta) \mid r \leq \rho \leq R, \ |\theta| \leq k\pi\}, \tag{3}$$

  k being a positive integer.

- The **separation** between consecutive sheets is $w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta) \in \mathbb{R}$.

- The surface

$$\boldsymbol{\Sigma}_g = \{(\rho \cos\theta, \rho \sin\theta, u(\rho, \theta)) \mid (\rho, \theta) \in S_{r,R}^{-k,k}\}$$

  is embedded if and only if $w > 0$ (or $w < 0$).

Definition

- Let γ be a piecewise-smooth 1-cycle in an **H**-surface **M**.
- The **flux** of γ is $\int_\gamma (\mathbf{H}\gamma + \xi) \times \dot{\gamma}$, where ξ is the unit normal to **M** along γ.
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Theorem (Meeks-Tinaglia)

$\exists\ \varepsilon > 0$ s.t. for **M** an **1**-disk with $\partial\mathbf{M} \subset (\mathbf{R}^3 - \mathbb{B}(\delta))$ with $\delta < \varepsilon$, then every component of $\mathbf{M} \cap \mathbb{B}(\delta)$ has at most **5** boundary components.

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Theorem (Curvature Estimates for Planar Domains with Zero Flux)

- Given $\varepsilon \in (0, \frac{1}{2})$ and $m \in \mathbb{N}$, there exists a constant $\mathbf{K} := \mathbf{K}(m, \varepsilon) > 0$ such that the following holds.
- Let $\mathbf{M} \subset \mathbb{B}(\varepsilon)$ be a compact, connected $\mathbf{1}$-surface of genus zero with m boundary components, $\vec{0} \in \mathbf{M}$, $\partial\mathbf{M} \subset \partial\mathbb{B}(\varepsilon)$ and $\mathbf{M}$ has zero flux. Then $|\mathbf{A}|_{\mathbf{M}}(\vec{0}) \leq \mathbf{K}$.

Steps/Outline of the Proof.

- Arguing by contradiction, suppose that the theorem fails.
- $\exists$ a sequence $\mathbf{M}_n$ of $\mathbf{1}$-surfaces satisfying the hypotheses and $|\mathbf{A}_{\mathbf{M}_n}|(\vec{0}) > n$.
- After replacing $\mathbf{M}_n$ with a subsequence composed by a fixed rotation fixing the origin, when n is sufficiently large we prove:

1. $\mathbf{M}_n$ is closely approximated by one or two vertical helicoids on a small scale around the origin.
2. $\exists$ a sequence of embedded stable minimal disks $E(n) \subset \mathbb{B}(\varepsilon)$ on the mean convex side of $\mathbf{M}_n$, where $E(n)$ contains a 10-sheeted multi-valued graph $\mathbf{E}_n^g$ of small gradient that starts near the origin and extends on a scale proportional to ε.
3. Use the minimal multivalued graph $\mathbf{E}_n^g$ to prove that $\mathbf{M}_n$ contains many $\mathbf{3}$-valued graphs $\mathbf{G}_n(\pm)$ of small gradient that starts near the origin and extend on a scale proportional to ε; $\pm$ refers the sign of the mean curvature as graphs.
4. Use the $\mathbf{3}$-valued graphs $\mathbf{G}_n(\pm) \subset \mathbf{M}(n)$ to obtain a contradiction.

Step 1: $\mathbf{M}_n$ is closely approximated by one or two vertical helicoids on a small scale around the origin.

$\exists$ sub-sequence (we still call) $\mathbf{M}_n$, points $\{p_n \in \mathbf{M}_n\}_n$ with $p_n \to \vec{0}$, numbers $\delta_n > 0$ with $\delta_n \to 0$, s.t. $\widehat{\mathbf{M}}_n = \mathbf{M}_n \cap \mathbb{B}(p_n, \delta_n)$ satisfy:

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3. The sequence of translated and rescaled surfaces
$$\mathbf{\Sigma}_n = \frac{1}{\sqrt{2}} |\mathbf{A}_{\mathbf{M}_n}(p_n)| \cdot (\widehat{\mathbf{M}}_n - p_n)$$
converges with multiplicity **1** or **2** to a properly embedded, nonflat, minimal surface $\mathbf{\Sigma}_\infty$ with
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5. Smooth loops α in $\mathbf{\Sigma}_\infty$ has normal lift $\alpha_n \subset \mathbf{M}_n$ such that the lifted loops converge with multiplicity **1** to α as $n \to \infty$; so the genus is **0** and zero flux condition implies $\mathbf{\Sigma}_\infty$ is a (vertical) helicoid.

Picture from Step 1.

- By Step 1, $\mathbf{M}_n$ contains approximated by a small vertical helicoid near $\vec{0}$. Given $\varepsilon_2 \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$, there exists $\overline{\omega} > 0$ such that for any $\omega_1 > \omega_2 > \overline{\omega}$ there exist an $n_0 \in \mathbb{N}$ and positive numbers r_n, with $r_n = \frac{\sqrt{2}}{|A_{\mathbf{M}_n}|(p_n)}$, such that for any $n > n_0$ the following statements hold.
- For the clarity of exposition we abuse the notations and we let $\mathbf{M} = \mathbf{M}_n$ and $r = r_n$.

1. The disk $\mathbf{M} \cap \mathbf{C}(\omega_1 r, 2\pi(N+2)r)$ contains the origin and we denote it by $\mathbf{M}(\omega_1 r)$.
2. $\mathbf{M}(\omega_1 r) \cap \mathbf{C}(\omega_2 r, 2\pi(N+2)r)$ is also a disk and we denote it by $\mathbf{M}(\omega_2 r)$.
3. $\mathbf{M}(\omega_1 r) \cap [\mathbf{C}(\omega_1 r, 2\pi(N+2)r) - \mathrm{Int}(\mathbf{C}(\omega_2 r, 2\pi(N+2)r))]$, that is
$$\mathbf{M}(\omega_1 r) - \mathrm{Int}(\mathbf{M}(\omega_2 r)),$$
contains two oppositely oriented N-valued graphs u_1 and u_2 over $A(\omega_1 r, \omega_2 r)$.
4. $|\nabla u_i| < \varepsilon_2$, $i = 1, 2$.

Simplifying Assumptions $m = 1$ and multiplicity of convergence is **1**: The planar domain $\mathbf{\Sigma}_n$ is a disk.

In what follows we use the following notation:

- For positive numbers, r, h and t,

  $$\mathbf{C}(r,h,t) = \{(x_1 - t)^2 + x_2^2 \leq r^2, \, |x_3| \leq h\},$$

  which is the vertical cylinder of radius r, height $2h$ and centered at the point $(t, 0, 0)$;

  $$\mathbf{C}(r,h) = \mathbf{C}(r,h,\vec{0}).$$

- For positive numbers $r_1 > r_2 > 0$, we let

  $$A(r_1, r_2) = \{r_2 < \sqrt{x_1^2 + x_2^2} < r_1, \, x_3 = 0\},$$

  which is the annulus in the plane $\{x_3 = 0\}$, centered at the origin with outer radius r_1 and inner radius r_2.

- Consider the intersection of

$$[\mathrm{graph}(u_1) \cup \mathrm{graph}(u_2)] \cap \mathbf{C}\left(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r\right);$$

recall that $\mathbf{C}(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ is the truncated vertical cylinder of radius $\frac{1}{2}$, centered at $(\frac{1}{2} + \omega_2 r, 0, 0)$ with $|x_3| \leq 1$.

- This intersection consists of a collection of disk components

$$\mathbf{\Delta} = \{\mathbf{\Delta}_1, \ldots, \mathbf{\Delta}_{2N}\},$$

and each $\mathbf{\Delta}_i$ is a graph over

$$\{x_3 = 0\} \cap \mathbf{C}(\omega_1 r, 1) \cap \mathbf{C}\left(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r\right),$$

- The mean curvature vectors of consecutive components $\mathbf{\Delta}_i$ and $\mathbf{\Delta}_{i+1}$ have oppositely signed x_3-coordinates.
- Let $\mathcal{F} = \{F(1), F(2), \ldots, F(2N)\}$ be the listing of the components of $M \cap \mathbf{C}(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ that intersect the union of $\mathbf{\Delta}$, and indexed so that $\mathbf{\Delta}_i \subset F(i)$.
- $\mathbf{\Delta}_i$ and $\mathbf{\Delta}_{i+j}$ may be contained in the same component of $M \cap \mathbf{C}(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ and so, $F(i)$ may equal $F(i+j)$.

Property

- Suppose $i \in \{1, 2, \ldots, 2N-1\}$. If $F(i) \cap \partial\mathbf{M} = \emptyset$ and the mean curvature vector of $\mathbf{\Delta}_i \subset F(i)$ is upward pointing, then $F(i) = F(i+1)$.

Property

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- Suppose $i \in \{2, 3, \ldots, 2N\}$. If $F(i) \cap \partial \mathbf{M} = \emptyset$ and the mean curvature vector of $\boldsymbol{\Delta}_i \subset F(i)$ is downward pointing, then $F(i) = F(i-1)$.

Property

- Suppose $i \in \{1, 2, \ldots, 2N-1\}$. If $F(i) \cap \partial\mathbf{M} = \emptyset$ and the mean curvature vector of $\mathbf{\Delta}_i \subset F(i)$ is upward pointing, then $F(i) = F(i+1)$.
- Suppose $i \in \{2, 3, \ldots, 2N\}$. If $F(i) \cap \partial\mathbf{M} = \emptyset$ and the mean curvature vector of $\mathbf{\Delta}_i \subset F(i)$ is downward pointing, then $F(i) = F(i-1)$.

Property

- There are at most $m-1$ indices i, such that $F(i) = F(i+1)$ and $F(i) \cap \partial\mathbf{M} = \emptyset$.
- There exists a simple closed curve $\mathbf{G} \subset \mathbf{M}$ like the drawn on the blackboard that bounds disk $D_{\mathbf{G}} \subset \mathbf{M}$ containing a "large" many sheeted multigraph $\mathbf{G}$ very small gradient over the annulus $A(r\omega_1, r\omega_2)$.

Step 2: $\exists$ a sequence of embedded stable minimal disks $E(n) \subset \mathbb{B}(\varepsilon)$ on the mean convex side of $\mathbf{M}_n$, where $E(n)$ contains a 10-sheeted multi-valued graph $\mathbf{E}_n^g$ of small gradient that starts near the origin and extends on a scale proportional to ε.

See the black board for arguments.

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See the black board for arguments.

Step 3: Use the minimal multi-valued graph $\mathbf{E}_n^g$ to prove that $\mathbf{M}_n$ contains many **3**-valued graphs $\mathbf{G}_n(\pm)$ of small gradient that starts near the origin and extend on a scale proportional to ε; $\pm$ refers the sign of the mean curvature as graphs.

See the black board for arguments.

Theorem (Extrinsic Radius Estimates for H-Disks, Meeks-Tinaglia 2014)

$\exists\ \mathbf{R_0} \geq \pi$ such that every embedded $\mathbf{1}$-disk in $\mathbf{R^3}$ has extrinsic radius $<$ $\mathbf{R_0}$.

Theorem (Extrinsic Radius Estimates for **H**-Disks, Meeks-Tinaglia 2014)

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Proof.

- Suppose that the **extrinsic** radius estimate fails.
- Then there exists a sequence of $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_n, \ldots$ of 1-disks passing through the origin such that for each n, $d_{\mathbb{R}^3}(\vec{0}, \partial\mathbb{D}_n) \geq n+1$.
- Let $\mathbf{\Delta}_n \subset \mathbb{D}_n \cap \mathbb{B}(n)$ be the component containing $\vec{0}$.
- Since $\mathbf{A}_{\mathbf{\Delta}_n} \leq \mathbf{C}$, after replacing by a subsequence, the $\mathbf{\Delta}_n$ converge with multiplicity **1** to a properly immersed strongly Alexandrov embedded **1**-surface $\mathbf{\Sigma}_\infty$ of genus **0** and zero flux.
- The Minimal Element Theorem implies that under a sequence of translations of $\mathbf{\Sigma}_\infty$ limits with multiplicity **1** to a Delaunay surface **D**.
- But a Delaunay surface has **non-zero** flux. $\square$